A Plumber's Approach to
Symplectic Geometry, Part I

Plumbing: Consider disk bundles \( D^n \to Q^n \) and \( D^n \to Q^n \).

Each has a neighborhood diffeo to \( D^n \times D^n \).

Pick a diffeo \( D^n \times D^n \to D^n \times D^n \) which exchanges the two factors, and use this to glue \( E_1 \) to \( E_2 \) making a plumbing of \( E_1 \) and \( E_2 \), \( E_1 \#_p E_2 \).

\[ \mathbb{E}_x: \quad T^* S^1 \#_p T^* S^1 : \]

\[ (\mathbb{A}_2) \]

\[ \mathbb{E}_x: \quad \text{Mirror fiber} \quad A_{m}^{2n} = \sum_{i \neq j} z_i^2 z_j^2 + \ldots + z_m^2 + 1 \geq 0 \]

is a plumbing of \( T^* S^{2n-2} \)'s according to the graph

\[ \begin{array}{cccccccc}
  & 1 & 0 & \cdots & 0 \\
-2 & -2 & -2 & \cdots & 0 \\
\end{array} \]

\[ \text{m copies} \]

\[ \mathbb{E}_x: \quad \mathbb{A}_3: \]

Fact: \( T^* Q^n \#_p T^* Q^n \) has a natural Liouville (and in particular symplectic) structure.
Main results (Abouzaid-Smith)

Let \((M, \omega)\) be a \(3\)-manifold, i.e. \(\Omega^3(M, \omega)\) is symplectic and the end is of the form \((\mathbb{R}^+ \times M, e^{-\lambda} \omega)\).

- In particular \((2M, \omega)\) is a \(C^1\) strat.
- For \(L, L' \subset M \) Lag\(\:\nabla\), define
  \[ CW(L, L') = \mathbb{T} \ominus \Omega(L) \wedge L'^{\perp}, \]
where \(\mathbb{T}\) is the Hamilton flow of a function \(H\) which looks like \((e^r)^2\) on the collar.

- Equip \(CW\) with the usual Floer differential, counting holomorphic disks.

Remk: For \(L, L' \subset \text{cpt}\), agrees with usual Floer homology.

Ex: For \(T_0^*Q \subset T^*Q\) a cot. fiber,
  \[ HW(T_0^*Q, T_0^*Q) \cong H_{-*}(\mathcal{L}_Q) \]
  \[ \text{based loop space} \]

In fact,
  \[ CW(T_0^*Q, T_0^*Q) \cong C_{-*}(\mathcal{L}_Q) \]
  \[ A_\infty - \text{algebra} \]
Local systems:

Def: A local system on a \( \mathcal{L} \) is a bundle of Abelian gps over \( \mathcal{L} \), i.e., a \( \mathbb{Z}[\mathcal{L}] \text{-module} \).

For \( \mathcal{L} \), \( \mathcal{L}' \), \( \mathcal{L} \) Lagns with local systems, can define \( \text{HF}(\mathcal{L}, \mathcal{L}') \), using \( \mathcal{L} \), \( \mathcal{L}' \) hom(\( \mathcal{L}, \mathcal{L}' \)) and using homology P.L.N.L along sides of a strip to define the skein.

Twisted complexes:

\( \text{Fuk}(M) \rightarrow \text{Tw Fk}(M) \) enlarged \( \infty \text{-category} \).

EBs of \( \text{Tw Fk}(M) \) consist of:

- a formal direct sum \( V_1 \oplus \mathcal{L}_1 \oplus \ldots \oplus V_k \oplus L_k \)
- \( V_i \) graded Ab. gp.
- \( L_i \) Lagns.
- an internal differential \( \delta_{ij} \), where for \( i \leq j \), \( \delta_{ij} \in \text{hom}(V_i, V_j) \oplus \mathcal{C}(L_i, L_j) \)
- \( \delta_{ij} \) satisfies a "generalized MC eqn" \( r \sum_{r=1}^s \mu^r(\delta_{i_1}, \ldots, \delta_{i_s}) = 0 \).

