COBORDISMS AND THE MONOPOLE SURGERY EXACT TRIANGLE

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Abstract. Given a cobordism W_0 between two closed oriented 3-manifolds, Y and Y_0, we give a relation between the maps on Monopole Floer homology, $\check{H}(W_0) : \check{H}(Y) \to \check{H}(Y_0)$ and maps from $\check{H}(Y)$ into $\check{H}(Y_1)$ and $\check{H}(Y_2)$, where Y_1 and Y_2 are 3-manifolds which fit into certain surgery exact triples with Y_0. Further, we show that the maps to $\check{H}(Y_1)$ and $\check{H}(Y_2)$ are natural in the sense that they are chain homotopic to maps induced by cobordisms. Similar statements hold for the $\widehat{H}$ and $\overline{H}$ theories.

1. Introduction

A surgery exact triangle was originally described by Floer in [5] for his instanton homology for 3-manifolds. Since then, both P. Ozvath and Z. Szabo's Heegaard Floer homology and P. Kronheimer and T. Mrowka's Monopole Floer homology have been shown to have similar surgery exact triangles. (in [15] and [8], respectively) Among other uses, the exact triangle is used in the proofs that the Euler characteristics of both the Heegaard and Monopole Floer homologies of zero surgery of a knot in S^3 is the Alexander polynomial of the knot. Here, the surgery exact triangle plays the role of the Conway skein relation. Corresponding results hold in the 4-manifold world where in [4], R. Fintushel and R. Stern's knot surgery technique used J. Morgan, T. Mrowka and Z. Szabo's formula [13] to prove that a skein relation for the Seiberg-Witten invariant holds.

Geometrically, the surgery exact triangle is constructed as follows: Let Y be a closed, oriented 3-manifold containing K, a smooth knot. Suppose that we have three simple closed curves $\gamma_0, \gamma_1, \gamma_2$ in $\partial\left(Y \backslash N(K)\right) \cong T^2$ with

$$\gamma_0 \cdot \gamma_1 = \gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_0 = -1.$$

Let Y_i be obtained by a Dehn filling of $Y \backslash N(K)$ along γ_i. We can think of $K \subset Y$ as sitting inside each of the Y_i as the core of the Dehn fillings.

Cobordisms W^n_{n+1} are obtained by taking $Y_n \times [0,1]$ and adding a 2-handle attached along $K \times \{1\}$ with framing given by the push off γ_{n+1}. The boundary of W^n_{n+1} is oriented so that $\partial W^n_{n+1} = -Y_n \sqcup Y_{n+1}$.

Theorem 2.4 of [8] states that the sequence

$$(1) \qquad \cdots \xrightarrow{\check{H}(W_{i+2})} \check{H}_\bullet(Y_i) \xrightarrow{\check{H}(W_i)} \check{H}_\bullet(Y_{i+1}) \xrightarrow{\check{H}(W_{i+1})} \check{H}_\bullet(Y_{i+2}) \xrightarrow{\check{H}(W_{i+2})} \cdots$$

is exact with $\mathbb{Z}/2\mathbb{Z}$ coefficients, *which we will assume throughout.* The same is true for the $\widehat{H}$ and $\overline{H}$ theories.

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Since the above sequence is exact, the group $\check{H}_\bullet(Y_0)$ is isomorphic to the direct sum of the kernel and cokernel of $\check{H}(W_1)$. More precisely, $\check{H}_\bullet(Y_0)$ is quasi-isomorphic to the mapping cone of $\check{H}(W_1)$. On the chain level, the mapping cone has chain groups equal to $\check{C}(Y_1) \oplus \check{C}(Y_2)$ together with differential:[1]

$$\check{\partial}_{MC} = \begin{pmatrix} \check{\partial}_1^1 & 0 \\ \check{m}_2^1 & \check{\partial}_2^2 \end{pmatrix}.$$

Such a mapping cone gives rise to a spectral sequence, with E_0 page equal to $\check{C}(Y_1) \oplus \check{C}(Y_2)$ and with E_0 differential $\check{\partial}^0_{MC} = \check{\partial}_1^1 \oplus \check{\partial}_2^2$. Then the E_1 page is equal to $\check{H}(Y_1) \oplus \check{H}(Y_2)$ and has differential $\check{\partial}^1_{MC} = \check{H}(W_2^1)$. This can be generalized to surgery on multiple link components and is described in [1].

The goal of this paper is to prove Theorem 1.

Theorem 1. *Suppose W_0 is a compact cobordism with oriented boundary $\partial W = -Y \sqcup Y_0$ and that K is a smoothly embedded copy of S^1 in Y_0 which bounds a punctured torus $\mathring{T}$ in W_0. For each orientation on K (equivalently on $\mathring{T}$) there is a surgery exact triangle obtained by surgery on $K \subset Y_0$ with 3-manifold terms $\{Y_n\}_{n \in \mathbb{Z}/3\mathbb{Z}}$. Then, under the quasi-isomorphism*

$$(\check{C}(Y_0), \check{\partial}_0^0) \cong (\check{C}(Y_1) \oplus \check{C}(Y_2), \check{\partial}_{MC})$$

given by the surgery exact triangle, there are cobordisms W_1 and W_2 with $\partial W_1 = -Y \sqcup Y_1$ and $\partial W_2 = -Y \sqcup Y_2$ such that the chain level maps

$$\check{m}_0 : \check{C}(Y) \to \check{C}(Y_0)$$

and

$$\check{m}_1 \oplus (q(U_\dagger)\check{m}_2) : \check{C}(Y) \to \check{C}(Y_1) \oplus \check{C}(Y_2),$$

where $q(U_\dagger) = \sum_{l \geqslant 0} U_\dagger^{l(l+3)/2}$ is an invertable power series in $U_\dagger$, satisfy

$$p(U_\dagger)(\check{m}_0)_* = (\check{m}_1 \oplus (q(U_\dagger)\check{m}_2))_*$$

where $p(U_\dagger) = \sum_{k \geqslant 0} U_\dagger^{k(k+1)/2}$ is an invertable formal power series in $U_\dagger$. Similar statements hold for the $\widehat{H}$ and $\overline{H}$ theories.

Computationally, the purpose of the theorem will be to determine the cobordism map for W in terms of cobordisms of lower "complexity" much in the same manner as computing the Alexander polynomial by using the skein relation to reduce a knot to unknotted components. For example, if W_1 and W_2 have positive scalar curvature, and hence only reducible solutions, we can compute the maps on Floer homology using only topological information.

During preparation of this paper, two relevant results were announced. The first is a series of papers by Li, Kutluhan, and Taubes showing the equivalence of the Heegaard Floer and Monopole Floer theories for 3-manifolds. (Currently [9], [10], and [11] are available. MORE NOW?) The second is [12] which proves similar results to ours in the world of Heegaard Floer homology.

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[1] Where convenient, we will adopt the general convention that for maps from cobordisms on the chain level, superscript labels denote incoming 3-manifolds and subscript labels denote outgoing 3-manifolds. $\check{m}_0 = \check{m}(W_0)$, $\check{m}_2^1 = \check{m}(W_2^1)$

2. Overview of Monopole Floer Homology

We will now give a "user's guide" to some important features of ($\mathbb{Z}/2\mathbb{Z}$-coefficient) Monopole Floer homology. For particulars, we defer readers to the detailed description of the theory in [7].

2.1. **Critical points and chain groups for 3-manifolds.** Let Y be a closed oriented 3-manifold with fixed data: a Riemannian metric g and closed 2-form ω. Let $\mathcal{B}^\sigma(Y)$ be the space consisting of 4-tuples $(\mathfrak{s}, B, s, \phi)$ where $\mathfrak{s}$ is a *spin*$^{\mathbb{C}}$ structure on Y with corresponding spinor bundle S, B is a *spin*$^{\mathbb{C}}$ connection on S, s is a non-negative real number, and ϕ is a section of S with unit norm. This configuration space splits into connected components $\bigsqcup_{\mathfrak{s}\in spin^{\mathbb{C}}(Y)} \mathcal{B}^\sigma(Y;\mathfrak{s})$. The boundary, $\partial\mathcal{B}^\sigma(Y;\mathfrak{s})$, is the set of *reducible configurations* – the set of triples with $s = 0$. The homotopy type of each $\mathcal{B}^\sigma(Y;\mathfrak{s})$ is the same as that of $T^{b_1(Y)} \times \mathbb{R}^{\geqslant 0} \times \mathbb{C}P^\infty$.

There is a "vector field" on $\mathcal{B}^\sigma(Y)$ coming from the gradient of the perturbed Chern-Simons-Dirac functional which is given by the following equations:

$$\frac{1}{2}\frac{d}{dt}B^t = -\frac{1}{2} * (F_{B^t} - 4\omega) - s^2\rho^{-1}(\phi\phi^*)_0 \tag{2}$$

$$\frac{d}{dt}s = -\Lambda(B, s, \phi)s \tag{3}$$

$$\frac{d}{dt}\phi = -D_B\phi + \Lambda(B, s, \phi)\phi \tag{4}$$

where $\Lambda(B, s, \phi) = \langle\phi, D_B\phi\rangle_{L^2(Y)}$. For each $\mathfrak{s} \in spin^{\mathbb{C}}(Y)$ we get three sets:

$$\mathfrak{C}^o(Y;\mathfrak{s}), \mathfrak{C}^s(Y;\mathfrak{s}), \text{ and } \mathfrak{C}^u(Y;\mathfrak{s})$$

which are the zeros of the above vector field over $\mathcal{B}^\sigma(Y;\mathfrak{s})$. The intersection with the zero section is not transverse in general and we will need to perturb the vector field.

The first set, $\mathfrak{C}^o(Y;\mathfrak{s})$, are the *irreducible* zeros. These points lie in the interior of $\mathcal{B}^\sigma(Y;\mathfrak{s})$.

The sets $\mathfrak{C}^s(Y;\mathfrak{s})$ and $\mathfrak{C}^u(Y;\mathfrak{s})$ comprise the reducible zeros of the vector field and lie in $\partial\mathcal{B}^\sigma(Y;\mathfrak{s})$. The $\mathfrak{C}^s(Y;\mathfrak{s})$ and $\mathfrak{C}^u(Y;\mathfrak{s})$ are the *boundary-stable* and *boundary-unstable* zeros, respectively.

Assume that $\omega \equiv 0$. Then when $c_1(\mathfrak{s})$ is non-torsion, the reducible sets are empty. In the case that $c_1(\mathfrak{s})$ is torsion then (after a perturbation which depends on a choice of Morse function f on the torus of flat connections $T^{b_1(Y)}$) we can assume that $\mathfrak{C}^s \cong \mathbb{Z}^{\geqslant 0} \times \mathrm{Crit}(f)$ and $\mathfrak{C}^u \cong \mathbb{Z}^{<0} \times \mathrm{Crit}(f)$. Here, the $\mathbb{Z}$ parametrizations of $\mathbb{Z} \times \{B\} \subset \mathbb{Z} \times \mathrm{Crit}(f)$ is an order-preserving $\mathbb{Z}$ parametrizations of the eigenvalues of the Dirac operator D_B with $0 \in \mathbb{Z}$ corresponding to the least positive eigenvalue.

Let $C^o(Y;\mathfrak{s}), C^s(Y;\mathfrak{s})$, and $C^u(Y;\mathfrak{s})$ be the $\mathbb{Z}/2\mathbb{Z}$ vector spaces generated by $\mathfrak{C}^o(Y;\mathfrak{s}), \mathfrak{C}^s(Y;\mathfrak{s})$ and $\mathfrak{C}^u(Y;\mathfrak{s})$. Define the chain groups

$$\begin{aligned}
\check{C}(Y;\mathfrak{s}) &= C^o(Y;\mathfrak{s}) \oplus C^s(Y;\mathfrak{s}),\\
\hat{C}(Y;\mathfrak{s}) &= C^o(Y;\mathfrak{s}) \oplus C^u(Y;\mathfrak{s}), \text{ and}\\
\overline{C}(Y;\mathfrak{s}) &= C^s(Y;\mathfrak{s}) \oplus C^u(Y;\mathfrak{s})
\end{aligned}$$

for the *to*, *from*, and *reducible* theories in the *spin*$^{\mathbb{C}}$ structure $\mathfrak{s}$. Finally, define $\check{C}(Y)$, $\hat{C}(Y)$, $\overline{C}(Y)$ by taking the direct sum over *spin*$^{\mathbb{C}}$ structures on Y.

2.1.1. *Gradings.* The monopole Floer chain groups admit a grading by $J(Y)$, the set of homotopy classes of 2-plane fields over Y. Since a 2-plane field determines a *spin*$^{\mathbb{C}}$ structure, $J(Y)$ splits into components $J(Y;\mathfrak{s})$. The set $J(Y)$ admits a $\pi_3(S^2) \cong \mathbb{Z}$ action which preserves, and is transitive on, each $J(Y;\mathfrak{s})$. (We orient the action of $\mathbb{Z}$ by letting the action of $+1$ correspond to connect summing with the lift of TS^2 to S^3 by the Hopf fibration.) For each $\mathfrak{s} \in spin^{\mathbb{C}}(Y)$, the stabilizer of the $\mathbb{Z}$ action is $d\mathbb{Z}$, where d is the divisibility of $c_1(\mathfrak{s})$ in $H^2(Y;\mathbb{Z})/\text{torsion}$ – an even number. If $c_1(\mathfrak{s})$ is torsion, we let $d = 0$.

The grading by $J(Y;\mathfrak{s})$ gives a relative $\mathbb{Z}/d\mathbb{Z}$ grading. Suppose that $\mathbf{a}, \mathbf{b} \in \mathfrak{C}(Y;\mathfrak{s})$ and $z \in \pi_0\Omega(\mathbf{a}, \mathbf{b}; \mathcal{B}^\sigma(Y;\mathfrak{s}))$ be a homotopy class of paths joining $\mathbf{a}$ to $\mathbf{b}$ in $\mathcal{B}^\sigma(Y;\mathfrak{s})$. Then the relative grading

$$\mathrm{gr}_z(\mathbf{a}, \mathbf{b})$$

is the index of the linearized Seiberg-Witten operator over z. See Definition 14.4.4 of [7]. This relative grading is additive in the sense that:

$$\mathrm{gr}_z(\mathbf{a}, \mathbf{c}) = \mathrm{gr}_{z_0}(\mathbf{a}, \mathbf{b}) + \mathrm{gr}_{z_1}(\mathbf{b}, \mathbf{c})$$

when, under concatenation of paths, $z = z_1 \circ z_0$. For the closed loop z_u corresponding to $u \in H^1(Y;\mathbb{Z})$,

$$\mathrm{gr}_{z_u}(\mathbf{a}, \mathbf{a}) = (u \cup c_1(\mathfrak{s}))[Y].$$

There is another flavor of the relative grading when both of $\mathbf{a}$ and $\mathbf{b}$ are reducible:

$$\overline{\mathrm{gr}}_z(\mathbf{a}, \mathbf{b}) = \mathrm{gr}_z(\mathbf{a}, \mathbf{b}) - o(\mathbf{a}) + o(\mathbf{b})$$

where $o(\mathbf{a}) = 0$ if $\mathbf{a}$ is boundary stable and $o(\mathbf{a}) = 1$ if $\mathbf{a}$ is boundary unstable.

There is also an absolute mod 2 grading $\mathrm{gr}^{(2)}$. The mod 2 grading $\mathrm{gr}^{(2)}(\mathbf{a})$ comes from computing the index of an operator over a path connecting $\mathbf{a}$ to some reducible configuration, not necessarily a solution. See Section 22.4 of [7] for details. This also comes with a reducible version

$$\overline{\mathrm{gr}}^{(2)}(\mathbf{a}) = \begin{cases} \mathrm{gr}^{(2)}(\mathbf{a}) & \text{if } \mathbf{a} \in \mathfrak{C}^s \\ \mathrm{gr}^{(2)}(\mathbf{a}) - 1 & \text{if } \mathbf{a} \in \mathfrak{C}^u \end{cases}$$

Suppose that W is a cobordism with $\partial W = -Y_- \sqcup Y_+$ and let $\iota(W)$ be defined by

$$\iota(W) = \frac{\chi(W) + \sigma(W) + b_1(Y_+) - b_1(Y_-)}{2}$$

The maps resulting from W will have even degree exactly when $\iota(W)$ is even. [2]

TALK ABOUT $\mathrm{gr}^{\mathbb{Q}}$

For each $j \in J(Y;\mathfrak{s})$, there are homogeneous subgroups

$$\begin{aligned} \check{C}_j(Y, \mathfrak{s}) &\subset \check{C}_*(Y;\mathfrak{s}) \\ \hat{C}_j(Y, \mathfrak{s}) &\subset \hat{C}_*(Y;\mathfrak{s}) \\ \overline{C}_j(Y, \mathfrak{s}) &\subset \overline{C}_*(Y;\mathfrak{s}). \end{aligned}$$

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[2] Move to section on cobordisms?

These give us internal direct sums:

$$\begin{aligned}\bigoplus_{j\in J(Y;\mathfrak{s})} \check{C}_j(Y,\mathfrak{s}) &\cong \check{C}_*(Y,\mathfrak{s})\\ \bigoplus_{j\in J(Y;\mathfrak{s})} \hat{C}_j(Y,\mathfrak{s}) &\cong \hat{C}_*(Y,\mathfrak{s})\\ \bigoplus_{j\in J(Y;\mathfrak{s})} \overline{C}_j(Y,\mathfrak{s}) &\cong \overline{C}_*(Y,\mathfrak{s})\end{aligned}$$

In the case that $c_1(\mathfrak{s})$ is torsion, it is often necessary to use a completion of these groups. The completions, $\check{C}_\bullet(Y)$, $\hat{C}_\bullet(Y)$ and $\overline{C}_\bullet(Y)$, are taking with respect to a filtration decreasing in $J(Y)$.

