Problem 1. If $n_5$ and $n_7$ denote the number of Sylow 5- and 7-subgroups of $G$ respectively, then we have $n_5 \equiv 1 \pmod{5}$ and $n_5 \mid 7 \implies n_5 = 1$, so the unique Sylow 5-subgroup $A$ is normal in $G$. Similarly $n_7 \equiv 1 \pmod{7}$ and $n_7 \mid 5 \implies n_7 = 1$ and again the unique Sylow 7-subgroup $B$ must be normal in $G$. Since $\gcd(5,7) = 1$, it follows that $A \cap B = \{e\}$ and now from Homework 5, Problem 1 we conclude that $G \cong A \times B \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$.

Problem 2. Let $S_q$ denote the actions of $S_q$ on its Sylow $q$-subgroups by conjugation. Because $q$ is prime, these all have order $q$ and are generated by any non-identity element. Therefore the number of (Sylow) subgroups of order $q$ is $\frac{q}{q(q-1)} = (q-2)!$; there are $q$! ways to order the elements of a $q$-cycle, with $q$ cyclic repetitions, and any of its $q-1$ non-trivial powers gives rise to the same subgroup of order $q$. Since the conjugation action is transitive, this quantity also equals the number of orbits $\implies$ by orbit-stabilizer theorem the normalizer $N_{S_q}(Q)$ of $Q$ has order $\frac{|S_q|}{(q-2)!} = q(q-1)$. Finally $p|q-1|q(q-1)$, so by Cauchy’s theorem $N_{S_q}(Q)$ has a subgroup $P$ of order $p$.

- PQ is a group thanks to $P \subseteq N_{S_q}(Q)$, and has order $\frac{|P||Q|}{|P\cap Q|} = \frac{pq}{1} = pq$ due to the fact that $\gcd(p,q) = 1$;
- Pick any $q \in Q$, then $pq = qp \iff pqp^{-1} = q \iff p \in C_{S_q}(q)$. Now the action of $S_q$ on the set of $q$-cycles by conjugation is transitive, so by orbit-stabilizer theorem $|C_{S_q}(q)| = \frac{|S_q|}{#q\text{-cycles}} = \frac{q!}{(q-1)!} = q$. But any $e \neq p \in P$ has order $p$ coprime to $q$, so we can’t have $p \in Q$ as well $\implies p = pe_q = e_{S_q}q$ and $q = e_pq = e_{S_q}q$ are non-commuting elements of $PQ$.

Problem 3. See here for Kyler’s solution.

Problem 4. We have $1365 = 3 \times 5 \times 7 \times 13$ and if $n_p = 1$ for any of those prime factors $p$, then the unique Sylow $p$-subgroup will be a proper normal subgroup, so $G$ automatically cannot be simple. Suppose therefore that $n_p > 1 \forall p|1365$; then together with the conditions $n_p \equiv 1 \pmod{p}$ and $n_p \mid \frac{1365}{p}$ we deduce that $n_{13} = 105, n_7 \geq 15, n_5 \geq 21$. This already implies that $G$ must have at least $(13-1) \times 105 + (7-1) \times 15 + (5-1) \times 21 = 1434 > 1365$ elements, contradiction.

Problem 5. Note that $\{e\}$ is always its own conjugacy class, so if $G$ were to have only 2 conjugacy classes, then $G \setminus \{e\}$ must constitute a conjugacy class. In such case, the orbit-stabilizer theorem tells us that $|G \setminus \{e\}||G|$, i.e. $n - 1|n$ which is only possible if $n = 2$. Conversely, the group of order 2 is clearly abelian and consequently has precisely 2 conjugacy classes.

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Problem 6. Using the prime factorization 203 = 7 \times 29, Sylow’s theorems imply that \( n_{29} \equiv 1 \pmod{29} \) and \( n_{29}|7 \implies n_{29} = 1 \), so the unique Sylow 29-subgroup \( K \) is normal in \( G \). Since \(|H| = \frac{203}{29} = 7\) is coprime to 29, it follows that \( H \cap K = \{e\} \) and we conclude from Homework 5, Problem 1 that \( G \cong H \times K \cong \mathbb{Z}_7 \times \mathbb{Z}_{29} \).

Problem 7. Note that since 168 = 2^3 \times 3 \times 7, a Sylow 7-subgroup has order 7 and so do all of its non-identity elements. Conversely, an element of order 7 generates a subgroup of order 7, so it must necessarily lie in a Sylow 7-subgroup. Therefore the number of elements of order 7 is given by the number of non-identity elements in Sylow 7-subgroups, i.e. \( 6 \times n_7 \). We know that \( n_7 \equiv 1 \pmod{7} \) and \( n_7|24 \), but also \( n_7 \neq 1 \) as our group is simple \( \implies \) the only possibility left is \( n_7 = 8 \), in which case the answer to the problem is 48.