Remark: Arrows look like \( \underbrace{\bullet} \text{ instead of } \text{ just} \)
Morphisms in $\text{Tufuk}$: made of morphisms of summands

$\text{Ex}: \operatorname{Mor}_{\text{Tufuk}}(V, \otimes L_1 \otimes V_2 \otimes L_2, W \otimes Q)$

$= \operatorname{hom}(V, W) \otimes \operatorname{CF}(L_1, Q)
\oplus \operatorname{hom}(V_2, W) \otimes \operatorname{CF}(L_2, Q)$

All maps in $\text{Tufuk}$ come from those of $\text{Fuk}$, twisted by internal differentials.

$M: \text{Tufuk} \to \text{Fuk}$

$\text{Main results: } (\text{Abova: Smith})$

Let $M = T^* Q^n \#_p T^* Q^n$

(C results will also hold for plumping along a tree)

Let $\mathcal{W}(M) = \text{wrapped Fuk cat}$

$\mathcal{I}(M) = \text{Fuk cat. of closed, exact Lgns}$

Then $\mathcal{L}(M)$ and $\mathcal{W}(M)$ is spanned by the

cat. fibers $T^*_0 Q_1$ and $T^*_0 Q_2$. 
Thm: If \( \dim M \geq 6 \), every closed exact Maslov zero Lagrangian is equivalent to a two-loop over the (petal cores equipped) local systems.

Now consider \( \mathbb{A}^2_\mathbb{Z} \).

Fact: \( \langle \mathbb{Q}_1, \mathbb{Q}_2 \rangle \cong \mathbb{B}_3 \).

Delan tauts

Thm: Every closed Maslov zero Lagrangian in \( \mathbb{L} (\mathbb{A}^n_\mathbb{Z}) \) lies in the orbit of one of the petal cores under \( \mathbb{B}_3 \) action.

Cor: Every closed exact Maslov zero Lagrangian in \( \mathbb{A}^n_\mathbb{Z} \) is a homology sphere and lies in a primitive homology class.

The fibers and the diagonal generate \( \mathbb{L} (M) \)

Let \( Q = Q_1 \# Q_2 \) and \( V \subset Q \) the belt
Sphere of the connect-sum. \( S^{n-1} \)

Pick a Morse function \( \phi : Q \rightarrow \mathbb{R} \) such that
- the \( Q_1 \) region maps to negative reals
- \( Q_2 \) region maps to positive reals
- \( V = \phi^{-1}(0) \) is a regular level set
Fukaya-Seidel-Smith: \( \Phi : Q \to R^n \) extends to a Lefschetz fibration

This means:
- \( \Phi \) is unique, as are the fibers of \( \Phi \)
- \( \Phi \) is a submersion onto its image, except for isolated critical points modeled on \( \mathbb{C}^n \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto z_1^2 + \cdots + z_n^2 \)

Image of \( \Phi \):

[Diagram of a circle with an annotation: can assume no critical points on y-axis]

Note: \( V \subset F = Q \times (0) \) is a Lagrangian \((n-1)\)-sphere

We now add a vanishing cycle to \( \Phi \) which corresponds to \( V \) to get a new L.C. \( \Phi : E \to C \)

Image of \( \Phi \):

[Diagram of a circle with an annotation: new critical point at (0,1)]

Let \( T \) be the set of points given by parallel transporting \( V \) along the segment \( \{ (0,y) : y \in [0,1] \} \).

Then \( T \) is a Lagrangian disk with \( T \cap F = V \).

\( T \) is called the thimble of the critical point at \((0,1)\)
Inside $E$ have a model for the plumbing.

\[ \text{should be horizontal lines near } (0, y) \]

Consider a general L.C. fiber of tangles $\Delta_1, \ldots, \Delta_m$.

Seidel's trick: Pull back $\mathbb{R}^2 \to \mathbb{D}^2$ along a branched double cover $\mathbb{D}^2 \to \mathbb{D}^2$. Get new L.C. $\mathbb{R}^2 \to \mathbb{R}^2$ with twice as many v.c.s.

5. e.

- \( \mathcal{L} \) is Lefschetz subdomain
- \( \mathcal{L} \) contains Lefschetz spheres (matching cycles)
  \[ \Delta_1, \ldots, \Delta_m, \quad \alpha_1 \wedge \beta = \Delta_i \]
  \[ T_{\Delta_1} \circ T_{\Delta_2} \circ \cdots \circ T_{\Delta_m} = \text{covering transf. of } \mathcal{L} \]

In particular, sends $E$ to a region disjoint from $\mathcal{L}$. 

