Midterm 1  
Modern Algebra 1  
Columbia University Fall 2019  
Instructor: Kyler Siegel

Exam instructions (these will also be printed on the actual exam):

- Please write your answers in this printed exam. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.

- Suspected cases of copying or otherwise cheating will be taken very seriously.

- Solve as many problems of the following problems as you can in the allotted time, which is one hour and fifteen minutes. I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.

- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!

- There are twenty true or false questions, four multiple choice questions, four short answer questions, and two short proof questions.

- For true or false questions, you will receive +2 points for a correct answer, 0 points for no answer, and −3 points for an incorrect answer. For multiple choice questions, you will receive +4 points for a correct answer, 0 points for no answer, and −2 points for an incorrect answer. This means you should not make random guesses unless you are reasonably sure that you know the answer.

- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.

- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.

Notation reminders:

- $N_G(A) := \{g \in G : gAg^{-1} = A\}$ denotes the normalizer of a subset $A \subset G$.

- $\ker(\Phi)$ and $\im(\Phi)$ denote the kernel and image respectively of a homomorphism $\Phi$.

- $D_{2n}$ denotes the dihedral group corresponding to the symmetries of the regular $n$-gon.

Some sample questions:

1. True or false questions. Circle one. You do not need to provide any justification.

   (I) Every subgroup of a cyclic group is normal.
   
   A. True    B. False
(II) There exists a homomorphism from the symmetric group $S_4$ to the symmetric group $S_5$ such that the kernel consists only of the identity element.
   A. True    B. False

(III) The symmetric group $S_4$ admits a surjective homomorphism to $\mathbb{Z}/(5\mathbb{Z})$
   A. True    B. False

(IV) If $G$ and $H$ are groups and $\Phi : G \to H$ is a homomorphism, then we have $N_H(\text{im}(\Phi)) = \text{im}(\Phi)$.
   A. True    B. False

(V) If $G$ is a group of order eight, then it is isomorphic to $C_8$, $C_2 \times C_4$, $C_2 \times C_2 \times C_2$, or $D_{2,4}$.
   A. True    B. False

(VI) The subgroup of $S_5$ generated by $(1 \ 3 \ 2)(4)(5)$ is normal.
   A. True    B. False

(VII) Let $G$ be a group with at least two distinct elements, whose only subgroups are $G$ itself and the trivial subgroup $\{e\}$. Then $|G|$ is a prime number.
   A. True    B. False

2. **Short answer questions.** You do not need to provide any justification for full credit. However, if you do you might receive some partial credit if your answer is incorrect but well-reasoned.
   (I) List all of the subgroups of $S_3$, and indicate which ones are normal.

3. **Short proofs.** Make your arguments as rigorous as possible. You may cite results covered in class provided you are completely clear about what you are citing.
   (I) Prove that every subgroup of a finite cyclic group is cyclic.