2.1.2. *Module structure.* The chain groups of monopole Floer homology are modules over the opposite ring of the configuration space's ordinary cohomology.

$$\mathbb{A}_\dagger(Y) = H^*(\mathcal{B}^\sigma(Y;\mathfrak{s}))^{opp} = \Lambda(H_1(Y)/\text{torsion}) \otimes \mathbb{Z}[U_\dagger]$$

via a $\cap$ product. Actions of an element $\gamma_\dagger \in H_1(Y)/\text{torsion}$ and of $U_\dagger$ are of degree -1 and -2 respectively. The action of $\mathbb{Z}[U_\dagger]$ extends to an action of the ring of formal power series, $\mathbb{Z}[[U_\dagger]]$, on the $\bullet$ versions.

This action is well understood for the $\overline{C}$ groups in general and for all three theories in the case that Y admits a flat or positive scalar curvature metric. See Section 35 of [7].

2.1.3. *Local Coefficients.* Omit?

2.1.4. *Positive Scalar Curvature.* Weitzenbock. Everything reducible.

2.1.5. *Cohomology and Duality.* Change in orientation for Y gives cohomology, switches $\check{C}$ and $\hat{C}$. Switches unstable and stable crit pts.

2.2. **Moduli spaces, Homology and Cobordisms, Families.** Let W be a compact oriented 4-manifold with boundary $\partial W = Y = \coprod_\alpha Y_\alpha$ oriented by the outward normal. Here we are thinking of W as a cobordism *from* the empty set *to* Y. Later it will be easier to reverse the orientation on some of the boundary components and view them as incoming.

Suppose W comes equipped with fixed data: a Riemannian metric g which is cylindrical near ∂W and a closed 2-form ω. Once this data is fixed, we obtain $M(W)$, the moduli space of monopoles. These are the solutions to:

$$\frac{1}{2}\rho_W(F^+_{A^t} - 4\omega^+) - s^2(\phi\phi^*)_0 = 0 \tag{5}$$

$$D^+_A\phi = 0 \tag{6}$$

[3] over the configuration space $\coprod_{\mathfrak{t}\in spin^{\mathbb{C}}(W)} \mathcal{B}^\sigma(W;\mathfrak{t})$ consisting of 4-tuples of $\mathfrak{t} \in spin^{\mathbb{C}}(W)$ with corresponding spinor bundles S^+, S^-, A a $spin^{\mathbb{C}}$ connection, s a non-negative real number, and ϕ a section of S^+ of unit norm. The moduli space splits over $spin^{\mathbb{C}}(W)$ structures as $M(W) = \coprod_{\mathfrak{t}\in spin^{\mathbb{C}}(W)} M(W;\mathfrak{t})$. Taking the intersection with $\partial\mathcal{B}^\sigma(W;\mathfrak{t})$, we obtain $M^{red}(W;\mathfrak{t})$ the moduli space of reducible monopoles. Each of these spaces is infinite dimensional, unlike the case of a closed 4-manifold.

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[3] Need perturbations. Cylinder functions KM 11.1

Instead of examining the moduli spaces over W, we will usually use the manifold W^*:

$$W^* = W \bigcup_{Y = Y \times \{0\}} (Y \times [0, \infty))$$

where the infinite ends are given the product metric. In this non-compact case, $M(W^*; \mathfrak{t})$ and $M^{red}(W^*; \mathfrak{t})$ are the *finite energy* monopole and reducible monopole moduli spaces. (see Definition 4.5.4 of [7])

Suppose that we choose $\mathbf{b}_\alpha \in \mathfrak{C}(Y_\alpha)$ for each boundary component Y_α. Let $\mathbf{b}$ be the product of the $\mathbf{b}_\alpha$.[4] Then we can consider the moduli spaces $M(\mathbf{a}, W^*, \mathbf{b}; \mathfrak{t})$ and $M^{red}(W^*, \mathbf{b}; \mathfrak{t})$: the subsets of $M(W^*; \mathfrak{t})$ and of $M^{red}(W^*; \mathfrak{t})$ which are asymptotic to $\mathbf{b}_\alpha$ on each of the infinite ends $Y_\alpha \times [0, \infty)$. Each of these moduli spaces further decompose as

$$\begin{aligned} M(W^*, \mathbf{b}; \mathfrak{t}) &= \coprod_{z \in \pi_0(\mathbf{b})} M_z(W^*, \mathbf{b}; \mathfrak{t}) \\ M^{red}(W^*, \mathbf{b}; \mathfrak{t}) &= \coprod_{z \in \pi_0(\mathbf{b})} M_z^{red}(W^*, \mathbf{b}; \mathfrak{t}) \end{aligned}$$

where $\pi_0(\mathbf{b})$ denotes the set of connected components of subset of the configuration space $\mathcal{B}^\sigma(W^*, \mathbf{b}; \mathfrak{t})$ consisting of elements of $\mathcal{B}^\sigma(W^*; \mathfrak{t})$ which are asymptotic to $\mathbf{b}$ on the infinite ends of W^*.

With generic data, the spaces $M_z(W^*, \mathbf{b}; \mathfrak{t})$ and $M_z^{red}(W^*, \mathbf{b}; \mathfrak{t})$ are regular – they are finite dimensional manifolds possibly with boundary. More specifically, we paraphrase Proposition 24.4.3 of [7]:

Proposition 2. *Suppose that the moduli space $M(W^*, \mathfrak{t}; \mathbf{b})$ is non-empty and regular. Then the moduli space is:*

1. *a smooth manifold consisting only of irreducibles, if any $\mathbf{b}_\alpha$ is irreducible.*
2. *a smooth manifold consisting only of reducibles, if any $\mathbf{b}_\alpha$ is reducible and boundary unstable.*
3. *a smooth manifold with (possibly empty) boundary if all the $\mathbf{b}_\alpha$ are reducible and boundary stable. In this case the boundary consists of the reducible solutions of the moduli space.*

In the case that the moduli spaces are of dimension zero, they are compact. Call a solution in $M(W^*, \mathbf{b}; \mathfrak{t})$ *boundary-obstructed* (of corank $c > 0$) if $c + 1$ of the $\mathbf{b}_\alpha$ are boundary unstable.

The decomposition of $M(W^*, \mathbf{b}; \mathfrak{t})$ into components $M_z(W^*, \mathbf{b}; \mathfrak{t})$ is a decomposition by dimension, as seen in Proposition 24.4.6 of [7]:

Proposition 3. *If the moduli space $M_z(W^*, \mathbf{b}; \mathfrak{t})$ is non-empty and regular, its dimension is $\mathrm{gr}_z(W; \mathbf{b})$, except in the boundary obstructed case. In the boundary obstructed of corank c case, the dimension is $\mathrm{gr}_z(W; \mathbf{b}) + c$*

For a moment, consider the case that $Y_\pm$ are two 3-manifolds with fixed orientations and that $\partial W = -Y_- \sqcup Y_+$. i.e. That W is a cobordism from Y_- to Y_+. Then as we have seen in section 2.1.5, boundary stable and unstable critical points reverse for Y_- with its original fixed orientation versus its orientation as part of the boundary of W. For $\mathbf{a} \in \mathfrak{C}(Y_-), \mathbf{b} \in \mathfrak{C}(Y_+)$, let $M_z(\mathbf{a}, W^*, \mathbf{b}; \mathfrak{t}) = M_z(W^*, \mathbf{b} \times \bar{\mathbf{a}})$. Then Proposition 2 becomes

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[4] pg 461 $\mathcal{B}^\sigma(Y, \mathfrak{s}) = \prod_\alpha \mathcal{B}^\sigma(Y_\alpha, \mathfrak{s}_\alpha)$. etc.

Proposition 4. *Suppose that* Y_- *and* Y_+ *are connected and that* $M_z(\mathbf{a}, W^*, \mathbf{b}; \mathfrak{t})$ *is regular, and let* $d = \mathrm{gr}_z(\mathbf{a}, \mathbf{b})$*. Then the moduli space is:*

- *a smooth* d*-manifold consisting entirely of irreducible solutions if either* $\mathbf{a}$ *or* $\mathbf{b}$ *is irreducible.*
- *a smooth* d*-manifold with boundary if* $\mathbf{a}, \mathbf{b}$ *are boundary-unstable and boundary-stable, respectively. The boundary consists of the reducible solutions.*
- *a smooth* d*-manifold consisting entirely of reducibles if* $\mathbf{a}, \mathbf{b}$ *are either both boundary-stable or boundary-unstable.*
- *a smooth* $(d+1)$*-manifold consisting entirely of reducibles in the boundary-obstructed case.*

The second and fourth cases justify the difference between the regular grading gr_z and the reducible grading $\overline{\mathrm{gr}}_z$: When both $\mathbf{a}$ and $\mathbf{b}$ are boundary stable or unstable $\mathrm{gr}_z(\mathbf{a}, \mathbf{b})$ and $\overline{\mathrm{gr}}_z(\mathbf{a}, \mathbf{b})$ agree. When $\mathbf{a}$ is boundary unstable and $\mathbf{b}$ is boundary stable, we are in the second case and the reducible moduli space (the boundary of the full space) is a manifold of dimension $d-1$. Here $\overline{\mathrm{gr}}_z(\mathbf{a}, \mathbf{b}) = \mathrm{gr}_z(\mathbf{a}, \mathbf{b}) - 1 = d-1$. In the boundary obstructed case, $\mathbf{a}$ is boundary stable and $\mathbf{b}$ is boundary unstable, and we are in the fourth case. Here the moduli space of reducibles is of dimension $d+1$ and $\overline{\mathrm{gr}}_z(\mathbf{a}, \mathbf{b}) = \mathrm{gr}_z(\mathbf{a}, \mathbf{b}) + 1 = d+1$.

When $\mathbf{a}$ is boundary stable and $\mathbf{b}$ is irreducible, $M(\mathbf{a}, W^*, \mathbf{b})$ is empty. (As it must both entirely consist of reducibles and of irreducibles.) Similarly, $M(\mathbf{a}, W^*, \mathbf{b})$ is empty when $\mathbf{a}$ is irreducible and $\mathbf{b}$ is boundary unstable.

2.2.1. *Moduli space on cylinders.* Consider the product cobordism $Y \times \mathbb{R}$ with the product metric. (As the set of *spin*$^{\mathbb{C}}$ structures for $Y \times \mathbb{R}$ is the same of as that of Y, we will omit them in the notation.) Each nontrivial (either $\mathbf{a} \neq \mathbf{b}$ or z is non-trivial) moduli space $M_z(\mathbf{a}, Y \times \mathbb{R}, \mathbf{b})$ or $M_z^{red}(\mathbf{a}, Y \times \mathbb{R}, \mathbf{b})$ comes with a fixed point free $\mathbb{R}$ action by reparametrization in the $\mathbb{R}$ coordinate. Let

$$\breve{M}_z(\mathbf{a}, \mathbf{b}) = M_z(\mathbf{a}, Y \times \mathbb{R}, \mathbf{b})/\mathbb{R} \text{ and } \breve{M}_z^{red}(\mathbf{a}, \mathbf{b}) = M_z^{red}(\mathbf{a}, Y \times \mathbb{R}, \mathbf{b})/\mathbb{R}$$

be the unparametrized moduli spaces on the cylinder.

An *unparametrized broken trajectory* with $n \geqslant 0$ components joining $\mathbf{a}$ to $\mathbf{b}$ consists of the following:

- a tuple $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n)$ of $n+1$ rest points: elements of $\mathfrak{C}(Y)$, where $\mathbf{a}_0 = \mathbf{a}$ and $\mathbf{a}_n = \mathbf{b}$.
- a tuple $(\breve{\gamma}_1, \ldots, \breve{\gamma}_n)$ of unparametrized trajectories in $\breve{M}_{z_1}(\mathbf{a}_0, \mathbf{a}_1) \times \cdots \times \breve{M}_{z_n}(\mathbf{a}_{n-1}, \mathbf{a}_n)$, where $z = z_n \circ \cdots \circ z_0$.

or the constant path if $n = 0$. Write $\breve{M}_z^+(\mathbf{a}, \mathbf{b})$ for the space of unparametrized broken trajectories with homotopy class z. Each $\breve{M}_z^+(\mathbf{a}, \mathbf{b})$ is compact and (assuming regularity) there are only finitely many z for which $\breve{M}_z^+(\mathbf{a}, \mathbf{b})$ is non-empty.

The spaces $\breve{M}_z^+(\mathbf{a}, \mathbf{b})$ are our natural compactification of $\breve{M}_z(\mathbf{a}, \mathbf{b})$ and are d-*dimensional spaces stratified by manifolds*, in the sense that there is a sequence of closed subsets:

$$\breve{M}_z^+(\mathbf{a}, \mathbf{b}) = N^d \supset N^{d-1} \supset \cdots \supset N^0 \supset N^{-1} = \varnothing$$

where the i-dimensional stratum, $N^i \backslash N^{i-1}$, is a (possibly empty) i-dimensional manifold. Like a 1-dimensional manifold, a 1-dimensional space stratified by manifolds has an even number of boundary components, counted appropriately. [5]

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[5] elaborate on δ-structure?

There is a similar construction for the reducible moduli spaces; involving a compactification by reducible broken trajectories.

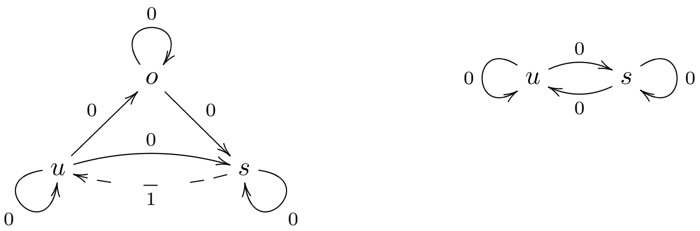

Figure 1. Trajectories connecting critical points

The reader may find the following description of the space of broken trajectories useful. Consider the oriented graph in the left part of Figure 1. Broken trajectories passing through irreducible, boundary stable and boundary unstable rest points correspond to paths in the graph passing through o, s, and u respectively. Each such path gets a weight w from summing the labels on edges traversed. A broken trajectory with $l+1$ components and path of weight w is then in the codimension-k strata if $l - w = k$.

For example, a 1-dimensional unparametrized moduli space on a cylinder connecting a u to an s has boundary of either of two types:

$$\begin{tikzcd} u \arrow[r] & o \arrow[r] & s \end{tikzcd}$$

$$\begin{tikzcd} u \arrow[r] & s \arrow[r, dashed] & u \arrow[r] & s \end{tikzcd}$$

A similar pattern holds for the reducible solutions and moduli space using the oriented graph in the left part of Figure 1.

2.2.2. *Perturbations.*

2.2.3. *Differential and Homology.* Let $n_z(\mathbf{a},\mathbf{b})$ be the mod 2 number of points of $\breve{M}_z(\mathbf{a},\mathbf{b})$ when $\breve{M}_z(\mathbf{a},\mathbf{b})$ is zero dimensional and zero otherwise. Similarly define $\bar{n}_z(\mathbf{a},\mathbf{b})$ for the reducible unparametrized moduli space. Then define:

$$\begin{aligned}
\partial^o_o(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^o}\ \sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} n_z(\mathbf{a},\mathbf{b})\mathbf{b} && \mathbf{a}\in\mathfrak{C}^o \\
\partial^o_s(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^s}\ \sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} n_z(\mathbf{a},\mathbf{b})\mathbf{b} && \mathbf{a}\in\mathfrak{C}^o \\
\partial^u_o(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^o}\ \sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} n_z(\mathbf{a},\mathbf{b})\mathbf{b} && \mathbf{a}\in\mathfrak{C}^u \\
\partial^u_s(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^s}\ \sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} n_z(\mathbf{a},\mathbf{b})\mathbf{b} && \mathbf{a}\in\mathfrak{C}^u
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{\partial}^s_s(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^s}\sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} \bar{n}_z(\mathbf{a},\mathbf{b})\mathbf{b} \qquad \mathbf{a}\in\mathfrak{C}^s\\
\overline{\partial}^u_u(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^u}\sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} \bar{n}_z(\mathbf{a},\mathbf{b})\mathbf{b} \qquad \mathbf{a}\in\mathfrak{C}^u\\
\overline{\partial}^s_u(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^u}\sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} \bar{n}_z(\mathbf{a},\mathbf{b})\mathbf{b} \qquad \mathbf{a}\in\mathfrak{C}^s\\
\overline{\partial}^u_s(\mathbf{a}) &= \sum_{\mathbf{b}\in\mathfrak{C}^s}\sum_{z\in\pi_0(\mathbf{a},\mathbf{b})} \bar{n}_z(\mathbf{a},\mathbf{b})\mathbf{b} \qquad \mathbf{a}\in\mathfrak{C}^u
\end{aligned}$$

Note that maps ∂^u_s and $\overline{\partial}^u_s$ are not equal and do not count points in the same spaces.[6] The case of $\overline{\partial}^s_u$ is special, in that it drops $\overline{\mathrm{gr}}_z$ by 1 but preserves gr_z. While these sums may not involve a finite number of terms, the zero dimensional spaces $\breve{M}_z(\mathbf{a},\mathbf{b})$ are compact, so the sums are well defined as formal power series. With $\check{C}(Y) = C^o(Y)\oplus C^s(Y)$, $\widehat{C}(Y) = C^o(Y)\oplus C^u(Y)$, and $\overline{C}(Y) = C^s(Y)\oplus C^u(Y)$ we form differentials as follows:

$$\begin{aligned}
\check{\partial} &= \begin{bmatrix} \partial^o_o & -\partial^u_o\overline{\partial}^s_u \\ \partial^o_s & \overline{\partial}^s_s - \partial^u_s\overline{\partial}^s_u \end{bmatrix}\\
\hat{\partial} &= \begin{bmatrix} \partial^o_o & \partial^u_o \\ -\overline{\partial}^s_u\partial^o_s & -\overline{\partial}^u_u - \overline{\partial}^s_u\partial^u_s \end{bmatrix}\\
\overline{\partial} &= \begin{bmatrix} \overline{\partial}^s_s & \overline{\partial}^u_s \\ \overline{\partial}^s_u & \overline{\partial}^u_u \end{bmatrix}
\end{aligned}$$

Each of $\check{\partial}, \hat{\partial}$, and $\overline{\partial}$ are square zero. These give our homology groups

$$\begin{aligned}
\check{H}_*(Y;\mathfrak{s}) &= H_*(\check{C}(Y;\mathfrak{s}),\check{\partial})\\
\widehat{H}_*(Y;\mathfrak{s}) &= H_*(\widehat{C}(Y;\mathfrak{s}),\hat{\partial})\\
\overline{H}_*(Y;\mathfrak{s}) &= H_*(\overline{C}(Y;\mathfrak{s}),\overline{\partial})
\end{aligned}$$

These groups are independent of the metric g on Y and independent of ω up to exact perturbations. (In the non-exact perturbation case, we have a wall-crossing phenomena. See Chapter VIII of [7] for details.) In the case that $c_1(\mathfrak{s})$ is torsion we naturally obtain the $\bullet$ versions of the above groups. The $*$ versions are obtained by the $J(Y;\mathfrak{s})$ filtration.

