1. Introduction

This is a talk about $\overline{M}_{0,n}$. So, what is this thing? Firstly, it’s a moduli space, so we’ll spend a bit of time discussing what that means; the idea is that it is a geometric object parameterising some family of geometric objects up to some kind of equivalence (isomorphism, projective equivalence, birational equivalence, etc...). More specifically, we’ll first discuss $M_{0,n}$, which parameterises $n$ ordered, marked points on $\mathbb{P}^1$ up to projective equivalence. We’ll then find a compactification of $M_{0,n}$ which arises as a moduli space of nodal curves, each irreducible component of which is isomorphic to $\mathbb{P}^1$, and this is what we will mean by $\overline{M}_{0,n}$. This is a very nice compactification, since we’ll be able to understand the boundary, i.e. added points, in a very geometric way.

2. So What’s a Moduli Space?

As mentioned above, a moduli space is a geometric object (ideally a variety, sometimes a scheme, hopefully at least a stack...) which “counts” equivalence classes of geometric objects given some kind of equivalence. Phrased differently, it is the “answer” to a moduli problem, i.e. the problem of determining a parameter space of some objects up to some kind of equivalence. In particular, we want there to be a bijection between our moduli space and the equivalence classes of objects we are considering, but we also want this bijection to fit with the geometry; there is, of course, a nice way of expressing that in terms of a universal property, but the details are not important right now.

3. Some Simple Moduli Spaces

A first example with a simple answer is the following moduli problem: Classify all circles in $\mathbb{R}^2$ up to equality (i.e. no circles are equivalent unless they are the same set of points). The moduli space in this case is $\mathbb{R}^2 \times \mathbb{R}_{>0}$, the real plane cross the positive real line. This is because a circle may be completely described as an ordered triple $(a, b, r)$, where the center of the circle is $(a, b)$ and the radius of the circle is $r$. It is then easy to see that the moduli space is $\mathbb{R}^2 \times \mathbb{R}_{>0}$, since $a$ and $b$ can be arbitrary but $r$ must be positive.

Another simple example is that of classifying conics in $\mathbb{P}^2$; since a conic is the vanishing set of a polynomial of the form $ax^2 + by^2 + cz^2 + dxy + exz + fyz$, we see that a conic is determined by the coefficients of this equation, i.e. the 6-tuple $(a, b, c, d, e, f)$. However, two equations which are scalar multiples of each other determine the same vanishing set, so we must mod out by the action of $k^\times$ by scaling; said differently, our moduli space should be $k^6/k^\times$, but this is simply $\mathbb{P}^5$. 
Of these two examples, the second is "nicer," since it is compact. This allows us to consider degenerations of families (for example, the familiar example of hyperbolas degenerating to a pair of crossed lines) without leaving our moduli space. This is a really nice feature that we want a moduli space to have, but not all moduli spaces have it.

4. Okay, so what’s $M_{0,n}$?

We now come to the main topic of the talk. $M_{0,n}$ is a moduli space which parameterises $n$ distinct, ordered, marked points on $\mathbb{P}^1$ up to projective equivalence, i.e. up to the action of $\mathbb{P}GL_2(\mathbb{C})$, which are also called Möbius transformations. This means we can pick any $n$ distinct points on $\mathbb{P}^1$, so we have an ordered $n$-tuple $(p_1, \ldots, p_n)$, and we have the equivalence relation generated by $(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n)$ if there is some $A \in \mathbb{P}GL_2(\mathbb{C})$ which sends $p_i \mapsto q_i$ for each $i$ (and order is important!). (We can also think of this as the space of genus 0 $n$-marked curves up to isomorphism; the 0 in the notation means genus 0, and since there is a unique genus 0 curve up to isomorphism, we may as well assume that we’re working on $\mathbb{P}^1$. Then the way that the isomorphisms move the points around correspond to automorphisms of $\mathbb{P}^1$, which are just Möbius transformations, so we’re back to where we started.)