Caption: Generating \( \mathcal{L} \) (Lefschetz fibration) (closed, exact Lag.)
Let \( L \subsetneq \mathcal{X} \) be a closed \( \mathcal{X} \)-in.

Recall: \[ T_{\Delta_0} L \overset{\simeq}{\longrightarrow} H^0(\widetilde{\Delta}_0, L) \otimes_{\Delta_0} \mathcal{O}_{\Delta_0} \to L \]

\( \Delta \)

\[ \text{a twisted } \mathcal{O}_X \]

Have \[ T_{\Delta_1} \circ \ldots \circ T_{\Delta_m} L = L \]

\[ \text{a } \mathcal{X} \text{-in. disjoint from } \mathcal{X} \]

Hence \[ \mathcal{X} \cdot L \overset{\simeq}{\longrightarrow} \text{Cone } (C \to L) \]

where \( C \) is in the support of \( \mathcal{X}(\mathcal{X}) \) given by \( \Delta_1, \ldots, \Delta_m \).

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Open string Viterbo

Let \( (W, \alpha) \) be \( \mathcal{X} \)-in. and assume \( \text{Win } CW \) is a codim 0 submanifold and \( (\text{Win }', \alpha) \) is also \( \mathcal{X} \)-in.

Prop (Abouzaid-Sadel): There is a \( \Lambda_0 \) functor \[ \mathcal{W}(W) \to \text{Win }' \text{Win } \]

which assigns to a \( \mathcal{X} \)-in. its intersection with \( \text{Win }' \text{Win } \).

-\( \text{Prop.}\)

\[ \text{Restrict to } \text{elts of } \text{Win }'(\text{Win }') \text{ having a locally constant primitive near the boundary and in } \text{Win } \text{ should also have a primitive which is constant near } \text{Win }'. \]

\[ \text{Hyp.}\]

Compare to Viterbo restr. for \( S H \).
Above, had to close Lagn in $\Delta$

\[ \text{cone } (C \to L) = L \]

\[ \text{generated by } \Delta_{i_1 \ldots i_m} \text{ from } L \]

Apply Viterbo:

- $L$ goes to 0
- $C$ goes to sum $\text{generated by } \Delta_{i_1 \ldots i_m}$
- $L$ goes to itself

Conclusion:

The $\text{Enkayn}$ cat. of \[ \text{closed Lagns} \]

\[ \{ (\overline{C}, C) \in \mathcal{N} \} \]

is generated by \[ \Delta_{i_1 \ldots i_m} \]

Back to our specific $L$: \[ \overline{\mathcal{N}}: \overline{E} \to \overline{C} \]

Consider the thimbles.

$\mathcal{E}$ is a weighted mod of $\mathcal{Q}_0 \cup \mathcal{Q}_1$

s.t. they intersect minimally.

- The $\mathcal{E}$ on $\mathcal{Q}_0 \cup \mathcal{Q}_1$ have thimbles
- The $\mathcal{E}$ on $\mathcal{Q}_0 \cup \mathcal{Q}_1$ have thimbles
- Near the $C$ at $(0,1)$, $\mathcal{Q}_0 \cup \mathcal{Q}_1$ looks like $M \cap C \cap C'$; the thimble looks like the diagonal $\{ t | t \in 1 \}$.
Apply \( V \) iterb to conclude:

**Prop** \( T^* (Q_0 \#_F Q_1) \) is gend by \( T^* (Q_0) T^* (Q_1) \) and the diagonal

Generating the diagonal

Lagrangian surgery:

\[
V_\sigma = U \cdot S^{-1} C C^T + \text{im}(\sigma)
\]

Lagun squared

\[
\text{agree with } H^\prime \text{ with }
\]

outside a cpt set

a positive Lag surgery

"negative Lag surgery"

we can resolve two Laggs intersecting transv.
at one pt to get a smooth Lag
Consider $L_p, L_q$ intersecting cat. fibers

\[ \text{(i.e., just } T^\times_{L_p} Q_0 \text{ and } T^\times_{L_q} Q_1 \text{)} \]

**Fact:** \( \text{Cone}(L_p \to L_0) \) and \( \text{Cone}(L_q \to L_p) \) are quasi-is to the two Lagrange surgeries

\[ \text{Proof: Seidel's } \]

Took together w/ relation bw B.T.s and Lagn Surgery

\[ \text{and apply Viterbo} \]

In model plumbing region:

\[ \begin{array}{ccc}
L_p & \text{to} & L_q \\
\text{Lag. surgery} & \rightarrow & \text{Lag. surgery}
\end{array} \]

Observe: Lag. surgery on \( L_0 \cup L_p \) agrees w/ diagonal near \( B^n \cup \text{IR}^n \).