We can also define three chain maps: $i : \overline{C}_*(Y) \to \check{C}_*(Y)$, $j : \check{C}_*(Y) \to \widehat{C}_*(Y)$ and $p : \widehat{C}_*(Y) \to \overline{C}_*(Y)$ by:

$$\begin{aligned}
i &= \begin{bmatrix} 0^s_o & \partial^u_o \\ 1^s_s & \partial^u_s \end{bmatrix}\\
j &= \begin{bmatrix} 1^o_o & 0^s_o \\ 0^o_u & \overline{\partial}^s_u \end{bmatrix}\\
p &= \begin{bmatrix} \partial^o_s & \partial^u_s \\ 0^o_u & 1^u_u \end{bmatrix}
\end{aligned}$$

---

[6] Note to self: Can an example be found where $\overline{\partial}^u_s$ is not zero?

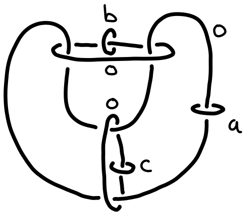

FIGURE 2. Dehn surgery diagram of T^3 with chosen generators of $H_1(T^3)$

The maps are of degree $0, 0$, and -1 respectively, also exist on the $\bullet$ groups, and induce a long exact sequence

$$\cdots \xrightarrow{i_*} \widetilde{HM}_k(Y) \xrightarrow{j_*} \widehat{HM}_k(Y) \xrightarrow{p_*} \overline{HM}_{k-1}(Y) \xrightarrow{i_*} \widetilde{HM}_{k-1}(Y) \xrightarrow{j_*} \cdots$$

analogous to the $B, (B, \partial B), \partial B$ sequence in singular homology.

2.2.4. *Example:* S^3 *and* $\#_n S^1 \times S^2$.

2.2.5. *Example:* T^3. As this case has been worked out in full in Section 37 of [7], we will simply summarize the results here.

The 3-torus can be described by zero surgery on the components of the Boromean rings. Choose 1-cycles a, b, c with some orientation representing a basis for $H_1(T^3; \mathbb{Z})$ as shown in Figure 2. Suppose that T^3 is given a flat metric. Then the Weitzenböck formula guarantees that there are no irreducible solutions to the unperturbed Seiberg-Witten equations. Thus, only the trivial *spin*$^{\mathbb{C}}$ structure $\mathfrak{s}_0$ may have non-zero Floer homology.

Let B_0 be the unique flat connection in $\mathfrak{s}_0$ with trivial holonomy. Let f be a perfect Morse function on the torus of flat *spin*$^{\mathbb{C}}$ connections (another T^3) with minimum occurring at B_0. We can arrange that all eight of the critical points of f correspond to the *spin* connections. This is a copy of the real points $(\mathbb{Z}/2\mathbb{Z})^3$ in $(S^1)^3 = T^3$. Note that this set is invariant under the conjugation automorphism of $\mathfrak{s}_0$ and the affine diffeomorphisms of T^3.

To setup notation, write

- x for the trivial connection B_0,
- y^a, y^b, y^c for the *spin* connections with holonomy -1 about a, b, and c respectively.
- z^{ab}, z^{ac}, z^{bc} for the *spin* connections with holonomy -1 about a and b, a and c, and b and c respectively.
- $w = w^{abc}$ for the *spin* connection with holonomy -1 about a, b, and c.

By choosing perturbation consisting of $-\frac{s^2\delta}{2}\|\phi\|$ and the above perfect Morse function f, all solutions remain reducible and correspond to the eight towers

$$\begin{array}{ccc} & w_i & \\ \\ z_i^{ab} & z_i^{ac} & z_i^{bc} \\ \\ y_i^a & y_i^b & y_i^c \\ \\ & x_i & \end{array}$$

with $i \in \mathbb{Z}$. There is a small sphere surrounding B_0 in the torus of flat connections over which D_B has kernel – these are the only flat connections with kernel. Then for $i \geqslant 0$

$$\begin{aligned}
\mathrm{gr}^{\mathbb{Q}}(w_i) &= \begin{cases} 2i & \text{for } i \geqslant 0 \\ 2i+1 & \text{for } i < 0 \end{cases} \\
\mathrm{gr}^{\mathbb{Q}}(z_i^j) &= \begin{cases} 2i-1 & \text{for } i \geqslant 0 \\ 2i & \text{for } i < 0 \end{cases} \\
\mathrm{gr}^{\mathbb{Q}}(y_i^j) &= \begin{cases} 2i-2 & \text{for } i \geqslant 0 \\ 2i-1 & \text{for } i < 0 \end{cases} \\
\mathrm{gr}^{\mathbb{Q}}(x_i^j) &= \begin{cases} 2i-1 & \text{for } i \geqslant 0 \\ 2i & \text{for } i < 0 \end{cases}
\end{aligned}$$

SUMMARIZE TRAJECTORIES.

All flows on the cylinder $T^3 \times \mathbb{R}$ are reducible and cover Morse flows on the torus of flat connections. Only non-trivial contribution is $\mathbf{w}_i \to \mathbf{x}_i$.

2.2.6. *Cobordism maps.* Suppose that W is a cobordism from Y_- to Y_+ ($\partial W = -Y_- \sqcup Y_+$) and that we are given $\mathbf{a} \in \mathfrak{C}(Y_-)$ and $\mathbf{b} \in \mathfrak{C}(Y_+)$. The moduli space over W^* admits a natural compactification, $M_z^+(\mathbf{a}, W^*, \mathbf{b})$, which is formed by adding broken trajectories. A broken trajectory consists of:

- $\gamma_- \in \breve{M}^+_{z_-}(\mathbf{a}, \mathbf{a}_0)$
- $\gamma_0 \in M_{z_0}(\mathbf{a}_0, W^*, \mathbf{b}_0)$
- $\gamma_+ \in \breve{M}^+_{z_+}(\mathbf{b}_0, \mathbf{b})$

where $z = z_- \circ z_0 \circ z_+$. Each $M_z^+(\mathbf{a}, W^*, \mathbf{b})$ is a compact space stratified by manifolds. If the dimension of $M_z^+(\mathbf{a}, W^*, \mathbf{b})$ is zero, then $M_z(\mathbf{a}, W^*, \mathbf{b}) = M_z^+(\mathbf{a}, W^*, \mathbf{b})$ is compact.

Let

$$m_z(\mathbf{a}, W, \mathbf{b}) = \begin{cases} |M_z(\mathbf{a}, W^*, \mathbf{b})| & \text{if } \dim M_z(\mathbf{a}, W^*, \mathbf{b}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

and define $\overline{m}_z(\mathbf{a}, W, \mathbf{b})$ similarly, using the moduli space of reducibles. We then get eight maps: $m^o_o, m^o_s, m^u_o, m^u_s$, and $\bar{m}^s_s, \bar{m}^u_u, \bar{m}^s_u, \bar{m}^u_s$ defined, analogous to the differential, by

$$\begin{aligned}
m^o_o(\mathbf{a}) &= \sum_{b \in \mathcal{C}^o} \sum_{z \in \pi_0(\mathbf{a}, \mathbf{b})} m_z(\mathbf{a}, W, \mathbf{b})\mathbf{b} \qquad a \in \mathcal{C}^o \\
&\text{etc.} \\
\bar{m}^s_s(\mathbf{a}) &= \sum_{b \in \mathcal{C}^s} \sum_{z \in \pi_0(\mathbf{a}, \mathbf{b})} \bar{m}_z(\mathbf{a}, W, \mathbf{b})\mathbf{b} \qquad a \in \mathcal{C}^s \\
&\text{etc.}
\end{aligned}$$

Each of these maps is not, in general, a finite sum. However, they are well defined as formal power series. This then gives us chain maps

$$\begin{aligned}
\check{m}(W) : \check{C}_\bullet(Y_-) &\to \check{C}_\bullet(Y_+) \\
\hat{m}(W) : \hat{C}_\bullet(Y_-) &\to \hat{C}_\bullet(Y_+) \\
\overline{m}(W) : \overline{C}_\bullet(Y_-) &\to \overline{C}_\bullet(Y_+)
\end{aligned}$$

defined by:

$$\begin{aligned}
(7)\ \check{m}(W) &= \begin{bmatrix} m_o^o & -m_o^u\overline{\partial}_u^s(Y_-) - \partial_o^u(Y_+)\overline{m}_u^s \\ m_s^o & \overline{m}_s^s - m_s^u\overline{\partial}_u^s(Y_-) - \partial_s^u(Y_+)\overline{m}_u^s \end{bmatrix} \\
(8)\ \hat{m}(W) &= \begin{bmatrix} m_o^o & m_o^u \\ \overline{m}_u^s\partial_s^o(Y_-)\sigma + \overline{\partial}_u^s(Y_+)m_s^o & \overline{m}_u^u - \overline{m}_u^s\partial_s^u(Y_-)\sigma - \overline{\partial}_u^s(Y_+)m_s^u \end{bmatrix} \\
(9)\ \overline{m}(W) &= \begin{bmatrix} \overline{m}_s^s & \overline{m}_s^u \\ \overline{m}_u^s & \overline{m}_u^u \end{bmatrix}
\end{aligned}$$

[7]

2.3. **Families.** The chain maps $\check{m}$ depend on a choice of metric and perturbation on W. Let P be a smooth oriented manifold, possibly with $\partial P = Q$, parametrizing a family of metrics and perturbations on W. Each member of the family is required to give equal data on some neighborhood of ∂W. Form

$$M(\mathbf{a}, W^*, \mathbf{b})_P = \bigcup_{p \in P} \{p\} \times M(\mathbf{a}, W^*, \mathbf{b})_p$$

where $M(\mathbf{a}, W^*, \mathbf{b})_p$ is the monopole moduli space for the metric and perturbation given by $p \in P$. We can similarly define $M(\mathbf{a}, W^*, \mathbf{b})_Q$; the definition of regularity for $M(\mathbf{a}, W^*, \mathbf{b})_P$ assumes regularity for $M(\mathbf{a}, W^*, \mathbf{b})_Q$ as well. When P is compact, we can define

$$m_z(\mathbf{a}, W, \mathbf{b})_P = \begin{cases} |M_z(\mathbf{a}, W^*, \mathbf{b})_P| \mod 2 & \text{if } \dim M_z(\mathbf{a}, W^*, \mathbf{b})_P = 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\overline{m}_z(\mathbf{a}, W, \mathbf{b})$ similarly, using the moduli space of reducibles. From these, we construct components:

$$m_o^o(W)_P, m_s^o(W)_P, m_o^u(W)_P, m_s^u(W)_P, \text{ and } \bar{m}_s^s(W)_P, \bar{m}_u^u(W)_P, \bar{m}_u^s(W)_P, \bar{m}_s^u(W)_P$$

as before. Finally, $\check{m}(W)_P, \hat{m}(W)_P, \overline{m}(W)_P$ are defined using (7), (8), and (9) as in the case of a single metric and perturbation.

When $\partial P = Q \neq \varnothing$, the maps $\check{m}(W)_P$ are not, in general, chain maps. Instead we have:

$$\check{\partial}\check{m}(W)_P + \check{m}(W)_P\check{\partial} = \check{m}(W)_Q \tag{10}$$

which is found by counting the boundary points of the compactifications of 1-dimensional moduli spaces $M_z(\mathbf{a}, W^*, \mathbf{b})_P$.

Taking $P \cong [0, 1]$ to be a path connecting two sets of metric and perturbation on W, (10) tells us that:

$$\check{\partial}\check{m}(W)_P + \check{m}(W)_P\check{\partial} = \check{m}(W)_0 - \check{m}(W)_1$$

verifying that, up to chain homotopy, cobordism maps are well defined independent of metric and perturbation.[8]

---

[7] Define σ in formula or omit.

[8] When considering a family using non-exact perturbations by 2-forms ω, the statement must qualified to hold only when we do not cross a wall. i.e. $4[\omega] \neq c_1(\mathfrak{t})$ in each *spin*$^{\mathbb{C}}$ structure.

2.3.1. *Stretching the neck in families.* Now, suppose that Y is a 3-manifold interior to W which separates W as $W \backslash Y = W_0 \sqcup W_1$ so that $\partial W_0 = -Y_- \sqcup Y$ and $\partial W_1 = -Y \sqcup Y_+$. Further suppose that we are given a metric and perturbation on W which are cylindrical near Y. We may form the Riemannian manifold $W(T)$ by *stretching* W *along* Y *by* $T \geqslant 0$ which consists of replacing an isometric copy of $(-\epsilon, \epsilon) \times Y$ by $(-\epsilon - T, \epsilon + T) \times Y$ inside of W. This gives us a family of data on W parametrized by $P = [0, \infty)$.

We compactify P to $\overline{P} = [0, \infty]$ by adding a fiber $W(\infty) = (W \backslash Y)^* = W_0^* \sqcup W_1^*$ over $p = \infty$. Then let $M_z(\mathbf{a}, W(\infty)^*, \mathbf{b})$ be

$$\bigcup_{\mathbf{c} \in \mathfrak{C}(Y)} \bigcup_{z_1 \circ z_0 = z} M_{z_0}(\mathbf{a}, W_0^*, \mathbf{c}) \times M_{z_1}(\mathbf{c}, W_1^*, \mathbf{b})$$

which compactifies to $M_z^+(\mathbf{a}, W(\infty)^*, \mathbf{b})$. An element of $M_z^+(\mathbf{a}, W(\infty)^*, \mathbf{b})$ consists of a tuple:

$$(\gamma_{Y_-}, \gamma_{W_0}, \gamma_Y, \gamma_{W_1}, \gamma_{Y_+})$$

where $\gamma_{Y_-}, \gamma_Y, \gamma_{Y_+}$ are broken trajectories on the corresponding 3-manifolds and $\gamma_{W_0}, \gamma_{W_1}$ are elements of the moduli spaces of the cobordisms. With the addition of the extra fiber, we obtain the compactification:

$$M_z^+(\mathbf{a}, W^*, \mathbf{b})_{\overline{P}} = \bigcup_{T \in [0, \infty]} \{T\} \times M_z^+(\mathbf{a}, W(T)^*, \mathbf{b})$$

We quote a structure result:

Lemma 5 (4.15 of [8])**.** *If $M_z(\mathbf{a}, W^*, \mathbf{b})_P$ is zero-dimensional, then it is compact. If $M_z(\mathbf{a}, W^*, \mathbf{b})_P$ is one-dimensional and contains irreducible trajectories, then the compactification $M_z^+(\mathbf{a}, W^*, \mathbf{b})_{\overline{P}}$ is a 1-dimensional space with a codimension-1 δ-structure*[9] *at all boundary points.*[10] *The boundary points are of the following types:*

1. *the fiber over $T = 0$: the space $M_z(\mathbf{a}, W(0)^*, \mathbf{b})$.*
2. *the fiber over $T = \infty$: the space $M_z(\mathbf{a}, W(\infty)^*, \mathbf{b})$.*
3. *broken trajectories escaping off of one of the two ends of W^*:*
$$\begin{array}{ccc} \breve{M}_{z_1}(\mathbf{a}, \mathbf{a}_1) & \times & M_{z_2}(\mathbf{a}_1, W^*, \mathbf{b})_P \\ M_{z_1}(\mathbf{a}, W^*, \mathbf{b}_1)_P & \times & \breve{M}_{z_2}(\mathbf{b}_1, \mathbf{b}) \end{array}$$
4. *three factor products with the middle term boundary obstructed:*
$$\begin{array}{ccccc} \breve{M}_{z_1}(\mathbf{a}, \mathbf{a}_1) & \times & \breve{M}_{z_2}(\mathbf{a}_1, \mathbf{a}_2) & \times & M_{z_3}(\mathbf{a}_2, W^*, \mathbf{b})_P \\ \breve{M}_{z_1}(\mathbf{a}, \mathbf{a}_1) & \times & M_{z_2}(\mathbf{a}_1, W^*, \mathbf{b}_1)_P & \times & \breve{M}_{z_3}(\mathbf{b}_1, \mathbf{b}) \\ M_{z_1}(\mathbf{a}, W^*, \mathbf{b}_1)_P & \times & \breve{M}_{z_2}(\mathbf{b}_1, \mathbf{b}_2) & \times & \breve{M}_{z_3}(\mathbf{b}_2, \mathbf{b}) \end{array}$$
5. *the part of $M_z^+(\mathbf{a}, W(\infty)^*, \mathbf{b})$ of the form:*
$$\begin{array}{ccccc} \breve{M}_{z_1}(\mathbf{a}, \mathbf{a}_1) & \times & M_{z_2}(\mathbf{a}_1, W_0^*, \mathbf{c}) & \times & M_{z_3}(\mathbf{c}, W_1^*, \mathbf{b}) \\ M_{z_1}(\mathbf{a}, W_0^*, \mathbf{c}_1) & \times & \breve{M}_{z_2}(\mathbf{c}_1, \mathbf{c}) & \times & M_{z_3}(\mathbf{c}, W_1^*, \mathbf{b}) \\ M_{z_1}(\mathbf{a}, W_0^*, \mathbf{c}) & \times & M_{z_2}(\mathbf{c}, W_1^*, \mathbf{b}_1) & \times & \breve{M}_{z_3}(\mathbf{b}_1, \mathbf{b}) \end{array}$$
with each the middle term boundary obstructed.

---

[9] define

[10] The signed count of boundary points for a one dimensional space with codimension-1 δ-structure is zero. See 19.5.3 and 21.3.2 of [7].