So what does this space look like? We’ll start with the case $n = 3$. If you play around with $2 \times 2$ matrices a little bit, it’s easy to show that any ordered triple of distinct points $(p_1, p_2, p_3)$ can be sent to the ordered triple $(0, 1, \infty)$, and via a unique Möbius transformation. But since Möbius transformations are automorphisms of projective space, we can invert the map, so if we find the map $\varphi$ sending $(p_1, p_2, p_3) \mapsto (0, 1, \infty)$ and the map $\psi$ sending $(q_1, q_2, q_3) \mapsto (0, 1, \infty)$, then $(p_1, p_2, p_3) \sim (q_1, q_2, q_3)$ via the map $\psi^{-1} \circ \varphi$. Thus any two ordered triples are equivalent, so the moduli space $M_{0,3}$ is just a point.

Since $M_{0,3}$ is not that interesting a space, let’s now look at $M_{0,4}$. Given an ordered quadruple $(p_1, p_2, p_3, p_4) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{\text{diagonals}\}$, we can do the same sort of thing as with three points — we can pick any three of them and send them to $(0, 1, \infty)$. For convenience, let’s do it with the first three. Then applying the unique Möbius transformation, $\lambda$, that does this to all four points sends

$$(p_1, p_2, p_3, p_4) \mapsto (0, 1, \infty, \lambda(p_4))$$

So, what is $\lambda(p_4)$? Well it’s not 0, 1, or $\infty$, since we started out with distinct points and just applied an automorphism. But it can be anything else, since we could have picked the quadruple $(0, 1, \infty, q)$ to start with (forcing $\lambda = \text{id}_{\mathbb{P}^1}$), and we can pick $q$ to be anything distinct from the first three points. With this in mind, we see that an ordered quadruple is completely determined by $\lambda(p_4)$, so our moduli space should be the set of all possible values of $\lambda(p_4)$, i.e. $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Now if we want to move on to $M_{0,5}$ or higher $n$, we can just consider the points four at a time — since

$$(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \iff (p_1, p_2, p_3, p_1) \sim (q_1, q_2, q_3, q_1) \forall i$$

we can see with a little bit of thought that

$$M_{0,n} \cong M_{0,4} \times M_{0,4} \times \ldots \times M_{0,4} \setminus \{\text{diagonals}\}$$

$\ (n - 3)$ copies
Since the diagonals are a closed subset, removing them does not affect the dimension of the space, so we see that
\[ \dim \mathcal{M}_{0,n} = n - 3 \]

All right, so now we have a moduli space for ordered \( n \)-tuples of distinct, marked points on \( \mathbb{P}^1 \). Is everything great? Not quite. Unfortunately, the moduli space we made (while a perfectly good moduli space) is not compact; we can see this since compact spaces contain limits of families, but the limit of the family \( C_t = (0, 1, \infty, t) \) as \( t \to 0 \) should be \( (0, 1, \infty, 0) \). Since the points are not distinct, this is not in our moduli space. What to do?

The first idea that comes to mind is to drop the condition that the points need to be distinct. This seems like a good idea, since then the moduli space would just be \( (\mathbb{P}^1)^{n-3} \), which is a nice, compact space. But when one considers the two families
\[
C_t = (0, 1, \infty, t) \quad D_t = (0, t^{-1}, \infty, 1)
\]
bad things start to happen. These families both make sense when \( t \neq 0 \). Since we can get from \( C_t \) to \( D_t \) by multiplying by the nonzero scalar \( t \), we see that \( C_t = D_t \) for all \( t \neq 0 \). But the families degenerate in different ways at 0; \( C_0 = (0, 1, \infty, 0) \), and \( D_t = (0, \infty, \infty, 1) \). These limits are not projectively equivalent, but each other member of the two families are equivalent. This basically means that the way we tried to compactify things lost some geometric information. We’ll have to try something else.