Then applying Viterbo to smaller nod of \( Q_0 \cup Q_1 \) in \( E \)

\[ \text{Cone}(L_p \to L_0) \subset L.S. (L_0 \cup L_p) \]

\[ \cong \text{diagonal spanned by } T^\times_{Q_0} Q_0 \text{ and } T^\times_{Q_1} Q_1 \]
Prop: A closed exact map is governed by finite rank local systems over $\mathbb{Q}_0$ and $\mathbb{Q}_1$.

Let $A^* = \oplus_{i,j} \mathbb{A}^*(L_{a_j}, L_{b_j})$.

Generation of $\mathcal{Y}(M)$ by cot. fibers

$\Rightarrow \mathcal{Y}(M) \subset \text{cat. at } A^*-\text{modules over } A^*$

So suffice to understand $A^*-\text{modules supp. in finitely many cohom. degrees}$.

Recall:

$\text{HW}(\text{cot. fiber in tangent bundle}) \cong H^* \left( \frac{\Omega}{\Omega^2} \right.$ loops $\left. \right)$ supp. in non-pos. degrees.

Similarly, have computation:

Prop.: $A^*$ is supp. in non-pos. degrees.

for $n \geq 3$ (false for $n = 1$ !)

$A^0 = \oplus_{i,j} \mathbb{A}^0(L_{a_j}, L_{b_j})$

$A^* \text{-module}$

Note: Local system $\mathcal{V} \leftarrow A^*$

(\text{Assume } n \geq 3 \text{ from now on})
Prop: \( C(W^0(L_{q_{i}}, L_{q_{j}})) \cong \mathbb{Z}[\text{Gr}_i(Q_i)] \)

\( \text{as rings} \)

\( C(W^0(L_{q_{i}}, L_{q_{j}})) \to \text{module} \quad C(W^0(L_{q_{i}}, L_{q_{j}})) \leftarrow \quad \text{module} \quad \mathbb{Z}[\text{Gr}_i(Q_i)] \)

Now consider an \( A^* \)-module \( P^* \) supp in f.m. carbon degrees.

* Since \( A^* \) is non-pos. graded, \( P^* \) has filtration \( P = \bigcup_{k \leq K} P^k \).

Since \( \mu^k \cdot P \otimes A^d \to P \) has degree \( 1 - \theta \)
and \( \theta \) can assume \( P \) minimal, i.e. \( \mu^{10} = 0 \)

The quotients \( P^k/P^k \) are precisely \( A^0 \)-modules.

Since \( A^0 \cong \bigoplus \mathbb{Z} [\text{Gr}_i(Q_i)] \),

\( \therefore \) an \( A^0 \)-module exactly corresponds to a local system over \( Q_0 \) and a local sys. over \( Q_1 \). \( \square \)
For Part III, quick review:

Let $A$ be a matrix with $\text{rank}(A) = n$. Then, $A$ is generated by $n$ linearly independent vectors.

**Proof:** Equip $\mathbb{R}^n$ with the standard inner product. General theory of Lagrange multipliers shows that the Lagrange multipliers $\lambda^*$ associated with $A = [v_1, \ldots, v_n]$ and the diagonal matrix $D = \text{diag}(\lambda_1^*, \ldots, \lambda_n^*)$ is redundant.

$\text{det}(A) = \prod_{i=1}^{n} \lambda_i^*$

Thus, $\text{det}(A)$ is non-zero if and only if each $\lambda_i^*$ is non-zero. This gives the result.

From now on, $A$ is not a diagonal matrix.
We've shown that any closed, exact Maslov zero Lagrangian \( L \subset T^*Q_0 \#_p T^*Q_1 \) is generated by finite \( \mathbb{R} \) local systems on \( Q_0 \) and \( Q_1 \).

We now explain: Fix a complex field \( \mathbb{K} \).

Let \( \mathbb{E}_0, \mathbb{E}_1 \) be \( \mathbb{K} \)-spheres and assume \( \pi_1(\mathbb{E}_i) \) has no non-trivial finite \( \mathbb{K} \)-linear repns.