(6) *the reducibles* $M_z^{red}(\mathbf{a}, W^*, \mathbf{b})_P$ *when* $\mathbf{a}$ *is boundary unstable and* $\mathbf{b}$ *is boundary-stable.*

Using the zero dimensional family moduli spaces, component maps K_o^o, K_s^o, etc. are defined by counting solutions as is usual. The maps $\check{K}, \hat{K}, \overline{K}$ on $\check{C}, \hat{C}, \overline{C}$ have slightly different formulas than the usual cobordism maps due to the inclusion of solutions on the broken cobordism. See the discussion surrounding Figure 1.

$$\begin{aligned}
\check{K} &= \begin{bmatrix} K_o^o & K_o^u \bar{\partial}_u^s + m_o^u(W_2)\overline{m}_u^s(W_1) + \partial_o^u \overline{K}_u^s \\ K_s^o & \overline{K}_s^s + K_s^u \bar{\partial}_u^s + m_s^u(W_2)\overline{m}_u^s(W_1) + \partial_s^u \overline{K}_u^s \end{bmatrix} \\
\hat{K} &= \begin{bmatrix} K_o^o & K_o^u \\ \bar{\partial}_u^s K_s^o + \overline{m}_u^s(W_2)m_s^o(W_1) + \overline{K}_u^s \partial_s^o & \overline{K}_u^u + \bar{\partial}_u^s K_s^u + \overline{m}_u^s(W_2)m_s^u(W_1) + \overline{K}_u^s \partial_s^u \end{bmatrix} \\
\overline{K} &= \begin{bmatrix} \overline{K}_s^s & \overline{K}_s^u \\ \overline{K}_u^s & \overline{K}_u^u \end{bmatrix}
\end{aligned}$$

Considering the structure of the boundary of 1-dimensional moduli spaces found in Lemma 5 gives us that

$$\check{\partial}\check{K} + \check{K}\check{\partial} = \check{m}(W_1)\check{m}(W_0) + \check{m}(W) \tag{11}$$

$$\hat{\partial}\hat{K} + \hat{K}\hat{\partial} = \hat{m}(W_1)\hat{m}(W_0) + \hat{m}(W) \tag{12}$$

$$\overline{\partial}\overline{K} + \overline{K}\overline{\partial} = \overline{m}(W_1)\overline{m}(W_0) + \overline{m}(W) \tag{13}$$

which verifies the composition law $\check{H}(W_1) \circ \check{H}(W_0) = \check{H}(W)$ on the level of homology.

Given several non-intersecting hypersurfaces $\{Y_n\}_{n=1}^N$ of W, we can continue this construction; parametrizing a family of metrics and perturbations by $[0, \infty)^N$ or its compactification $[0, \infty]^N$. In fact, if we allow the $\{Y_n\}_{n=1}^N$ to intersect, it is possible to build a convex polytope parametrizing the various ways of stretching along non-intersecting subsets. In this paper we will use two combinatorially identical families shown in Figure 7, based on [8], and in Figure 10. A more general discussion of the combinatorics is in [1].

In fact, in these higher dimensional families compactified by adding in (family) solutions on broken 4-manifolds, we may generalize 10 in a similar spirit to equations (11)–(13). This generalization requires the concept of a "good break". Consider, for example, the maps on $\check{H}$ which involve critical points in $\mathfrak{C}^o \cup \mathfrak{C}^s$ and look at critical points with $\mathrm{gr}(\mathbf{a}, W^*, \mathbf{b}) = 1 - \dim(P)$ so that the expected dimension of the moduli spaces is -1. We say that a solution has a good break if it

(1) is a solution over W broken along a single Y_n,
(2) if Y_n bounds in W the solution breaks along $\mathfrak{C}^o \cup \mathfrak{C}^u$
(3) if Y_n does not bound in W the solution breaks along $\mathfrak{C}^o \cup \mathfrak{C}^s$

Similar statements hold for the $\hat{H}, \overline{H}$ theories. As argued on page 511 of [8] and more explicitly in Appendices I and II of [1], the multiplicity of solutions which do not have good breaks is even. Therefore, at least mod 2, our formulas will only require us to keep track of good breaks.[11]

---

[11] At the moment it seems reasonable to believe that the mod 2 requirement is unnecessary.

2.4. **Negative Indefinite Cobordisms.** On a connected 4-manifold W with boundary there is a non-degenerate pairing

$$H^2(W, \partial W; \mathbb{R}) \otimes H^2(W; \mathbb{R}) \to H^4(W, \partial W; \mathbb{R}) \cong \mathbb{R}.$$

Using the restriction map $j^* : H^2(W, \partial W; \mathbb{R}) \to H^2(W; \mathbb{R})$, we get a pairing on $H^2(W, \partial W; \mathbb{R})$ which is degenerate on the kernel of j^*. (By exactness, the kernel of j^* is the image of $H^1(\partial W) \to H^2(W, \partial W; \mathbb{R})$.) Then on $I^2(W) = \mathrm{Image}(j^*)$, the above pairing is non-degenerate. Note, on a 4-manifold with cylindrical ends, $I^2(W^*)$ is isomorphic to the space of L^2-integrable harmonic 2-forms.[12]

Recall that a closed, oriented 4-manifold W is *negative definite* if the square of any non-zero second cohomology class is negative. i.e. $b_2(W) = b_2^-(W)$. Similarly, if W has boundary or cylindrical ends, we say that W is *negative indefinite* if the square of any element of $I^2(W)$ is negative.

Let A_0 be a reference connection for the *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}$ with $F_{A_0^t}$ harmonic. Then the self- and anti-self-dual parts $F^{\pm}_{A_0^t}$ are closed and thus harmonic.[13] Up to gauge transformation any other connection may be written as $A = A_0 + a$ with $\delta a = 0$. Now, using Lemma 3 of [2], we see that A_0 is unique up to addition of an L^2 harmonic 1-form.

If W is negative indefinite, $\omega^+ = d^+ a$ for any closed, L^2-integrable 2-form ω. In this case, equations (5) and (6), the 4-dimensional Seiberg-Witten equations on the blown up configuration space, reduce to

$$\begin{aligned} \rho(d^+ a) - s^2(\phi\phi^*)_0 &= 0 \\ D_A^+ \phi &= 0 \end{aligned}$$

in each *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}$. By integrating the norm squared of $d^+ a$ and $s^2(\phi\phi^*)_0$ and equating using the first equation, we find that $s = 0$ and so all solutions are reducible. Then the moduli space consists of the solutions (A, ϕ) to $D_A^+ \phi = 0, da = 0, \delta a = 0$ modulo gauge. Let $d = \frac{c_1^2(\mathfrak{t}) - \sigma(W)}{8}$ be the complex index of the Dirac operators D_A^+ in the *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}$.

In the case that $H_1(W)$ and $H_1(Y)$ are zero or torsion, much is known about negative definite 4-manifolds. See Donaldson's famous paper [3], and Frøyshov's [6]. In the next section, we consider some examples with positive b_1.

2.5. **Examples: Surgeries on the Boromean rings.** Up to diffeomorphism, there are three 4-manifolds which have Kirby diagrams equal to the Boromean rings and where each component is either a 0-framed 2-handle or a 1-handle. (At least one handle must be a 2-handle.) Each of these have vanishing L^2 2nd cohomology ($I^2 = 0$) and so are negative indefinite.

2.5.1. $T^2 \times \mathbb{D}^2$. The case of $T^2 \times \mathbb{D}^2$ has been worked out in the well-known papers [13] and [16]. Our discussion will be a partial conversion of the latter to the language of Floer homology.

Let $X = T^3 \times [0, \infty)$ and suppose that X is given a product metric which is flat on T^3. We identify X with $(T^2 \times \mathbb{D}^2)^*$ minus a small open neighborhood of the central T^2. Let the Seiberg-Witten equations be unperturbed near $T^3 \times 0$ and be our standard perturbation for T^3 on the infinite end. Further suppose that all

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[12] See the discussion preceding 24.8.2

[13] On a 4-manifold, any closed (anti-)self-dual 2-form is harmonic since $d\omega^{\pm} = 0, \omega^{\pm} = \pm * \omega^{\pm}$ implies that $d^* \omega^{\pm} = *d * \omega^{\pm} = \pm * d\omega^{\pm} = *0 = 0$.

metrics are invariant under rotation in the $\partial\mathbb{D}^2$ factor of T^3. Note that the moduli spaces $M(X;\mathbf{b})$ on X are identified with the stable manifolds for $\mathbf{b}$.

There is a map $X \to (T^2 \times \mathbb{D}^2)^*$ which collapses $T^3 \times 0$ to the central T^2 of $T^2 \times \mathbb{D}^2$ and is a diffeomorphism elsewhere. Then pullback of any solution on $(T^2 \times \mathbb{D}^2)^*$ corresponds to a solution on X which is rotationally invariant in the $\partial\mathbb{D}^2$ factor on $T^3 \times 0$.

Following [16], these rotationally invariant solutions are the 0-vortices[14] on T^2 – themselves a copy of T^2. Lifted to $T^3 \times 0$, each solution is of the form $(A, 0, \phi)$ with $A \in A_0 + \mathcal{H}^1(T^2) \subset \mathcal{H}^1(T^2 \times \partial\mathbb{D}^2)$ and $D_A\phi = 0$.

It then follows that the moduli spaces $M((T^2 \times \mathbb{D}^2)^*;\mathbf{b})$ are given by intersections of the stable manifold of $\mathbf{b}$ with this 2-torus.

Proposition 6. *The relative invariant of $T^2 \times \mathbb{D}^2$ obtained by counting zero dimensional moduli spaces is*

$$\begin{aligned} n_o &= 0 \in C_o(T^3) \\ n_s &= 0 \in C_s(T^3) \\ \overline{n}_s &= 0 \in C_s(T^3) \\ \overline{n}_u &= x_{-1} + z^{bc}_{-1} \in C_u(T^3) \end{aligned}$$

Each of $M((T^2 \times \mathbb{D}^2)^*;\mathbf{x}_{-1})$ and $M((T^2 \times \mathbb{D}^2)^*;\mathbf{z}^{bc}_{-1})$ consist of a single point. The moduli spaces $M((T^2 \times \mathbb{D}^2)^*;\mathbf{y}^{b}_{-1})$ and $M((T^2 \times \mathbb{D}^2)^*;\mathbf{y}^{c}_{-1})$ are open intervals and are the only other non-empty moduli spaces.

2.5.2. *Montesinos Twin Neighborhood.* A Montesinos twin is a pair of 2-spheres smoothly embedded in S^4 which meet transversely in two points; once positively, once negatively. Write $N\mathbb{T}\mathrm{w}$ for the regular neighborhood of a Montesinos twin. This manifold can also be realized as the complement of an unknotted torus in $\S^4$.

The 4-manifold $N\mathbb{T}\mathrm{w}$ can be given by the Kirby diagram on the left hand side of Figure 3. Observe that $\partial N\mathbb{T}\mathrm{w} = T^3$.

There is an alternate description of $N\mathbb{T}\mathrm{w}$ coming from the right hand side of Figure 3. Namely, $N\mathbb{T}\mathrm{w}$ is the composition of two cobordisms $T^2 \times \mathbb{D}^2$ and W' where W' is the cobordism $\partial W' = -T^3 \sqcup T^3$ shown in Figure 4. We will use the latter description for computation.

Applying standard homological algebra we find that $I^2(N\mathbb{T}\mathrm{w}) = 0$ and $I^2(W') = 0$. Thus only the trivial *spin*$^{\mathbb{C}}$ structures on $N\mathbb{T}\mathrm{w}$ and W' restrict to the trivial *spin*$^{\mathbb{C}}$ structure on $\partial N\mathbb{T}\mathrm{w} = T^3$.

Since we computed the relative invariant of $(T^2 \times \mathbb{D}^2)^*$ in the previous section, we now compute the endomorphism on the Floer chains of T^3 induced by $(W')^*$.

First, we compute the formal dimensions of the irreducible moduli spaces:

$$\begin{aligned} \dim M_z(a,(W')^*,b) &= \mathrm{gr}^{\mathbb{Q}}(a) - \mathrm{gr}^{\mathbb{Q}}(b) - \iota(W') \\ &= \mathrm{gr}^{\mathbb{Q}}(a) - \mathrm{gr}^{\mathbb{Q}}(b) - \frac{2+0+3-3}{2} \\ &= \mathrm{gr}^{\mathbb{Q}}(a) - \mathrm{gr}^{\mathbb{Q}}(b) - 1 \end{aligned}$$

and the reducible moduli spaces:

$$\begin{aligned} \dim M^{red}_z(a,(W')^*,b) &= \overline{\mathrm{gr}}^{\mathbb{Q}}(a) - \overline{\mathrm{gr}}^{\mathbb{Q}}(b) - \iota(W') \\ &= \overline{\mathrm{gr}}^{\mathbb{Q}}(a) - \overline{\mathrm{gr}}^{\mathbb{Q}}(b) - 1 \end{aligned}$$

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[14] The other vortex moduli spaces occur for non-exact perturbations.

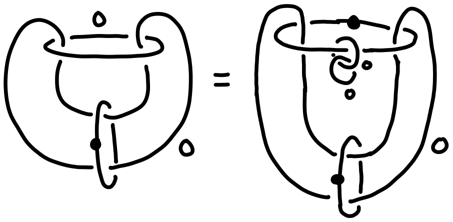

FIGURE 3. The Montesinos Twin Neighborhood, $N\mathbb{T}\mathrm{w}$

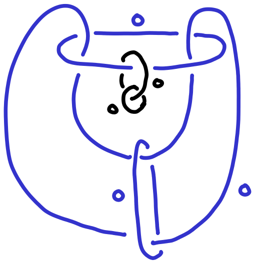

FIGURE 4. W', blue curves indicate 3-manifold surgeries

So W' drops each of the $\mathrm{gr}^{\mathbb{Q}}$ and $\overline{\mathrm{gr}}^{\mathbb{Q}}$ gradings by 1.

Recall that $N\mathbb{T}\mathrm{w}$, and thus W' embeds in S^4. This can be seen explicitly as follows: Take the left picture in Figure 3 and add a zero framed 2-handle to a meridian of the 1-handle. Then the diagram separates into two unknotted zero framed 2-handles. Cancel each with a 3-handle and attach a final 4-handle. The resulting manifold is S^4. Thus $N\mathbb{T}\mathrm{w}$ and W' each have a (usually non-complete) metric of positive scalar curvature. *We then claim that cylindrical end manifolds* $(N\mathbb{T}\mathrm{w})^*$ *and* $(W')^*$ *each have metrics of non-negative scalar curvature which are asymptotic to flat metrics on* T^3 *on the cylindrical ends.*

Proposition 7. *The map* $\overline{C}(T^3;\mathfrak{s}_0) \to \overline{C}(T^3;\mathfrak{s}_0)$ *induced by* W' *is given by:*

$$\begin{array}{rcl}
w_i^{abc} & \to & z_i^{ab} \\
z_i^{ab} & \to & y_i^{a} \\
z_i^{ac} & \to & 0 \\
z_i^{bc} & \to & y_i^{c} \\
y_i^{a} & \to & 0 \\
y_i^{b} & \to & x_{i-1} \\
y_i^{c} & \to & 0 \\
x_i & \to & 0
\end{array}$$

Corollary 8. *The relative invariant of* $N\mathbb{T}\mathrm{w}$ *is*

$$\begin{array}{rcl}
n_o & = & 0 \in C_o(T^3) \\
n_s & = & 0 \in C_s(T^3) \\
\overline{n}_s & = & 0 \in C_s(T^3) \\
\overline{n}_u & = & y_{-1}^{c} \in C_u(T^3)
\end{array}$$

2.5.3. *Three Zero Surgeries.* This case may be deduced from the previous discussion, since three zero-framed 2-handles attached to the Boromean rings can also be described as $N\mathbb{T}\mathrm{w} \cup_{T^3} W'$, using a gluing which interchanges two of the rings. Then:

Corollary 9. *The relative invariant of* B*, with* B *equal to three zero-framed* 2*-handles attached to the Boromean rings, is*

$$\begin{array}{rcl}
n_o & = & 0 \in C_o(T^3) \\
n_s & = & 0 \in C_s(T^3) \\
\overline{n}_s & = & 0 \in C_s(T^3) \\
\overline{n}_u & = & x_{-1} \in C_u(T^3)
\end{array}$$

3. Overview of the surgery exact triangle

We start our overview of the surgery exact triangle with the following algebraic result:

Lemma 10. *For* n *in* $\mathbb{Z}/3\mathbb{Z}$*, let* (C_n, ∂_n) *be a collection of chain complexes and* $f^n_{n+1} : C_n \to C_{n+1}$ *a collection of chain maps with the following properties:*

1. *the composition* $f^{n+1}_{n+2} f^n_{n+1} : C_n \to C_{n+2}$ *is chain-homotopic to zero, by a chain homotopy* H^n_{n+2}*. i.e.* $\partial^{n+2}_{n+2} H^n_{n+2} + H^n_{n+2} \partial^n_n = f^{n+1}_{n+2} f^n_{n+1}$
2. *the sum* $\psi^n_n = f^{n+2}_n H^n_{n+2} + H^{n+1}_n f^n_{n+1} : C_n \to C_n$*, which is a chain map, induces isomorphisms on homology:* $(\psi^n_n)_* : H_*(C_n) \to H_*(C_n)$

Then the sequence

$$\cdots \longrightarrow H_*(C_{n-1}) \xrightarrow{(f^{n-1}_n)_*} H_*(C_n) \xrightarrow{(f^n_{n+1})_*} H_*(C_{n+1}) \longrightarrow \cdots$$

is exact. Additionally, the mapping cone of f^{n+1}_{n+2} *is quasi-isomorphic to* C_n *by the map* $(H^{n+1}_n + f^{n+2}_n) : C_{n+1}[1] \oplus C_{n+2} \to C_n$*, where* $C_{n+1}[1]$ *denotes the complex with shifted grading* $(C_{n+1}[1])_j = (C_{n+1})_{j-1}$*.*

In [8], the authors show that the cobordism maps from the surgery exact triangle, together with maps from homotopies of metrics on these cobordisms, satisfy these conditions. To setup notation and the objects of study, we sketch the proof below.