5. \( \overline{M}_{0,n} \) to the Rescue

The something else we need to come up with is what we will call \( \overline{M}_{0,n} \). We’ve seen that bad things happen when we let points coincide, so instead let’s say the limit of two points colliding is another copy of \( \mathbb{P}^1 \) (which we’ll call a new twig) with the two points on it. In pictures:

![Diagram showing two points colliding and forming a new twig]

This has a nice intuitive interpretation: when two points collide, they simply pop out of the space on a new twig. In fact, we don’t even need to worry about where to put the points, since we only have three points (the two points colliding plus the node at which the new twig is attached) — any configuration is equivalent to any to any other. (We can also deal with more complicated collisions and things don’t get too bad). We’ll need a bit more language to talk about all the new objects we’ll have to worry about:

**Definition.** A tree of projective lines is a variety which:

- is a tree in the graph theoretic sense, i.e. it is connected and has no loops,
- has each irreducible component (called a twig) isomorphic to \( \mathbb{P}^1 \),
- has no more than two twigs crossing at any point.
An **n-marked tree of projective lines** is a tree of projective lines with \( n \) chosen points, subject to the condition that each twig must have at least three "special points," i.e. either marks or nodes.

So \( \overline{M}_{0,n} \) will be the moduli space for these \( n \)-marked trees of projective lines. Are we justified in calling this a compactification of \( M_{0,n} \), though? Yes, because of our condition on the marks; \( n \)-marked trees of projective lines as we’ve defined them are the ones that can arise as degenerations of configurations in \( M_{0,n} \).

6. **The Boundary (i.e. What exactly did we add?)**

So \( \overline{M}_{0,n} \) contains \( M_{0,n} \) as a dense subvariety, since all the new points we’ve added correspond to degenerations of configurations in \( M_{0,n} \). And it turns out that there is a very nice stratification of the boundary, i.e. added points. For example, a point in the boundary might correspond to a tree like this:

```
1 2 3 4
  5 6
   7
```

We could also consider the subvariety of all points corresponding to curves of a certain form, which we will denote by a tree in brackets, i.e:

```
1 2 3 4
  5 6
   7
```

The presence of brackets indicates that we mean a whole subset determined by the varying positions of the points. These subsets are called **boundary strata**, since we can express the whole boundary as a union of these subvarieties in a nice way.

A natural question to ask is, "What is the dimension of one of these boundary strata?" In order to answer that, we’ll need a fact about automorphisms of \( n \)-marked trees of projective lines – an automorphism of a tree is just a bunch of automorphisms of the twigs, “glued together.” This means that we can think of an automorphism as happening on each twig separately. This, in turn, means that on each twig, we can move around three points however we want by Möbius transformations, but once we’ve chosen the positions of those points, the rest are determined. This allows us to put “coordinates” on a boundary stratum – we can just use a product of smaller moduli spaces. In the example above, we can see that this stratum corresponds to \( \overline{M}_{0,8} \times \overline{M}_{0,3} \times \overline{M}_{0,3} \times \overline{M}_{0,3} \) (recall that nodes count as marked points on both twigs). The dimension of this product is easy to compute, however, since it is just the sum of the dimensions of each factor, and the dimension of the space of possible coordinates is the same as the dimension of the space. Thus our example has dimension 5, since all but one factor has dimension 0.

In general, every time we add a node to a boundary stratum, we have points on a new twig which can be moved around by automorphisms, so the dimension (which counts degrees of freedom
of points after automorphisms) should drop. In fact, the codimension of a boundary stratum is always equal to the number of nodes of the tree, as one can see in the example we were using: the dimension is 5 in a dimension \( 11 - 3 = 8 \) space. Every time points "crash together" and we get a new node, we are effectively working in a special case of the old configuration, so it is natural that the dimension should drop. Also, since we can think of a configuration with an extra node due to points crashing as a special case, it is natural that we should get certain inclusions among the boundary strata, such as

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{pmatrix}
\subseteq
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{pmatrix}
\]