Have a \( Br_3 \) action on \( \text{Tw}_L(T^*\mathbb{E}_0 \#_p T^*\mathbb{E}_1) \) (recall \( \theta = \text{closed Lagrs} \)

by \( \langle T_1, T_2 \rangle \)

Here \( T_i \) is the twist functor

\[
T_i : L \mapsto \text{CF}(\mathbb{E}_i, L) \otimes \mathbb{E}_i \to L
\]

\( \text{tw. cpx} \)

Remark: Can define twist functor \( T_A \) of object \( A \)

If \( H^*(\mathbb{K}A, A) = H^*(S^n) \) (i.e. \( A \) is an \( \mathbb{K} \)-sphere)

then \( T_A \) is necessarily

\[
T_A = : L \mapsto L^{ev} \to \text{CF}(L, A) \otimes A
\]

\( \text{tw. cpx} \)

If \( A \) is a Lagrn sphere

\[
T_A = T_A
\]

Thm: Modulo quasi-equivalence and grading shift in \( \text{Tw}_L \) every closed Lagrn \( L \) lies in the \( Br_3 = \langle T_1, T_2, T_3 \rangle \) orbit of \( Q_0 \).

Ex.: \( A_2 \) - Milnor fiber.
Picking gradings, can assume
\[ HF(Q_0 Q_1) \cong K \otimes E_{0, j, i} \]
\[ deg \theta \]
\[ deg \psi \]
\[ deg \zeta \]
\[ \text{identifying morphism} \]
\[ \text{fund. class} \]
\[ HF(Q_0 Q_1) \cong K \otimes p \]
\[ \text{deg } 1 \]
\[ HF(Q_0 Q_2) \cong K \otimes p \]
\[ \text{deg } -n-1 \]
\[ \text{Poincare duality for } HF \]
\[ \text{Assume } k \text{ is minimal. Also, strictly unitary.} \]

Using the fact that \( A^k \) is non-pos graded, can prove

**Lemma:** Any LCP \( L \in S \) is equiv to a tw. CPO built from \( Q_0, Q_1 \) (no l.c.s)

such that none of the arrows are \( e_0 \) or \( e_1 \).

**Note:** The higher products \( M^{k,2,3} \) \( 6 \) in \( Q_0, Q_1 \)

vanish for degree reasons. Hence only have \( M^2 \).

\[ E^2 : M^3(p, p', p) \otimes HF^{(1+n+1)-1}(Q_0 Q_1) \]

\[ HF^n(Q_0 Q_1) = 0 \]

Can now assume:
\[ L \text{ is a tw. CPO } \]
\[ \text{with degrees in } E_{0, j, N, J} \]
\[ \text{and } n \text{ arrows} \]
Ex: For $n = 4$, \[ h_3 \circ f_0 \]

\[ L \sim U_0 \oplus Q_0 \downarrow t_0 \circ p \quad U_1 \oplus Q_0 \downarrow t_0 \circ p \quad U_2 \oplus Q_0 \downarrow t_0 \circ p \quad U_3 \oplus Q_0 \downarrow t_0 \circ p \]

\[ V_0 \oplus Q_1 \quad V_1 \oplus Q_1 \quad V_2 \oplus Q_1 \quad V_3 \oplus Q_1 \]

i.e. \[ j \circ p \] labeled by $p$

Ex: Check. All arrows have degree 1.

Idea: Apply $T_0, T_1, T_0^{-1}$, or $T_1^{-1}$

to simplify the tree for $L$.

That is, we'll apply induction on the complexity.$$
\text{Def: Let } |U_i| = 2^{i+1}, \quad |V_j| = 2^j \\
C^T(U) = \max (\max_{u_i \to U_i} |U_i|, \max_{v_i \to V_i} |V_i|) - \min (\min_{u_i \to U_i} |U_i|, \min_{v_i \to V_i} |V_i|)
$$

Note: $C^T$ is the longest zig-zag with nonzero starting and ending pts.

Ex: $C^T(\downarrow V) = \#(\downarrow V) = 3$

Ex: $C^T(L) = 7$
Important obs: \( HF^*(L, L) \) is non-neg. graded.

\[ H^*(C^L L) \]

Therefore an elt at \( CF^*(L, L) \) which is closed, non-exact, and has neg degree

**contradiction**

Claim: Assume \( V_0 \neq 0 \).