Suppose that we are given a surgery triangle with 3-manifolds $\{Y_n\}_{n\in\mathbb{Z}/3\mathbb{Z}}$, and cobordisms $\{W^n_{n+1}\}_{n\in\mathbb{Z}/3\mathbb{Z}}$ as in the introduction. Each composite cobordism $X^n_{n+2} = W^n_{n+1} \cup_{Y_{n+1}} W^{n+1}_{n+2}$ contains a smoothly embedded 2-sphere of square -1, called E_n, formed from the cocore of the 2-handle in W^n_{n+1} and the core of the 2-handle in W^{n+1}_{n+2}. Let Z_n be a small closed regular neighborhood of E_n and $B_n = X^n_{n+2}\backslash\mathring{Z}_n$. See Figure 5. Note that Z_n is diffeomorphic to $\overline{\mathbb{C}P}^2\backslash\mathring{\mathbb{D}}^4$. Let S_n be a copy of S^3 with orientation given by viewing it as ∂Z_n. Then, as oriented manifolds, $\partial B_n = -(Y_n \sqcup S_n) \sqcup Y_{n+2}$ and $\partial Z_n = S_n$.

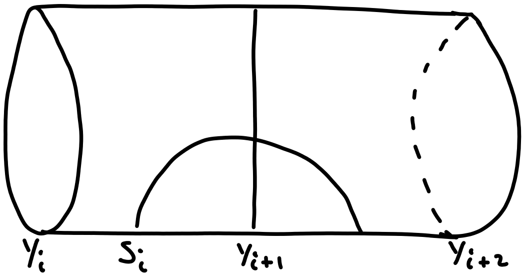

FIGURE 5. X^n_{n+2}

Similarly, the composite cobordism $V^n_n = W^n_{n+1} \cup_{Y_{n+1}} W^{n+1}_{n+2} \cup_{Y_{n+2}} W^{n+2}_n$ contains the pair E_n, E_{n+1} of -1 spheres. By construction, $E_n \cap E_{n+1}$ is exactly the index 2 critical point of the Morse function on W^{n+1}_{n+2}. Hence the two exceptional curves intersect transversely and positively exactly once. Let N_n be a small closed regular neighborhood of E_n and E_{n+1}, large enough to properly contain Z_n and Z_{n+1}. Let $U_n = V^n_n\backslash\mathring{N}_n$. Notice that U_n is diffeomorphic to the manifold obtained from $[0,1]\times Y_n$ by removing a neighborhood of $\left\{\frac{1}{2}\right\}\times K$. Let R_n be a copy of $S^1\times S^2$ with orientation given by viewing it as the boundary ∂N_n. Then, as oriented manifolds, $\partial U_n = -(Y_n \sqcup R_n) \sqcup Y_n$ and $\partial N_n = R_n$.

Within V^n_n, these 5 distinguished hypersurfaces intersect either in 2-tori or vacuously. The non-empty intersections are: (Y_{i+1}, S_i) for $i = n, n+1$, (Y_i, R_n) with $i = n+1, n+2$, and (S_n, S_{n+1}). See Figure 6

Pick a generic metric and perturbation on V^0_0 which is cylindrical on the following submanifolds:

- the surgered 3-manifolds: Y_1, Y_2 and the two copies of Y_0 from $\partial V^0_0 = -Y_0 \sqcup Y_0$.
- the two 3-spheres: S_0 and S_1.
- the $S^1\times S^2$: R_0

Additionally, we require that the metrics on the S_n and R_n have positive scalar curvature and are close to the round/standard metrics. Each of $W^n_{n+1}, X^n_{n+2}, B_n, Z_n, U_n, N_n$ inherit metrics and perturbations from V^0_0, which we will omit unless changed.

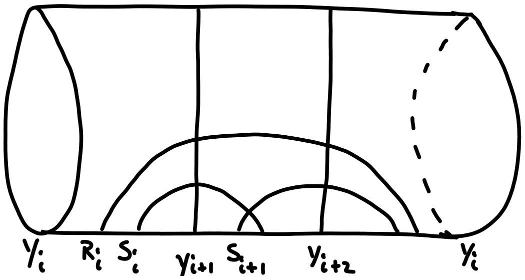

FIGURE 6. V_n

Let $\check{m}_{n+1}^{n}$ be the chain map $\check{m}_{n+1}^{n} : \check{C}(Y_n) \to \check{C}(Y_{n+1})$ induced by $(W_{n+1}^{n})^*$. We now find the chain homotopy $\check{H}_{n+2}^{n}$ that fulfills the conditions of part 1 of Lemma 10 i.e. we show that $\check{\partial}_{n+2}^{n+2}\check{H}_{n+2}^{n} + \check{H}_{n+2}^{n}\check{\partial}_{n}^{n} = \check{m}_{n+2}^{n+1}\check{m}_{n+1}^{n}$.

For the map $\check{m}_{n+2}^{n+1}\check{m}_{n+1}^{n}$, the underlying (unstretched, glued-up) cobordism X_{n+2}^{n} contains two distinguished hypersurfaces Y_{n+1} and S_n. See Figure 5. Given either Y_{n+1} or S_n, we can parametrize stretching metrics on X_{n+2}^{n} along these manifolds by $[0, \infty)$ and $(-\infty, 0]$, respectively, as in Section 2.3.1. [15] The compactifications of the parametrizations glue together at $T = 0$ to get a parametrization $\overline{P} = [-\infty, \infty]$ where $(X_{n+2}^{n})^*$ splits into $(W_{n+1}^{n})^* \sqcup (W_{n+2}^{n+1})^*$ over $r = \infty$ and into $B_n^* \sqcup Z_n^*$ over $r = -\infty$. Here $\check{m}_{n+2}^{n+1}\check{m}_{n+1}^{n}$ is given by counting solutions on $X_{n+2}^{n}(\infty)^*$. As in Section 2.3.1, we have a map $\check{H}_{n+2}^{n} : \check{C}(Y_n) \to \check{C}(Y_{n+2})$ which is a chain homotopy:

$$\begin{aligned}
\check{\partial}_{n+2}^{n+2}\check{H}_{n+2}^{n} + \check{H}_{n+2}^{n}\check{\partial}_{n}^{n} &= \check{m}(X_{n+2}^{n}(\infty)^*) - \check{m}(X_{n+2}^{n}(-\infty)^*) \\
&= \check{m}_{n+2}^{n+1}\check{m}_{n+1}^{n} - \check{m}(X_{n+2}^{n}(-\infty)^*)
\end{aligned}$$

We now claim that the map $\check{m}(X_{n+2}^{n}(-\infty)^*)$ is zero. This map counts elements of zero dimensional moduli spaces $M_z^+(\mathbf{a}, X_{n+2}^{n}(-\infty)^*, \mathbf{b})_P$ which consist of quintuples

$$(\check{\gamma}_{Y_n}, \check{\gamma}_{S_n}, \check{\gamma}_{Y_n+2}, \gamma_{B_n}, \gamma_{Z_n})$$

where the first three are (possibly empty) broken trajectories on cylinders and the latter two are solutions on B_n^* and Z_n^*.

Let $n_o \in C_\bullet^o(S_n), n_s \in C_\bullet^s(S_n), \overline{n}_u \in C_\bullet^u(S_n), \overline{n}_s \in C_\bullet^s(S_n)$ be the elements obtained by counting zero dimensional moduli spaces on Z_n. Similarly, we construct

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[15] Note the reversed orientation on the second interval which introduces a minus sign below.

the components of the irreducible

$$\begin{aligned}
m^{oo}_o &: \quad C^o_\bullet(Y_n) \otimes C^o_\bullet(S_n) \to C^o_\bullet(Y_{n+2})\\
m^{uo}_o &: \quad C^u_\bullet(Y_n) \otimes C^o_\bullet(S_n) \to C^o_\bullet(Y_{n+2})\\
m^{ou}_o &: \quad C^o_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^o_\bullet(Y_{n+2})\\
m^{oo}_s &: \quad C^o_\bullet(Y_n) \otimes C^o_\bullet(S_n) \to C^s_\bullet(Y_{n+2})\\
m^{uu}_o &: \quad C^u_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^o_\bullet(Y_{n+2})\\
m^{uo}_s &: \quad C^u_\bullet(Y_n) \otimes C^o_\bullet(S_n) \to C^s_\bullet(Y_{n+2})\\
m^{ou}_s &: \quad C^o_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^s_\bullet(Y_{n+2})\\
m^{uu}_s &: \quad C^u_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^s_\bullet(Y_{n+2}),
\end{aligned}$$

reducible

$$\begin{aligned}
\overline{m}^{uu}_u &: \quad C^u_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^u_\bullet(Y_{n+2})\\
\overline{m}^{us}_s &: \quad C^u_\bullet(Y_n) \otimes C^s_\bullet(S_n) \to C^s_\bullet(Y_{n+2})\\
\overline{m}^{su}_s &: \quad C^s_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^s_\bullet(Y_{n+2})\\
\overline{m}^{uu}_s &: \quad C^u_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^s_\bullet(Y_{n+2}),
\end{aligned}$$

reducible boundary obstructed

$$\begin{aligned}
\overline{m}^{us}_u &: \quad C^u_\bullet(Y_n) \otimes C^s_\bullet(S_n) \to C^u_\bullet(Y_{n+2})\\
\overline{m}^{su}_u &: \quad C^s_\bullet(Y_n) \otimes C^u_\bullet(S_n) \to C^u_\bullet(Y_{n+2})\\
\overline{m}^{ss}_s &: \quad C^s_\bullet(Y_n) \otimes C^s_\bullet(S_n) \to C^s_\bullet(Y_{n+2}),
\end{aligned}$$

and reducible double boundary obstructed

$$\overline{m}^{ss}_u : \quad C^s_\bullet(Y_n) \otimes C^s_\bullet(S_n) \to C^u_\bullet(Y_{n+2})$$

moduli spaces on B_n.

Then for $X^n_{n+2}(-\infty)^*$ we define the irreducible

$$\begin{aligned}
m^o_o : C^o_\bullet(Y_n) \to C^o_\bullet(Y_{n+2}) &\quad \text{by} \quad m^o_o = m^{oo}_o(\cdot \otimes n_o) + m^{ou}_o(\cdot \otimes \overline{n}_u)\\
m^o_s : C^o_\bullet(Y_n) \to C^s_\bullet(Y_{n+2}) &\quad \text{by} \quad m^o_s = m^{oo}_s(\cdot \otimes n_o) + m^{ou}_s(\cdot \otimes \overline{n}_u)\\
m^u_o : C^u_\bullet(Y_n) \to C^o_\bullet(Y_{n+2}) &\quad \text{by} \quad m^u_o = m^{uo}_o(\cdot \otimes n_o) + m^{uu}_o(\cdot \otimes \overline{n}_u)\\
m^u_s : C^u_\bullet(Y_n) \to C^s_\bullet(Y_{n+2}) &\quad \text{by} \quad m^u_s = m^{uo}_s(\cdot \otimes n_o) + m^{uu}_s(\cdot \otimes \overline{n}_u) + \overline{m}^{us}_s(\cdot \otimes n_s)
\end{aligned}$$

and reducible counts of moduli spaces over the broken manifold.

$$\begin{aligned}
\overline{m}^u_u : C^u_\bullet(Y_n) \to C^u_\bullet(Y_{n+2}) &\quad \text{by} \quad \overline{m}^u_u = \overline{m}^{uu}_u(\cdot \otimes \overline{n}_u)\\
\overline{m}^u_s : C^u_\bullet(Y_n) \to C^s_\bullet(Y_{n+2}) &\quad \text{by} \quad \overline{m}^u_s = \overline{m}^{uu}_s(\cdot \otimes \overline{n}_u) + \overline{m}^{us}_s(\cdot \otimes \overline{n}_s)\\
\overline{m}^s_u : C^s_\bullet(Y_n) \to C^u_\bullet(Y_{n+2}) &\quad \text{by} \quad \overline{m}^s_u = \overline{m}^{us}_u(\overline{\partial}^s_u \cdot \otimes \overline{n}_s) + \overline{m}^{su}_u(\cdot \otimes \overline{\partial}^s_u \overline{n}_s) + \overline{\partial}^s_u \overline{m}^{ss}_s(\cdot \otimes \overline{n}_s)\\
\overline{m}^s_s : C^s_\bullet(Y_n) \to C^s_\bullet(Y_{n+2}) &\quad \text{by} \quad \overline{m}^s_s = \overline{m}^{su}_s(\cdot \otimes \overline{n}_u)
\end{aligned}$$

Then we define $\check{m}(X^n_{n+2}(-\infty))$ by (7), as before.

We have oriented $S_n \cong S^3$ so that $\partial Z_n = S_n$. (So S_n is an incoming end for B_n.) As S_n has positive scalar curvature and is simply connected then, under small perturbations, there are no irreducible generators and the reducible generators correspond to eigenvalues λ_i of the Dirac operator, indexed by $\mathbb{Z}$. Thus $n_o = 0$.

Label the generator corresponding to λ_i by $\mathbf{a}_i$. All critical points differ by an even grading, so the space of $\breve{\gamma}_{S_n}$ is empty or even dimensional. Then as we are interested in zero dimensional moduli space, these must be empty.

Consider $\gamma_{Z_n} \in M_z(Z_n, \mathbf{a}_i)$. As Z_n is simply connected, choice of path z corresponds exactly to choice of *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}$ on Z_n. Write z_k for the path corresponding to $\mathfrak{t}_k$ with $\langle c_1(\mathfrak{t}_k), E_n\rangle = 2k+1$. Note that the conjugation action gives $\bar{\mathfrak{t}}_k = \mathfrak{t}_{-1-k}$.

Lemma 11 (5.3 of [8]). *The following hold for a sufficiently small perturbation on* S_n*:*

(1) *The moduli spaces* $M_{z_k}(Z_n, \mathbf{a}_i)$ *contain no irreducibles. They are empty for* $i \geqslant 0$*.*
(2) *For* $i < 0$*, the moduli space* $M_{z_k}(Z_n, \mathbf{a}_i)$ *consists of a single point when it has formal dimension equal to zero.*
(3) *The formal dimensions of* $M_{z_k}(Z_n, \mathbf{a}_i)$ *and* $M_{z_{-1-k}}(Z_n, \mathbf{a}_i)$ *are the same.*

Proof. Using Lemma 27.4.2 and Definition 28.3.1 of [7], we obtain

$$\begin{aligned}\dim M_{z_k}(Z_n, \mathbf{a}_i) &= \dim M((\mathbb{D}^4)^*, \mathbf{a}_0) + \mathrm{gr}_{z_k}(\mathbf{a}_0, Z_n\backslash\mathbb{D}^4, \mathbf{a}_i)\\ &= -1 - \mathrm{gr}^{\mathbb{Q}}(\mathbf{a}_i) + \frac{c_1^2(\mathfrak{t}_k) - \sigma(Z_n\backslash\mathbb{D}^4)}{4} - \iota(Z_n\backslash\mathbb{D}^4)\\ &= -1 - \mathrm{gr}^{\mathbb{Q}}(\mathbf{a}_i) - k(k+1)\end{aligned}$$

Then since $\mathrm{gr}^{\mathbb{Q}}(\mathbf{a}_i)$ is $2i$ when $i \geqslant 0$ and $2i+1$ when $i < 0$, the formal dimension $M_{z_k}(Z_n, \mathbf{a}_i)$ is $-1-2i-k(k+1)$ if $i \geqslant 0$ and $-2-2i-k(k+1)$ if $i < 0$. This proves (3).

Arithmetic guarantees that in the case that $\mathbf{a}_i$ is boundary stable ($i \geqslant 0$), the formal dimension of $M_{z_k}(Z_n, \mathbf{a}_i)$ is negative and so the space is empty for generic data.

Since we have chosen a metric of positive scalar curvature on S^3, there are no irreducible critical points. Thus $n_o = 0$ automatically.

When $i < 0$, $\mathbf{a}_i$ is boundary unstable, and so there are at most reducible solutions. Since Z_n is simply connected and negative definite, for each *spin*$^{\mathbb{C}}$ structure on Z_n there is, up to gauge equivalence, a unique *spin*$^{\mathbb{C}}$ connection A_0 with $F^+_{A_0^t} = 0$. Then $M^{red}_{z_k}(Z_n, \mathbf{a}_i) \cong \mathbb{C}P^{l/2}$ where $l \geqslant 0$ is the formal dimension of $M_{z_k}(Z_n, \mathbf{a}_i)$. Thus when the formal dimension is zero, $M_{z_k}(Z_n, \mathbf{a}_i)$ consists of a single point. □

Thus $\breve{m}(X_n(-\infty)^*) \equiv 0 \mod 2$ and

$$\breve{\partial}^{n+2}_{n+2}\breve{H}^n_{n+2} + \breve{H}^n_{n+2}\breve{\partial}^n_n = \breve{m}^{n+1}_{n+2}\breve{m}^n_{n+1}$$

Now we wish to consider part 2 of Lemma 10. Out of our five distinguished hypersurfaces Y_1, Y_2, S_1, S_2, and R in V_0^0, there are five pairs of disjoint manifolds:

$$(Y_1, Y_2), (Y_1, S_2), (R, S_2), (R, S_1), \text{ and } (Y_2, S_1) \tag{14}$$

Since each pair consists of disjoint hypersurfaces, we may stretch on either member independently. Thus for each such pair (A, B), we obtain a square $P(A, B) = [0, \infty) \times [0, \infty)$ of metrics. As before, we can add in the broken cobordisms at the boundary to get a parameter space $\overline{P}(A, B) = [0, \infty] \times [0, \infty]$. At (∞, ∞) we get cobordism broken at both A and B.

Suppose that we are given two pairs of non-intersecting hypersurfaces, (A, B) and (B, C), where A and C may intersect or not. Then in $\overline{P}(A, B)$ and $\overline{P}(B, C)$, $\{0\} \times [0, \infty]$ and $[0, \infty] \times \{0\}$ parametrize the same family of metrics. This allows us to form a larger parameter space by gluing on the boundary. Denote by $Q(B)$, the face of this new space given by the union of all parameters for which B is fully stretched. i.e. $Q(B) = [0, \infty] \times \{\infty\} \cup \{\infty\} \times [0, \infty]$. The space $Q(B)$ interpolates between breaking the cobordism – already broken at B – at A and at C. Note that in (14), each hypersurface occurs exactly twice and there is a cyclic arrangement of the pairs. This allows us to glue the parameter space into the pentagon $\overline{P}$ shown in Figure 7.