7. Intersection Theory on \( \overline{M}_{0,n} \)

An interesting result of Keel shows that all the cohomology classes of \( \overline{M}_{0,n} \) (say elements of \( H^k(\overline{M}_{0,n}) \)) can be thought of as codimension \( k \) boundary strata. Thus to completely understand the intersection theory of \( \overline{M}_{0,n} \), all we need to do is understand how different boundary strata intersect. Luckily, intersections among boundary strata behave very nicely. For example,

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{pmatrix} = 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{pmatrix}
\]

What we see in this example is that the curves in the intersection of the two strata are the curves which are "specialised" in way required by both strata, i.e. curves in which both the 2nd and 3rd marks have collided and the 5th, 6th, and 7th marks have collided. This is a nice, transverse intersection, since it's easy to see that the codimensions add properly (since the number of nodes adds up correctly). In general, whenever one has a transverse intersection, we can just think of the intersection in this way, as the set of curves with all the degenerations of the curves being intersected.

However, the codimensions do not always add properly, corresponding to non-transverse intersections. This is most obvious when taking an intersection of a boundary stratum with itself, since it is clear that intersecting as above will only return the original stratum. This winds up not being the correct notion of intersection, so we'll have to think of something a bit more subtle in this case. We need to keep in mind that we are really interested in intersecting cohomology classes, so the intuition of how to fix the problem is to say, "Well, transversality is a generic sort of condition, so we should pick different representatives of the cohomology classes which are transverse and intersect those. The correct intersection of the two classes should be the class of that intersection."

Unfortunately, it's not clear what "wiggling" a boundary stratum should do to it. Fortunately, there are wonderful things called \( \psi \)-classes which we can use to fix our non-transverse intersections. We'll illustrate with an example; say we want to compute the following intersection.
Since the left boundary stratum is a degeneration of the right one, the set-theoretic intersection will just be the first stratum, so it is clear that the intersection is not transverse. The proper way to "wiggle" into transversality is to first intersect the stratum with higher codimension with the $\psi$-class at all the nodes which both strata have in common; in our case, if we label the common node by •, then the common node is the one marked on each stratum:

We next look locally at the intersection at the marked node, separating all the twigs but remembering the points at which they were attached (in the figure we label points of attachment similarly):

Now the $\psi$-class with which we want to intersect is $\psi_{\bullet}$, which is defined to be $-\psi_{\bullet} - \psi_{\circ}$, where $\psi_{\bullet}$ is the $\psi$-class at the mark • on the left twig and the other class is defined similarly. Now we never defined what these $\psi$-classes are, but in this case we have $\psi_{\bullet} = 54$ and $\psi_{\circ} = 0$. Gluing things back together, this means that we must first intersect with the $\psi$-class $\psi_{\bullet}$.

Since this $\psi$-class is contained in both strata we are intersecting, we see that the intersection will be precisely the $\psi$-class.

These intersections seem complex at first, but they are actually quite straightforward. Similarly, actually defining and computing all possible $\psi$-classes looks hard and intimidating at first, but it is neither.
8. A Correspondence with Admissible Covers

First, by admissible cover we will mean a degree 2 cover of a \( n \)-pointed tree of projective lines (i.e. a map \( \varphi : C \rightarrow T \)) which is ramified and singular over the marks on the tree and smooth elsewhere. Basically, these are degree 2 "nodal covers," which makes sense since we are considering nodal curves. It turns out that there is a moduli space for all such covers of a given genus, denoted by \( \text{Adm}_g \). Such a cover is determined up to isomorphism by its ramification points on the base, so this gives an "almost-isomorphism" \( \pi : \text{Adm}_g \rightarrow \overline{M}_{0,2g+2} \), the map which sends a cover to the corresponding configuration of marks. (What does that mean? This is actually a degree \( \frac{1}{2} \) map in some stack-theoretic sense, because admissible covers have a non-trivial automorphism. It’s a bit technical and not essential.) Since we understand the intersection theory of \( \overline{M}_{0,2g+2} \) very well, we can use this correspondence to study the intersection theory of \( \text{Adm}_g \).