Then \( U_0 \xrightarrow{u_1} U_1 \xrightarrow{u_2} U_2 \xrightarrow{u_3} U_3 \)

\( \xrightarrow{v_0} V_0 \xrightarrow{v_1} V_1 \xrightarrow{v_2} V_2 \xrightarrow{v_3} V_3 \)

\( \xrightarrow{\text{subject to hms}} \)

pf: Suppose say \( U_3 \rightarrow V_3 \) not surj.

Then \( V_3 \cong \text{im}(U_3) \oplus V_3' \).

Consider elt \( q \) of \( \text{Hom}(V_3', V_0) \otimes e_1 \)

Then \( \mu^1_{\text{fw}}(q) = \mu^2(S, \text{fp}) + \mu^2(q, S) \)

by defn of \( \mu^1_{\text{fw}} = 0 \) since no arrows

hit \( V_3' \) or emanate from \( V_0 \).

Also, \( q \) not exact since for degree reasons

\( \mu^2(a, b) = e_1 = 0 \Rightarrow a = b = e_1 \)

But no \( e_1's \) in the two cp at \( L \! \)

But \( q \) has degree \(-3 < 0 \) \( \Rightarrow \) contradiction.
Can similarly show that

\[ q_i \in \mathbb{R} \text{ for } i = 0, 1, 2 \]

and \( u_i \) are zero.

So have

\[
\begin{array}{cccc}
  u_0 & u_1 & u_2 & u_3 \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  v_0 & v_1 & v_2 & v_3 \\
\end{array}
\]

Recall

\[
T_i(q_i) \sim Q_i \oplus \mathbb{R} \]

\[
\Rightarrow \oplus T_i^{-1}(Q_i) \sim Q_i \oplus \mathbb{R}
\]
Now apply $T_0^{-1}$ to $L$.

- $T_0^{-1}(U_3 \otimes Q_0) = U_3' \otimes Q_0[3]$, i.e. $U_3'$ moves to the left 3 positions.

- $T_0^{-1}(V_3 \otimes (Q_0 \rightarrow Q_1))$
  
  $\sim T_0^{-1}(V_3 \otimes T_0 Q_1)$
  
  $\sim V_3 \otimes Q_1$.

  i.e. $V_3$ becomes $V_3$

- Similarly, $U_1' \sim 0$

  $U_1 \rightarrow U_1$

  and $U_2' \rightarrow 0$

  $U_2 \rightarrow U_2$

Have $T_0^{-1}(Q_1) \sim Q_1 \rightarrow Q_0[3]$

so $U_0 \otimes Q_0$

$\rightarrow (U_0 \otimes V_0' \otimes Q_1) \sim U_0 \otimes Q_1 \otimes V_0' \otimes Q_1$.
So \( T_{0}^{-1}(L) \approx \)

\[
\begin{array}{cccc}
& V_1 & V_2 & V_3 \\
\longrightarrow & 0 & U_1 & U_2 & U_3 \\
\end{array}
\]

\( U_1 \oplus V_1 \)

\( 0 \to U_0 \oplus V_0 \to U_1 \oplus V_1 \to U_2 \oplus V_2 \to U_3 \oplus V_3 \to \) No

\( C_1(T_{0}^{-1}(L)) \) is not \( \leq C_1(L) \)

but it is if \( V_2 = V_3 = 0 \).

Note: For \( c_1 \), then \( V_3 \neq 0 \Rightarrow U_0 = 0 \).

since one can take a nonzero eff \( \hom(V_3, V_0) \otimes \mathbb{Q}_1 \)

not supposed to be obvious!

Claim: If \( V_2, V_3 \) not both zero

\( C_1(T_{0}^{-1}(L)) \leq C_1(L) \)

Finally, if \( V_0 = 0 \), there's a similar analysis in which we apply \( T_{-1} \) or \( T_0 \) instead.

Eventually, get \( c_1 = 1 \Rightarrow U_1 \oplus Q_0 \) or \( V_0 \oplus Q_1 \).

But \( HF^0(L, L) \) is \( 1 \)-dim.

\( \Rightarrow U \) or \( V \) is \( 1 \)-dim, too.

Also, note that \( Q_0, Q_1 \) lie in some \( B_{-3} \) abit!