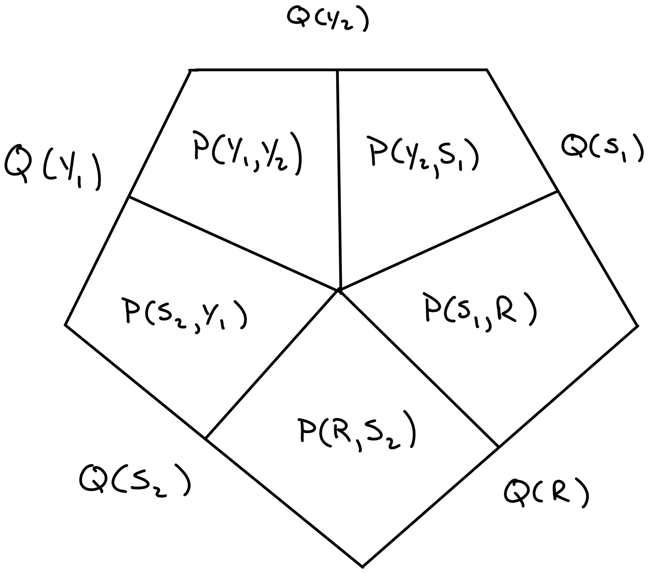

Figure 7. 2-parameter Homotopy of length 3 cobordism

Let $\check{G}^n_n$ be the map given by counting zero dimensional moduli spaces $M_z(\mathbf{a}, V_n^*, \mathbf{b})_{\overline{P}}$. Then looking at the boundary of 1-dimensional moduli spaces:

$$\check{\partial}^n_n \check{G}^n_n + \check{G}^n_n \check{\partial}^n_n = \check{m}_{Q(Y_{n+1})} + \check{m}_{Q(Y_{n+2})} + \check{m}_{Q(S_n)} + \check{m}_{Q(S_{n+1})} + \check{m}_{Q(R_n)}$$

Two of maps from the boundary components are easily identified:

$$\begin{aligned} \check{m}_{Q(Y_{n+1})} &= \check{H}^{n+1}_n \check{m}^n_{n+1} \\ \check{m}_{Q(Y_{n+2})} &= \check{m}^{n+2}_n \check{H}^n_{n+2} \end{aligned}$$

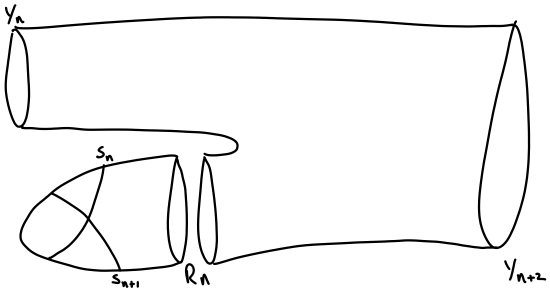

FIGURE 8. The cobordism $U_n^* \sqcup N_n^*$

Two more are known to be zero mod 2 from the previous discussion of the moduli spaces over Z_i:

$$\begin{aligned} \check{m}_{Q(S_n)} &= 0 \\ \check{m}_{Q(S_{n+1})} &= 0 \end{aligned}$$

Then if we write $\check{L}_n^n = \check{m}_{Q(R_n)}$ we have:

$$\check{\partial}_n^n \check{G}_n^n + \check{G}_n^n \check{\partial}_n^n = \check{m}_n^{n+2} \check{H}_{n+2}^n + \check{H}_n^{n+1} \check{m}_{n+1}^n + \check{L}_n^n \tag{15}$$

Now, to show that the surgery triangle satisfies part 2 of Lemma 10, we need to show that $\check{m}_n^{n+2} \check{H}_{n+2}^n + \check{H}_n^{n+1} \check{m}_{n+1}^n$ induces an isomorphism on homology. By (15), this is equivalent to showing that $\check{L}_n^n$ does. We quote the relevant proposition:

Proposition 12 (5.6 of [8]). *The map $\check{L}_n^n : \check{C}_\bullet(Y_n) \to \check{C}_\bullet(Y_n)$ induces isomorphisms on homology. The resulting map on $\check{H}_\bullet(Y_n)$ is multiplication by the power series*

$$p(U) = \sum_{k \geqslant 0} U^{k(k+1)/2},$$

which has leading coefficient 1. *(Recall that such a formal power series is invertable exactly when it has an invertable degree zero coefficient.)*

We omit the proof here and simply discuss the setup; a similar proposition will be proved later. In the relevant parametrization space $Q(R_n)$, the homotopy of metrics occurs between stretching on S_n and S_{n+1}. As R_n is fully stretched out in $Q(R_n)$, the homotopy is non-trivial on only the component N_n^* of the broken cobordism, $U_n^* \sqcup N_n^*$. See Figure 8. Through careful analysis of an explicit model in Section 5.3 of [8], it is then determined that $\check{m}(N_n^*)_{Q(R_n)}$ is $\sum_{k \geqslant 0} U_\dagger^{k(k+1)/2}$ times $\check{m}(S^1 \times \mathbb{D}^3)$. Then since $U_n \cup_{R_n} S^1 \times \mathbb{D}^3 \cong Y_n \times I$, the proposition follows.

Now let us mention a slight enhancement to the statement of the theorem: Suppose that ω_{Y_0} is a closed 2-form on Y_0 (though of as the the incoming boundary of V_0), then since $H^2(V_0) \cong H^2(Y_0) \oplus \mathbb{Z}^2$ we can extend ω_{Y_0} over V_0 to a closed 2-form ω_{V_0} which has $\int_{E_i} \omega_{V_0} = 0$. Let ω_{Y_1} and ω_{Y_2} be the corresponding restrictions to Y_1 and Y_2. Finally, let ω'_{Y_0} be the restriction of ω_{V_0} to the outgoing end of V_0.

Since ω_{V_0} pairs trivially with $I^2(V_0)$, the restrictions to the 4-manifolds Z_n, N_n and the 3-manifolds S_n, R_n are exact and the above results go through with the modification that we are to take the perturbed moduli spaces and perturbed Floer groups throughout.

4. The main theorem

Let Y and Y_0 be closed oriented 3-manifolds and $K \subset Y_0$ a smoothly embedded copy of S^1. Suppose that W_0 is a compact oriented cobordism with $\partial W_0 = -Y \sqcup Y_0$ and that K bounds a punctured torus $\mathring{T}$ in W_0. A framing for K is a isotopy class of push-off inside of $N(K) \subset Y_0$. Choosing a framing for K determines a relative Euler number $e(\mathring{T}, K)$ for the normal bundle $N(\mathring{T}) \subset W_0$. We will call the framing for which $e(\mathring{T}, K) = 0$, the *null-homologous framing*. In the case that K is null-homologous in Y_0, this corresponds to the usual notion. The null homologous framing determines, up to isotopy, a splitting $N(K) \cong \partial \mathring{T} \times \mathbb{D}^2$.

Let γ_0 be a copy of $\{pt\} \times \partial\mathbb{D}^2$ in $\partial N(K) \cong \partial\mathring{T} \times \partial\mathbb{D}^2$. Using the outward normal, a choice of orientation for $\mathring{T}$ is equivalent to an orientation for K; this then orients γ_0. We wish to find a surgery triangle with 3-manifolds $\{Y_n\}_{n\in\mathbb{Z}/3\mathbb{Z}}$ and cobordisms $\{W^n_{n+1}\}_{n\in\mathbb{Z}/3\mathbb{Z}}$ for which $\mathring{T}$, together with the core of the 2-handle of W^0_1, forms a 2-torus T in $W_0 \cup_{Y_0} W^0_1$. Note that such a T has $[T]^2 = -1$ in $H_2(W_0 \cup W^0_1; \mathbb{Z})$.

The closure requirement, together with $\gamma_0 \cdot \gamma_1 = \gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_0 = -1$, forces $\gamma_1 = -\partial\mathring{T}$ and γ_2 to be the diagonal with homology class $-[\partial\mathbb{D}^2] + [\partial\mathring{T}] \in H_1(\partial N(K))$. There are exactly two such choices corresponding to the orientations on K. However each gives identical surgery triangles, differing only by the choice of orientation of a homology class of negative square.

Consider the diagram of maps in (16). The surgery triangle gives the maps $\{\check{m}^n_{n+1}\}_{n\in\mathbb{Z}/3\mathbb{Z}}$, $\{\check{H}^n_{n+2}\}_{n\in\mathbb{Z}/3\mathbb{Z}}$, and $\{\check{L}^n_n\}_{n\in\mathbb{Z}/3\mathbb{Z}}$ along the top row. Let $\check{m}_0$ be the chain map $\check{m}_0 : \check{C}_\bullet(Y) \to \check{C}_\bullet(Y_0)$ induced by W_0.

$$\begin{tikzcd}
\cdots \arrow[r] & \widecheck{CM}(Y_0) \arrow[r, "\check{m}^0_1"] \arrow[rr, bend left, "\check{H}^0_2"] \arrow[rrr, bend left=40, "\check{L}^0_0"] & \widecheck{CM}(Y_1) \arrow[r, "\check{m}^1_2"] \arrow[rr, bend left, "\check{H}^1_0"] & \widecheck{CM}(Y_2) \arrow[r, "\check{m}^2_0"] & \widecheck{CM}(Y_0) \arrow[r] & \cdots \\
& \widecheck{CM}(Y) \arrow[u, "\check{m}_0"] \arrow[ur, "\check{m}_1"'] \arrow[urr, "\check{m}_2"'] \arrow[urrr, bend right, "\check{L}^0_0\check{m}_0"'] & & & &
\end{tikzcd} \tag{16}$$

Previously, we saw that the maps $\check{L}^n_n : \check{C}(Y_n) \to \check{C}(Y_n)$ induce isomorphisms on homology and in (15) that they are chain homotopic to

$$\check{H}^{n+1}_n \check{m}^n_{n+1} + \check{m}^{n+2}_n \check{H}^n_{n+2}. \tag{17}$$

Since are interested in computing $(\check{m}_0)_*$, we can consider the composition $\check{L}^0_0\check{m}_0$ which, under the isomorphism $(\check{L}^0_0)_*$, induces the same map as $\check{m}_0$. From (15), the map $\check{L}^0_0\check{m}_0$ is chain homotopic to

$$\check{H}^1_0 \check{m}^0_1 \check{m}_0 + \check{m}^2_0 \check{H}^0_2 \check{m}_0 \tag{18}$$

From the mapping cone construction in the surgery exact triangle (See Lemma 10), we obtain

$$\check{H}(Y_0) \cong \ker(\check{m}^1_2)_* \oplus \left(\check{H}(Y_2)/\mathrm{im}(\check{m}^1_2)_*\right) \tag{19}$$

where the isomorphism is induced by $\check{H}_0^1 + \check{m}_0^2$ on the chain level. Thus, if we are given the isomorphism in (19) and can compute $(\check{m}_1^0\check{m}_0)_*$ and $(\check{H}_2^0\check{m}_0)_*$, we can determine $(\check{L}_0^0\check{m}_0)_*$ (and hence $(\check{m}_0)_*$.)

Of course, we cannot compute the maps $\check{m}_1^0\check{m}_0$ and $\check{H}_2^0\check{m}_0$ directly since we would *a priori* need to know $\check{m}_0$. Hence we wish to find maps

$$\check{m}_1 : \check{C}(Y) \to \check{C}(Y_1) \text{ and } \check{m}_2 : \check{C}(Y) \to \check{C}(Y_2)$$

so that Equation 18 is chain homotopic to

$$\check{H}_0^1\check{m}_1 + \check{m}_0^2\,(q(U_\dagger)\check{m}_2) \tag{20}$$

where $q(U_\dagger)$ is some invertable formal Laurent series in $U_\dagger$. Further, we will want $\check{m}_1, \check{m}_2$ to be natural in the sense that they are induced by cobordisms W_1 and W_2 from Y to Y_1 and Y_2 respectively.

Lemma 13. *Let* $\check{m}_1 : \check{C}(Y) \to \check{C}(Y_1)$ *be the map induced by* $(W_0 \underset{Y_0}{\cup} W_1^0)^*$. *Then* $\check{m}_1$ *is chain homotopic to* $\check{m}_1^0\check{m}_0$.

Proof. This follows directly from the gluing result. See Section 2.3.1 or section 26 of [7]. □

Now consider $\check{H}_2^0\check{m}_0$. The map $\check{H}_2^0$ is induced by the 1-parameter family of metrics on W_2^0 which interpolate between stretching on S_0 and Y_1. The underlying, unstretched cobordism is $V_T = W_0 \cup_{Y_0} W_1^0 \cup_{Y_1} W_2^1$, as seen in Figure 9. This cobordism has several distinguished interior hypersurfaces: $Y_0, Y_1, S_T = \partial N(T), S_0 = \partial N(E_0), R_T = \partial N(T \cup E_0)$. The normal neighborhoods $N(T), N(E_0)$ are understood to be small and the regular neighborhood $N(T \cup E_0)$ is small but large enough to contain $N(T)$ and $N(E_0)$. Here S_0 is a copy of the 3-sphere as before and R_T is a 3-torus.

The manifold $Z_T = N(T)$ is diffeomorphic to an Euler number -1 disc bundle over the 2-torus T. As a disc bundle, both Z_T and its boundary S_T admit a S^1 action which has fixed point set T. This gives S_T the structure of an Euler number -1 Seifert-fibered space over the torus with no singular fibers.

Using the outward normal, orient S_T as the boundary of Z_T, S_0 as the boundary of $Z_0 = N(E_0)$, and R_T as the boundary of $N_T = N(T \cup E_0)$. Let

- $B_T = (W \cup_{Y_0} W_0)\backslash\mathring{Z}_T$ with boundary $\partial B_T = -(Y \sqcup S_T) \sqcup Y_1$,
- $B_0 = (W_0 \cup_{Y_1} W_1)\backslash\mathring{Z}_0$ with boundary $\partial B_0 = -(Y_0 \sqcup S_0) \sqcup Y_2$,
- $U_T = V_T\backslash\mathring{N}_T$ with boundary $\partial U_T = -(Y \sqcup R_T) \sqcup Y_2$

Fix a metric on V_T so that each of the listed hypersurfaces and boundary have tubular neighborhoods on which the metric is a product. Additionally, we will require that the metric on S_0 have positive scalar curvature and be close to the standard round metric and that the metric on R_T be flat. *There is also a requirement for the metric on S_T which will be made more clear below.*

There are five tuples of non-intersecting hypersurfaces:

$$(Y_0, Y_1), (Y_0, S_0), (R_T, S_0), (R_T, S_T), \text{ and } (S_T, Y_1).$$

Note that the pattern of intersection is exactly that appearing in the surgery sequence. As a result, we can form a pentagon $\overline{P}_T$ of metrics and perturbations as

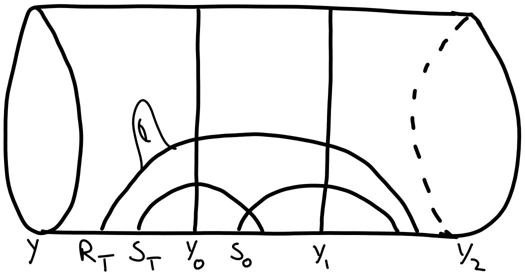

FIGURE 9. Hypersurfaces in V_T

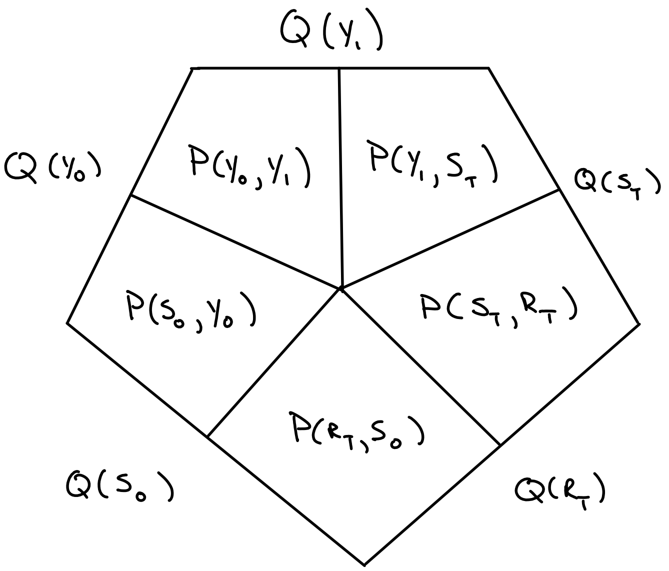

FIGURE 10. 2-parameter Homotopy of length 3 cobordism

shown in Figure 10. Let the maps $G_o^o, G_s^o, G_o^u, G_s^u, \overline{G}_s^s, \overline{G}_u^s, \overline{G}_s^u, \overline{G}_u^u$ be given by counting zero dimensional moduli spaces $M_z(\mathbf{a}, V_T^*, \mathbf{b})_{\overline{P}_T}$ and $\check{G}_2 : \check{C}_\bullet(Y) \to \check{C}_\bullet(Y_2)$ formed as below:

$$\check{G}_2 = \left[\begin{array}{cc} G_o^o & \partial_o^u \overline{G}_u^s + G_o^u \overline{\partial}_u^s + m_o^u \overline{H}_u^s + H_o^u \overline{m}_u^s + \partial_o^u (\overline{m}_u^{ss}(n_s \otimes \cdot)) \\ G_s^o & \overline{G}_s^s + \partial_s^u \overline{G}_u^s + G_s^u \overline{\partial}_u^s + m_s^u \overline{H}_u^s + \partial_s^u \overline{m}_u^{ss}(n_s \otimes \cdot) + \overline{m}_s^{ss}(n_s \otimes \cdot) \end{array} \right]$$

[16] Then looking at the boundary of 1-dimensional moduli spaces we obtain:

$$\check{\partial}_2^2 \check{G}_2 + \check{G}_2 \check{\partial} = \check{m}_{Q(Y_0)} + \check{m}_{Q(Y_1)} + \check{m}_{Q(S_T)} + \check{m}_{Q(R_T)} + \check{m}_{Q(S_0)}$$

---

[16] this need decorating with $0, 1, 2$ see analogous comment on pg 510 of Lens space surgeries

As in the surgery triangle, the component Z_0 has an even number of solutions for each solution on S_0. So

$$\check{m}_{Q(S_0)} \equiv 0 \mod 2$$

One of the other components is identifiable as:

$$\check{m}_{Q(Y_0)} = \check{H}^0_2 \check{m}_0$$

and another has a decomposition:

$$\check{m}_{Q(Y_1)} = \check{m}^1_2 \check{J}_1$$

where $\check{J}_1$ is determined by the homotopy of metrics on $W \cup_{Y_0} W_0$ which interpolates between stretching on Y_0 and S_T.

Lemma 14. *$\check{m}_{Q(S_T)}$ is chain homotopic to zero.*

Proof. We begin by following [14] to find the Floer chain groups for S_T. As previously stated, Z_T is a disc bundle over the torus T with projection π and boundary S_T, a circle bundle over T by π. Let $i\eta$ be a connection 1-form for a constant curvature connection on S_T. Fix a metric g_T on T which has constant scalar curvature equal to zero. Then S_T can be given the metric:

$$g_{S_T} = \eta^2 + \pi^*(g_T)$$

and the tangent bundle splits $T(S_T) \cong \mathbb{R} \oplus T(T)$. The Levi-Civita connection on T induces a reducible connection ${}^\circ\nabla$ which respects this splitting. This is not necessarily the Levi-Civita connection. Let ${}^\circ\hat{\nabla}$ be the Levi-Civita connection. With A a connection on a complex line bundle, we then get two Dirac operators: D_A and $\hat{D}_A$ on the corresponding *spin*$^{\mathbb{C}}$ bundle. Lemma 5.2.1 of [14] then tells us that:

$$\hat{D}_A = D_A - \frac{1}{2}\xi$$

where $\xi = -\frac{\pi \deg(Z_T)}{\mathrm{Vol}(T)} = \frac{\pi}{\mathrm{Vol}(T)}$. This constitutes an admissible perturbation. We then consider the solutions to the perturbed equations, on the *pre-blown up* configuration space $\mathcal{B}(S_T; \mathfrak{s})$.

Theorem 15 (Theorem 1 of [14])**.** *Let M be a Seifert fibered space of non-zero degree over the orbifold Σ. The moduli space of solutions to the Seiberg-Witten equations on M with metric g_M and connection ${}^\circ\nabla$ is naturally identified with the moduli space of effective orbifold divisors over Σ with orbifold degree no greater than $-\chi(\Sigma)/2$.*

Thus $\mathcal{M}_{sw}(S_T)$ is simply the torus of flat connections. Further, by 5.8.4 of [14], the Dirac operator has trivial kernel over this space.

Corollary 16. *With the above perturbation and a choice of perfect Morse function on the torus of flat connection of S_T (a T^2), the Monopole Floer critical points are $\mathbf{x}_i$, $\mathbf{y}^0_i, \mathbf{y}^1_i$ and $\mathbf{z}_i$, all reducible and corresponding to the critical points x, y^j, z of index $0, 1, 2$ (respectively) of the Morse function. (As before the if $i \geqslant 0$, the critical point is boundary stable and if $i < 0$ boundary unstable.) Further, there is no spectral flow along any trajectory.*

Since S_T can also be described as the boundary of a 4-manifold obtained by attaching zero framed 2-handles to both components of the Whitehead link, we find that, in the boundary stable case (when $i \geqslant 0$)

$$\begin{aligned} \mathrm{gr}^{\mathbb{Q}}(x_i) &= 2i-2 \\ \mathrm{gr}^{\mathbb{Q}}(y_i^0) = \mathrm{gr}^{\mathbb{Q}}(y_i^1) &= 2i-1 \\ \mathrm{gr}^{\mathbb{Q}}(z_i) &= 2i \end{aligned}$$

and in the boundary unstable case (when $i < 0$),

$$\begin{aligned} \mathrm{gr}^{\mathbb{Q}}(x_i) &= 2i-1 \\ \mathrm{gr}^{\mathbb{Q}}(y_i^0) = \mathrm{gr}^{\mathbb{Q}}(y_i^1) &= 2i \\ \mathrm{gr}^{\mathbb{Q}}(z_i) &= 2i+1 \end{aligned}$$

Since $Z^2 \cong H_1(S_T) \to H_1(Z_T)$ is an isomorphism, a choice of path z is determined by a choice of *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}$ on Z_T. Let $\mathfrak{t}_K$ be the *spin*$^{\mathbb{C}}$ structure with $\langle c_1(\mathfrak{t}_k), T\rangle = 2k+1$. Then the conjugation action sends $\bar{\mathfrak{t}}_k = \mathfrak{t}_{-1-k}$.

As in Lemma 11, using Lemma 27.4.2 and Definition 28.3.1 of [7], we compute formal dimensions

$$\begin{aligned} \dim M_{z_k}(Z_T, \mathbf{a}) &= \dim M((\mathbb{D}^4)^*, \mathbf{a}_0) + \mathrm{gr}_{z_k}(\mathbf{a}_0, Z_T\backslash\mathbb{D}^4, \mathbf{a}) \\ &= -1 - \mathrm{gr}^{\mathbb{Q}}(\mathbf{a}) + \frac{c_1^2(\mathfrak{t}_k) - \sigma(Z_T\backslash\mathbb{D}^4)}{4} - \iota(Z_T\backslash\mathbb{D}^4) \\ &= -1 - \mathrm{gr}^{\mathbb{Q}}(\mathbf{a}) - k(k+1) \end{aligned}$$

We then have the following cases: When $i \geqslant 0$:

$$\begin{aligned} \dim M_{z_k}(Z_T, \mathbf{x}_i) &= 1 - 2i - k(k+1) \\ \dim M_{z_k}(Z_T, \mathbf{y}_i^{\mu}) &= -2i - k(k+1) \\ \dim M_{z_k}(Z_T, \mathbf{z}_i) &= -1 - 2i - k(k+1) \end{aligned}$$

when $i < 0$

$$\begin{aligned} \dim M_{z_k}(Z_T, \mathbf{x}_i) &= -2i - k(k+1) \\ \dim M_{z_k}(Z_T, \mathbf{y}_i^{\mu}) &= -1 - 2i - k(k+1) \\ \dim M_{z_k}(Z_T, \mathbf{z}_i) &= -2 - 2i - k(k+1) \end{aligned}$$

As in Lemma 11, we see that formal dimensions are equal for conjugate *spin*$^{\mathbb{C}}$ structures. (i.e. $\mathfrak{t}_k$ and $\mathfrak{t}_{-1-k}$)

By arithmetic we see that, in the boundary stable case ($i \geqslant 0$), the only moduli spaces with non-zero formal dimension occur for $\mathbf{x}_0$ and $\mathbf{y}_0^{\mu}$ in the *spin*$^{\mathbb{C}}$ structures $\mathfrak{t}_0$ and $\mathfrak{t}_1$. The formal dimensions are 1 and 0, respectively.

In the boundary unstable case ($i < 0$), the zero dimensional moduli spaces in the *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}_k$ occur for $\mathbf{x}_{i_k}$ and $\mathbf{z}_{i_k-1}$ with $i_k = -\frac{k(k+1)}{2}$.

Now Z_T is negative indefinite, so all solutions over Z_T^* have $F_A^+ = 0$ and so are reducible. In fact, Theorem 10.0.15 (and following remarks) of [14], give us that the moduli space over Z_T for the non-blown up equations is the 2-torus $A_0 + \mathcal{H}^1(Z_T)/2\pi i H^1(Z_T; \mathbb{Z})$ with a vanishing spinor for each connection. (The blown-up finite-energy moduli space also includes the projectivization of the negative eigenspaces for D_A of each connection.)

Thus the contribution of each *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}_k$ to the relative invariant is $\mathbf{x}_{i_k}$ and $\mathbf{z}_{i_k-1}$ to $\overline{n}_u$. However, since $\overline{\mathfrak{t}_k} = \mathfrak{t}_{-1-k}$ contribute identically, $\overline{n}_u \equiv 0 \mod 2$.

Along the way we have also showed $n_o = n_s = \overline{n}_s = 0$.

$\square$

At this point,

$$\check{\partial}_2^2 \check{G}_2 + \check{G}_2 \check{\partial} = \check{H}_2^0 \circ \check{m}_0 + \check{m}_2^1 \circ \check{J}_1 + \check{m}_{Q(R_T)}$$

Let X be the manifold obtained by blowing down V_T along E_0. Equivalently, X is $U_T \underset{R_T = T^2 \times \partial\mathbb{D}^2}{\cup} T^2 \times \mathbb{D}^2$ where the identification of R_T and $T^2 \times \partial\mathbb{D}^2$ identifies $\partial\mathbb{D}^2$ with the kernel of the inclusion $H_1(R_T) \to H_1(N_T)$. Then $\partial X = -Y \sqcup Y_2$.

Lemma 17. *$\check{m}_{Q(R_T)}$ is chain homotopic to* $\left(\sum_{k \geqslant 0} U^{k(k+1)/2}\right) \check{m}(X)$

Proof. We proceed in a fashion similar to Proposition 12. Let P be the homotopy of metrics on N_T which interpolates between stretching fully on S_T to stretching fully on S_0. Let $n_s, n_o, \overline{n}_s$, and $\overline{n}_u$ be defined by the 0-dimensional counts of solutions over this family of metrics. These counts lie in $C^s_\bullet(R_T), C^o_\bullet(R_T), C^s_\bullet(R_T)$ and $C^u_\bullet(R_T)$, respectively.

Also, by counting points in 0-dimensional moduli spaces over U_T, we get several families of maps. The first set count irreducible solutions.

$$\begin{aligned}
m_o^{uo} : C^u_\bullet(R_T) \otimes C^o_\bullet(Y) &\quad\to\quad C^o_\bullet(Y_0)\\
m_o^{uu} : C^u_\bullet(R_T) \otimes C^u_\bullet(Y) &\quad\to\quad C^o_\bullet(Y_0)\\
m_s^{uo} : C^u_\bullet(R_T) \otimes C^o_\bullet(Y) &\quad\to\quad C^s_\bullet(Y_0)\\
m_s^{uu} : C^u_\bullet(R_T) \otimes C^u_\bullet(Y) &\quad\to\quad C^s_\bullet(Y_0)
\end{aligned}$$

The second set count reducible, *not*-boundary obstructed solutions.

$$\begin{aligned}
\overline{m}_u^{su} : C^s_\bullet(R_T) \otimes C^u_\bullet(Y) &\quad\to\quad C^u_\bullet(Y_0)\\
\overline{m}_s^{us} : C^u_\bullet(R_T) \otimes C^s_\bullet(Y) &\quad\to\quad C^s_\bullet(Y_0)\\
\overline{m}_s^{uu} : C^u_\bullet(R_T) \otimes C^u_\bullet(Y) &\quad\to\quad C^s_\bullet(Y_0)\\
\overline{m}_u^{uu} : C^u_\bullet(R_T) \otimes C^u_\bullet(Y) &\quad\to\quad C^u_\bullet(Y_0)
\end{aligned}$$

The third counts reducible boundary obstructed solutions. (For these, 0-dimensional moduli spaces have $\mathrm{gr}_z(a', a, U_T^*, b) = -1$.)

$$\begin{aligned}
\overline{m}_s^{ss} : C^s_\bullet(R_T) \otimes C^s_\bullet(Y) &\quad\to\quad C^s_\bullet(Y_0)\\
\overline{m}_s^{su} : C^s_\bullet(R_T) \otimes C^u_\bullet(Y) &\quad\to\quad C^s_\bullet(Y_0)\\
\overline{m}_u^{us} : C^u_\bullet(R_T) \otimes C^s_\bullet(Y) &\quad\to\quad C^u_\bullet(Y_0)
\end{aligned}$$

The fourth counts reducible doubly boundary obstructed solutions. (For these, 0-dimensional moduli spaces have $\mathrm{gr}_z(a', a, U_T^*, b) = -2$.)

$$\overline{m}_u^{ss} : C^s_\bullet(R_T) \otimes C^s_\bullet(Y) \quad\to\quad C^u_\bullet(Y_0)$$

WRITE HOW TO GET THE MAPS ON COMPOSITE

L_T defined by sum over good breaks on R_T.

BEGIN MESSY NOTE, CHECK THESE

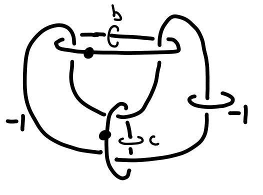

FIGURE 11. N_T with 1-cycles b, c identified

$$\begin{aligned}
(L_T)^o_o &= m^{ou}_o(\cdot \otimes \overline{n}_u) \\
(L_T)^o_s &= m^{ou}_s(\cdot \otimes \overline{n}_u) \\
(L_T)^u_o &= m^{uu}_o(\cdot \otimes \overline{n}_u) \\
(L_T)^u_s &= m^{uu}_s(\cdot \otimes \overline{n}_u)
\end{aligned}$$

$$\begin{aligned}
(\overline{L}_T)^u_u &= \overline{m}^{uu}_u(\cdot \otimes \overline{n}_u) \\
(\overline{L}_T)^u_s &= \overline{m}^{uu}_s(\cdot \otimes \overline{n}_u) \\
(\overline{L}_T)^s_u &= \overline{m}^{su}_u(\cdot \otimes \overline{n}_u) + \overline{\partial}^s_u \overline{m}^{su}_s(\cdot \otimes \overline{n}_u) \\
(\overline{L}_T)^s_s &= \overline{m}^{su}_s(\cdot \otimes \overline{n}_u)
\end{aligned}$$

END MESSY NOTE

THEN L IS LINEAR IN $\overline{n}_u$.

Before we examine the family moduli spaces over N_T further, recall the solutions on $R_T \cong T^3$ from Section 2.2.5. Note that the inclusion $H^1(N_T) \to H^1(R_T)$ gives us a 2-torus $\mathbb{T}$ in the 3-torus of flat connections on R_T.

To fix notation, let us say that it is x_i, y^b_i, y^c_i, and z^{bc}_i which may be the limit of a solution on N_T. Equivalently, we identify N_T with the Kirby diagram in Figure 11 and see that if we blow down the -1 sphere, we get a $T^2 \times \mathbb{D}^2$ with 1-cycles b and c the same as in Figure 2.

Now we consider the moduli spaces over N_T. The second homology of N_T, $H_2(N_T)$, is generated by T and exceptional sphere E_0. Each of these is a class of square -1 and with their given orientations, $[T] \cdot [E_0] = 1$. If $\mathfrak{t} \in spin^{\mathbb{C}}(N_T)$ with $\mathfrak{t}|_{R_T} = \mathfrak{s}_0$, then $\mathfrak{t}$ is determined by $c_1(\mathfrak{t}) \cdot T$. Write $\mathfrak{t}_k$ for such a $spin^{\mathbb{C}}$ structure with $c_1(\mathfrak{t}_k) \cdot T = 2k+1$. With this identification we see that $\bar{\mathfrak{t}}_k = \mathfrak{t}_{-1-k}$.

The paths in $\pi(N^*_T, \mathbf{a})$ are an affine space over $\pi(R_T)$, under the operation of right-composition. Since $\mathrm{gr}_{z_u}(a,a) = (u \cup c_1(\mathfrak{s}_0))[R_T] = 0$, all moduli spaces $M_{z \circ z_u}(N^*_T, \mathbf{a})$ have the same formal dimension, with z a path for $\mathfrak{t}_k$. Form $M_k(N^*_T, \mathbf{a})$ as union of $M_{z \circ z_u}(N^*_T, \mathbf{a})$ over all u – which is still a compact manifold when zero dimensional.

Now, let is compute the formal dimensions of the moduli spaces. We begin with the moduli space for a single metric:

$$\begin{aligned}
\dim M_k(N_T^*, \mathbf{a}) &= \dim M(\mathbb{D}^4, \mathbf{a}_0) + \mathrm{gr}_{z_k}(\mathbf{a}_0, N_T \backslash D^4, \mathbf{a}) \\
&= -1 + \Big(-\mathrm{gr}^{\mathbb{Q}}(\mathbf{a}) + \frac{c_1^2(\mathfrak{t}_k) - \sigma(N_T \backslash \mathbb{D}^4)}{4} - \iota(N_T \backslash \mathbb{D}^4) \Big) \\
&= -1 + \Big(-\mathrm{gr}^{\mathbb{Q}}(\mathbf{a}) + \frac{c_1^2(\mathfrak{t}_k) - \sigma(N_T \backslash \mathbb{D}^4)}{4} - \iota(N_T \backslash \mathbb{D}^4) \Big) \\
&= -2 - \mathrm{gr}^{\mathbb{Q}}(\mathbf{a}) - k(k+1)
\end{aligned}$$

So, for our 1-parameter family of metrics:

$$\begin{aligned}
\dim M_k(N_T^*, \mathbf{a})_Q &= \dim M_k(N_T^*, \mathbf{a}) + 1 \\
&= -1 - \mathrm{gr}^{\mathbb{Q}}(\mathbf{a}) - k(k+1)
\end{aligned}$$

Then in terms of the x_i, y_i^*, z_i^*, w_i:

$$\begin{aligned}
\dim M_k(N_T^*, \mathbf{x}_i)_Q &= \begin{cases} -2i - k(k+1) & \text{if } i \geqslant 0 \\ -1 - 2i - k(k+1) & \text{if } i < 0 \end{cases} \\
\dim M_k(N_T^*, \mathbf{y}_i^*)_Q &= \begin{cases} 1 - 2i - k(k+1) & \text{if } i \geqslant 0 \\ -2i - k(k+1) & \text{if } i < 0 \end{cases} \\
\dim M_k(N_T^*, \mathbf{z}_i^*)_Q &= \begin{cases} -2i - k(k+1) & \text{if } i \geqslant 0 \\ -1 - 2i - k(k+1) & \text{if } i < 0 \end{cases} \\
\dim M_k(N_T^*, \mathbf{w}_i)_Q &= \begin{cases} -1 - 2i - k(k+1) & \text{if } i \geqslant 0 \\ -2 - 2i - k(k+1) & \text{if } i < 0 \end{cases}
\end{aligned}$$

Suppose that $i \geqslant 0$ so that the critical point is boundary stable. By arithmetic, we find that the only moduli spaces with non-negative formal dimension occur for

$$\begin{array}{rlll}
\mathbf{x}_0 \text{ and } \mathbf{z}_0^* & \text{when} & k = 0, \text{ or } -1 & \text{with dimension} = 0 \\
\mathbf{y}_0^* & \text{when} & k = 0, \text{ or } -1 & \text{with dimension} = 1
\end{array}$$

Now, if the moduli space for the $\mathbf{y}_0^*$ were non-empty we would see, via Lemma 5, that one of N_T^*, Z_T, or Z_0 has an irreducible solution. However, each is negative indefinite and there are no irreducible solutions. As the reducible moduli spaces for a boundary stable critical point occur as the boundaries of these spaces, $M_k^{red}(N_T^*, \mathbf{y}_0^*)_Q$ contributes zero to n_s and $\overline{n}_s$.