(since \( T_{-3} Q_1 = T_{-3} Q_0 \), at least 4 to shift) \( \square \)
Spherical twists

Recall: $T^* S^n$ admits a geometric Dehn twist $\theta_c$, $S^n$ admits periodic geodesic flow.

Also have such twists for $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{QP}^2$.

Thm (Botto) $M$ admits periodic geodesic flow $\iff H^*(M)$ is a truncated polynomial.

For $E$ not diffeo to $S^n$, $T_E$ has no geometric candidate.

Expect: $\pi_0 \text{Symp}_c (T^* E)$ image in $\text{Aut}_c \mathcal{C}$

For $E$ $\mathbb{H}$-sphere:

$L T_E$

intersect only in identity.

Problem: $\gamma (T^* E)$ has only one object, so need to enlarge somehow.

Approach: Nadler (2005)

Let $M_\xi = T^* E \#_p T^* S^h = T^* E \cup$ handle

Lemma: $T_\xi \in \text{Aut}_c (M_\xi) / \mathcal{C} E 13$, has infinite order.

$\text{rk} T_\xi (S^{h}) \to \infty$ as $k \to \infty$. 
Theorem: For $\pi_1(M) \neq 1$,

$$\pi_0 \text{Symplect}^* \subseteq \text{Aut} \mathcal{W}(M) / \text{C}$$

meet only \(\alpha\) in the identity.

(i.e., \(T_\alpha\) is not geometric!)

will use:

**Theorem (Abouzaid)**: Let \(M\) be Liouville and \(\tilde{M} \to M\) the universal cover. There is an \(\infty\)-category \(\mathcal{W}(\tilde{M}, j^*\pi)\) and a pullback functor

$$\pi^*: \mathcal{W}(M) \to \mathcal{W}(\tilde{M}, j^*\pi)$$

- sending \(L\) to \(\pi^{-1}(L)\)
- and s.t.

$$HF^*(L_1, L_2) \to HF^*(\pi^{-1}(L_1), \pi^{-1}(L_2))$$

for \(L\) closed

$$\pi^* (L) \to H^*(\pi^{-1}(L))$$

- \(\pi_1(M)\) acts by automorphisms of \(\mathcal{W}(\tilde{M}, j^*\pi)\)

**Remark**: \(\mathcal{W}(\tilde{M}, j^*\pi)\) agrees with \(\mathcal{W}(\tilde{M})\) when \(\pi_1(M) \leq 1\).
pf of thrn about $T^3$.

Let $\tilde{\tau} \rightarrow \sim \rightarrow M \rightarrow M \sim$.

Suppose by contr. that $T^3$ is geometric.

Then

$L := T^2(S^n) \simeq \tilde{T}^2(S^n) \subseteq \tilde{T}^2(S^n)$

geometric

Then

Pick coeff. field $K$ s.t. Char $(K)$ divides $\mid \text{tr}_1(\mathcal{L})\mid = \infty$.

Applying $\pi^{-1}$, get

$\pi^{-1}(L) \cong \tilde{\mathcal{L}}^{n-1} \leftarrow \tilde{T}^{-1}(S^n)$

Note: $\tilde{\mathcal{L}}$ connected $\Rightarrow$ indecomposable.

Claim: $\tilde{\mathcal{L}} \leq \pi^{-1}(S^n)$ also indecomp.

Lemma: In $W(\tilde{M}_{n+1}^k)$, $\tilde{\mathcal{L}}^{n-1}$ and

$\tilde{\mathcal{L}} \leq \pi^{-1}(S^n)$ are not in same deck transcr orbit.

PB: Applying $HF(\_)$ a component of $\pi^{-1}(S^n)$ get different ranks.
But the components \( E_0 \) of \( \pi^*(L) \) are all related by deck transformation of each indecomp. 

Claim: over \( k \), the indecomp. decomposition is unique.

This contradicts \( \pi^{-1}(L) = \bigoplus \bigoplus \mathfrak{c}_{\text{indecomp.}} \), not related by deck transformation.

With similar techniques, can prove:

Thm: Let \( Q \) be a simply-connected \( k \)-module. Suppose \( T^\ast Q \) has a coisotropic symplectic action non-trivially on objects of \( \mathfrak{g}(M_Q) \).

Then \( Q \cong S^4 \) or \( \text{Sp}_2 \).