Now suppose that $i < 0$ so that the critical point is boundary unstable. Then any solution is reducible and we verify by arithmetic that the spaces with formal dimension 0 are $M_k^{red}(N_T^*, \mathbf{y}^n_{-k(k+1)/2})_Q$ and $M_k^{red}(N_T^*, \mathbf{w}_{-1-k(k+1)/2})_Q$.

At this point we have,

$$\begin{aligned}
n_o &= 0\\
n_s &= \#\left(M_0(N_T^*,\mathfrak{t}_0;\mathbf{x}_0)_Q \cup M_{-1}(N_T^*,\mathfrak{t}_{-1};\mathbf{x}_0)_Q\right)\mathbf{x}_0\\
&\quad +\#\left(M_0(N_T^*,\mathfrak{t}_0;\mathbf{z}_0^{ab})_Q \cup M_{-1}(N_T^*,\mathfrak{t}_{-1};\mathbf{z}_0^{ab})_Q\right)\mathbf{z}_0^{ab}\\
&\quad +\#\left(M_0(N_T^*,\mathfrak{t}_0;\mathbf{z}_0^{ac})_Q \cup M_{-1}(N_T^*,\mathfrak{t}_{-1};\mathbf{z}_0^{ac})_Q\right)\mathbf{z}_0^{ac}\\
&\quad +\#\left(M_0(N_T^*,\mathfrak{t}_0;\mathbf{z}_0^{bc})_Q \cup M_{-1}(N_T^*,\mathfrak{t}_{-1};\mathbf{z}_0^{bc})_Q\right)\mathbf{z}_0^{bc}\\
\overline{n}_s &= 0\\
\overline{n}_u &= \sum_{k\in\mathbb{Z}} b_k\mathbf{w}_{-1-k(k+1)/2} + \sum_{k\in\mathbb{Z}\setminus\{0,-1\}}\ \sum_{*\in\{a,b,c\}} a_{k,*}\mathbf{y}^*_{-k(k+1)/2}\\
&= \sum_{k=0}^{\infty}(b_k+b_{-1-k})\mathbf{w}_{-1-k(k+1)/2} + \sum_{k=1}^{\infty}\ \sum_{*\in\{a,b,c\}}(a_{k,*}+a_{-1-k,*})\mathbf{y}^*_{-k(k+1)/2}
\end{aligned}$$

where $a_{k,*} = \#\left(M_k^{red}(N_T^*,\mathfrak{t}_k;\mathbf{y}^*_{-k(k+1)/2})\right)$ and $b_k = \#\left(M_k^{red}(N_T^*,\mathfrak{t}_k;\mathbf{w}_{-1-k(k+1)/2})\right)$.

Lemma 18. *For all $k\in\mathbb{Z}$, we have:*

$$\begin{aligned}
a_{k,*}+a_{-1-k,*} &\equiv \begin{cases} 1 \mod 2 & \text{if } *=a\\ 0 \mod 2 & \text{otherwise}\end{cases}\\
b_k+b_{-1-k} &\equiv 1 \mod 2
\end{aligned}$$

So that:

$$\begin{aligned}
\overline{n}_u &\equiv \sum_{k=0}^{\infty}\mathbf{w}_{-1-k(k+1)/2} + \sum_{k=1}^{\infty}\mathbf{y}^a_{-k(k+1)/2} \mod 2\\
&\equiv \left(\sum_{k=0}^{\infty}U_\dagger^{k(k+1)/2}\right)\mathbf{w}_{-1} + \left(\sum_{k=1}^{\infty}U_\dagger^{k(k+1)/2}\right)\mathbf{y}^a_{-1} \mod 2
\end{aligned}$$

Note that

$$\sum_{k=1}^{\infty}U_\dagger^{k(k+1)/2} = \sum_{l=0}^{\infty}U_\dagger^{l(l+3)/2}$$

whose leading term is 1. Thus the coefficient of y^a_{-1} is an *invertable* power series in $\mathbb{Z}[[U_\dagger]]$.

Proof. We will make use of our results from section 2.5.2. We can assume that the perturbation 2-form ω vanishes, since this is not needed for regularity on R_T. (So reducible connections are actually flat.)

Fix a *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}_k$ on N_T^* which restricts to $\mathfrak{s}_0$ on R_T and let A_0 be a *spin*$^{\mathbb{C}}$ connection on $\mathfrak{t}_k$ with harmonic curvature. Since N_T^* is negative indefinite $F^+_{A_0^t} = 0$ and $F^+_{(A_0+a)^t} = 0$ for all $a\in\mathcal{H}^1(N_T)$. Thus there is a $T^2 = H^1(N_T;\mathbb{R})/2\pi i H^1(N_T;\mathbb{Z})$ worth of solutions to the abelian anti-self-dual equation $F^+_{A^t} = 0$ in each *spin*$^{\mathbb{C}}$ structure on N_T, for any cylindrical end metric.

Let $\mathcal{S}_{R_T}$ be the 3-torus of flat connections on R_T and $\mathcal{S}_{N_T}(k,g)$ the 2-torus of abelian anti-self-dual connections on N_T for the cylindrical end metric g in the *spin*$^{\mathbb{C}}$ structure $\mathfrak{t}_k$. (i.e. solutions to $F^+_{A^t} = 0$) Since each of the connections $A^t\in\mathcal{S}_{N_T}(k,g)$ is asymptotically flat, it defines a point in $\mathcal{S}_{R_T}$. This gives a map $\theta_k(g):\mathcal{S}_{N_T}\to\mathcal{S}_{R_T}$ which depends only on g,k.

Complex conjugation on $\mathfrak{s}_0$ acts on the torus of flat connections by $\sigma : (S^1)^3 \to (\overline{S^1})^3$. The 8 fixed points correspond to the *spin* structures on R_T and are the connection parts of the reducible critical points for R_T. The conjugation action extends to N_T by $\bar{\mathfrak{t}}_k = \mathfrak{t}_{-1-k}$. Then we have

$$\sigma\theta_k(g) = \theta_{-1-k}(g).$$

Now consider the 1-parameter family of metrics $\overline{Q} = \overline{Q}(S_T, S_1)$. As t goes to each of $\pm\infty$, N_T^* breaks into two parts. In the case of $t = -\infty$, we get Z_T^* and a manifold C_T^* which has cylindrical ends $\mathbb{R}^- \times S_T$ and $\mathbb{R}^+ \times R_T$. In the case of $t = +\infty$, we get Z_1^* and a manifold C_1^* which has cylindrical ends $\mathbb{R}^- \times S_1$ and $\mathbb{R}^+ \times R_T$. Here C_1^* is a punctured $T^2 \times \mathbb{D}^2$ with cylindrical ends.

In either case, the $C_{\square}^*$'s have $I^2 = 0$ and so carry no L^2 harmonic 2-forms. Thus the maps θ_K extent continuously to $\pm\infty$ and $\phi_k(\pm\infty)$ is invariant under σ. This gives a map

$$\theta_k : \overline{Q} \times T^2 \to \mathcal{S}_{R_T}.$$

Also, $\theta_k(+\infty) = \theta_{-1-k}(+\infty)$ and $\theta_k(-\infty) = \theta_{-1-k}(-\infty)$ so the union

$$\theta_k(\overline{Q}) \cup \theta_{-1-k}(\overline{Q})$$

defines a $\mathbb{Z}/2\mathbb{Z}$ 3-cycle Θ_k in $\mathcal{S}_{R_T}$. Now, for generic x in $\mathcal{S}_{R_T}$, $\Theta_k^{-1}(x)$ is cobordant to

$$\left(M_k^{ab}(N_T^*, \mathcal{T})_{\overline{Q}} \bigcup M_{-1-k}^{ab}(N_T^*, \mathcal{T})_{\overline{Q}}\right) \times M^{ab}(\mathcal{T}, ([0,1] \times R_T)^*, \mathcal{S}_{R_T}) \times_{\mathcal{S}} \{x\}$$

with $\mathcal{T}$ the 2-torus of flat connections in $\mathcal{S}_{R_T}$ with holonomy -1 around the loop a, the M^{ab} the moduli spaces for the perturbed abelian anti-self-duality equations, and $M^{ab}(w, ([0,1] \times R_T)^*, \mathcal{S})$ with perturbed at $+\infty$ and no perturbation at $-\infty$.

This cobordism comes from taking 1-parameter family of perturbations $t \in \mathbb{R}^+$ on N_T^* which are supported on ever longer pieces of attached cylinder. The fiber over $t = 0$ is $\{x\} \times_{\mathcal{S}} \left(M_k^{ab}(N_T^*, \mathcal{S}_{R_T})_{\overline{Q}} \bigcup M_{-1-k}^{ab}(N_T^*, \mathcal{S}_{R_T})_{\overline{Q}}\right)$.

These moduli space are pertinent since the are related to the zero dimensional Seiberg-Witten moduli spaces as follows:

$$M_k^{ab}(N_T^*, \mathcal{T}) \times_{\mathcal{S}_{R_T}} \mathbf{a} \equiv M_k(N_T^*, a_{i_k})$$

for $\mathbf{a} \in \left\{\mathbf{y}^a, \mathbf{y}^b, \mathbf{y}^c, \mathbf{w}\right\}$ and

$$i_k = \begin{cases} -\frac{k(k+1)}{2} & \text{if } \mathbf{a} \in \{\mathbf{y}^a, \mathbf{y}^b, \mathbf{y}^c\} \\ -1 - \frac{k(k+1)}{2} & \text{if } \mathbf{a} = \mathbf{w} \end{cases}$$

and where $M_k^{ab}(N_T^*, \mathcal{T}) \times_{\mathcal{S}_{R_T}} \mathbf{a}$ denotes the fiber product.

We immediately see that

$$a_{k,*} + a_{-1-k,*} \equiv 0 \mod 2 \text{ if } * = b, c$$

since $\mathbf{y}^b, \mathbf{y}^c$ are not in $\mathcal{T}$. The remainder of the theorem will follow if the mod 2 degree of Θ_k is non-zero. This is equivalent to $\theta_k(\infty) \neq \theta_k(-\infty)$.

Observe that the loop a corresponds to the generator of kernel $H_1(R_T) \to H_1(N_T)$. Let $\Sigma \subset N_T^*$ a disc with cylindrical boundary $\mathbb{R}^+ \times a$. Let $A : \mathcal{S}_{R_T} \to S^1$ be the map which takes a flat connection in $\mathcal{S}_{R_T}$ and sends it to its holonomy around the loop a.

Now, there is an explicit model of $A \circ \theta_k$ given by

$$A \circ \theta_k(g) = \exp \frac{1}{2} \int_\Sigma F_{A^t}$$

When $t = -\infty$, Σ is in T_1^* and A^t is flat. so $A \circ \theta_k(-\infty) = 1$. However, when $t = +\infty$, Σ decomposes into a cylinder in T_2^* which has zero contribution to integral and a disc Δ with cylindrical end in Z_T^*.

Now, $\displaystyle\int_\Delta F_{A^t} = (2\pi/i)(2k+1)$ so $A \circ \theta_k(\infty) = -1$. This completes the proof of Lemma 18.

□

Thus we have completed the proof of Lemma 17.

□

Let W_2 be $U_T \cup_\phi N\mathbb{T}\mathrm{w}$ where ϕ is an orientation reversing diffeomorphism of T^3 which sends the curve c in $\partial N\mathbb{T}\mathrm{w}$ to a generator of $\ker(H_1(T^3) \to H_1(N_T))$. [17] Equivalently this may be accomplished by blowing down the -1 sphere in N_T then exchange •s and 0s in the Kirby diagram – all within $V_T = U_T \cup N_T$.

Then, for some chain homotopy G'',

$$\check{\partial}\check{G}'' + \check{G}''\check{\partial} = \check{H}_2^0 \check{m}_0 + \check{m}_2^1 \check{J} + \left(\sum_{l=0}^{\infty} U^{l(l+3)/2} \right) \check{m}(W_2)$$

Now, $\check{m}_2^1$ is the d_1 differential in the mapping cone's 2 page spectral sequence. Thus $\check{m}_1 \check{J}$ is chain homotopic to zero in the mapping cone. (Although not necessarily in the chain complexes for Y_2.) Therefore, $\check{L}_0^0 \check{m}_0$ is chain homotopic to $\check{m}(W_1) + \left(\sum_{l=0}^{\infty} U^{l(l+3)/2}\right) \check{m}(W_2)$ as a map to the mapping cone. Notice that $\check{m}(W_1)$ and $\check{m}(W_2)$ are cobordism maps, not maps from homotopies. This proves the main theorem.

As in the case of the surgery exact sequence, the result continues to hold in the presence of a perturbation by a non-exact 2-form. Suppose that ω_{W_0} is a closed 2-form on W_0 which pairs trivially with the punctured torus $\mathring{T}$ ($\int_{\mathring{T}} \omega_{W_0} = 0$), then ω_{W_0} extends over $W_0 \cup_{Y_0} V_0$ to a closed 2-form $\omega_{W_0 \cup V_0}$ with $\int_{E_i} \omega_{W_0 \cup V_0} = 0$. Since W_1 is a submanifold of $W_0 \cup_{Y_0} V_0$, this defines ω_{W_1} on W_1.

Now, to define ω_{W_2} on W_2, we note that $\omega_{W_0 \cup V_0}$ pairs trivially with both T and E_0 by construction. Then the restriction of $\omega_{W_0 \cup V_0}$ is exact on R_T. Thus, we can extend the restriction of $\omega_{W_0 \cup V_0}$ to U_T over $N\mathbb{T}\mathrm{w}$ by an exact 2-form. This defines ω_{W_2}. The perturbed version of the theorem then follows using these perturbations.

5. 2-HANDLE COBORDISMS AND CROSSING CHANGES

Let W be the cobordism corresponding to a 2-handle addition to $[0,1] \times Y$. We can obtain a relative Kirby calculus diagram for W by first constructing a Dehn surgery diagram for Y (we will use thin lines to denote these) then in this diagram drawing the attaching region of the 2-handle as a knot K with framing indicated by the difference from the Seifert framing of K as a knot in S^3. For convenience, fix an orientation for K corresponding to an orientation of the core of the 2-handle.

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[17] Need to make this more concrete for the skein sequence. Linking number will matter.

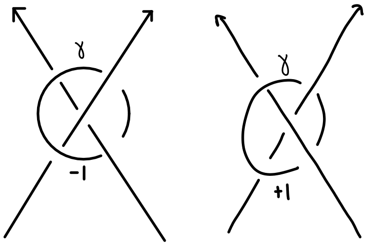

Figure 12. Surgeries which change crossing type

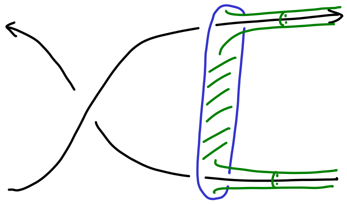

Figure 13. Punctured torus

Let Y_0 be obtained by the corresponding integral Dehn surgery on K in Y. Then $\partial W = -Y \sqcup Y_0$. Now suppose that γ is a null-homologous loop in the neighborhood of a crossing of K, as shown in Figure 12 for a positive crossing (left) and a negative crossing (right).

For either type of crossing, γ is the boundary of a punctured torus as shown in Figure 13 in the case of a positive crossing. Explicitly, we take a torus which is the boundary of a tubular neighborhood of K then the Seifert disc D for γ meets K in two points, once positively and once negatively, and so divides the torus into two pieces. Select one of these annular pieces A and form the punctured torus $\mathring{T} = (D \backslash N(K)) \cup A$. By construction, $\partial \mathring{T} = \gamma$.

Note that the two versions of $\mathring{T}$ obtained by the two choices of annulus are isotopic rel γ. (Slide the annular part over the 2-handle.)

5.1. **Positive Crossings.** Suppose that the crossing in question is positive and let Y_1 be the manifold obtained by -1 surgery on γ as in the left diagram of Figure 12. Similarly, let Y_2 be obtained by zero surgery on γ. Then Y_0, Y_1 and Y_2 fit into a surgery exact triangle which is of the type dealt with in Section 4.

We now wish to construct relative Kirby diagrams for the cobordisms W_1 and W_2.

5.2. **Negative Crossings.** For a negative crossing, the usual skein sequence involves doing the surgery in the right diagram of Figure 12. As this sequence involves $+1$ surgery on γ and thus a torus of square 1 instead of a torus of square -1, our techniques fail here. That is, the algebraic decomposition of the map $\check{L}_0^0\check{m}(W_0) \sim \check{H}_0^1\check{m}_1^0\check{m}_0 + \check{m}_0^2\check{H}_2^0\check{m}_0$ into pieces $\check{m}_1^0\check{m}_0$ and $\check{H}_2^0\check{m}_0$ which land in the

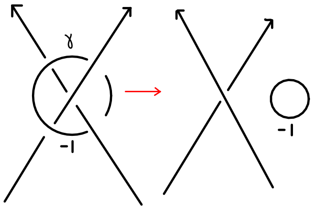

FIGURE 14. Handle slide for a crossing change (positive to negative)

direct summands $\check{C}(Y_1) \oplus \check{C}(Y_2)$ of the mapping which was discussed at the beginning of Section 4 still holds. It remains unclear if these homomorphisms are induced by cobordisms.

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