Final Examination
Math 2030: Ordinary Differential Equations
Columbia University Spring 2018
Instructor: Kyler Siegel

Instructions:

- Please write your answers in this printed exam. You may use the back of pages for additional work. You may also use blue books or white paper for scratch work, but these are not to be handed in.

- Solve as many problems of the following problems as you can in the allotted time, which is two hours and 50 minutes. I recommend first solving the problems are you most comfortable with before moving on to the more challenging ones. Note that many of the problems contain more or less independent subproblems, and you do not need to answer all the subproblems to get partial credit.

- You do not need to justify your answers for full credit (unless explicitly asked to do so), although it may help you get partial credit if some of your final answers are incorrect or incomplete. Suspected cases of copying or otherwise cheating will be taken very seriously.

- At the end of the exam you will find bonus problems. These problems are more challenging and worth proportionally fewer points, so you should only attempt them after you are confident with your answers to the main problems. You can safely skip them if you’re out of steam by then.

- Best of luck!!

Name: ____________________________________________

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1. **Multiple choice questions. Note that in some cases you may have to choose more than one letter.**

(I) (2 points) Which of the following ordinary differential equations is linear? Select all that apply.

A. $y'(t) + 3\sin(t)y(t) = 0$  
B. $y''(t) + 3\sin(t)y(t) = 0$  
C. $y''(t) + \sin(3y(t)) = 0$  
D. $y'(t) + y(t)^2 = 0$  
E. $e^t y'(t) + e^{\sin(t)} y(t) = 0$

(II) (2 points) Which of the following ordinary differential equations has a solution which is defined for all $t \in (-\infty, \infty)$ and satisfies the conditions $y(1) = 0$ and $y'(1) = 7$? Select all that apply.

A. $y''(t) + 2y'(t) + 3\sin(t)y(t) = 4$  
B. $y''(t) = 0$  
C. $y'(t) - 7y(t) = 0$  
D. $y'(t) - 7 = 0$

(III) (2 points) Given the system

\[
\begin{align*}
    x_1'(t) &= x_2(t) \\
    x_2'(t) &= -x_1(t) - 3x_2(t),
\end{align*}
\]

which of the following pictures most likely represents the phase portrait of solutions in the $(x_1, x_2)$ plane? Select one: A.  B.  C.  D.  E.
(IV) (2 points) Answer the same multiple choice question as in the previous part, but now for the system
\[
\begin{align*}
    x_1'(t) &= x_2(t) \\
    x_2'(t) &= -2x_1(t) - 1.5x_2(t).
\end{align*}
\]
Select one: A. B. C. D. E.

(V) (2 points) Answer the same multiple choice question as in the previous part, but now for the system
\[
\begin{align*}
    x_1'(t) &= x_2(t) \\
    x_2'(t) &= x_1(t).
\end{align*}
\]
Select one: A. B. C. D. E.

(VI) (2 points) Answer the same multiple choice question as in the previous part, but now for the system
\[
\begin{align*}
    x_1'(t) &= x_2(t) \\
    x_2'(t) &= -2.5x_1(t) + 1.5x_2(t).
\end{align*}
\]
Select one: A. B. C. D. E.

(VII) (2 points) Answer the same multiple choice question as in the previous part, but now for the system
\[
\begin{align*}
    x_1'(t) &= x_2(t) \\
    x_2'(t) &= -x_1(t) + 3x_2(t).
\end{align*}
\]
Select one: A. B. C. D. E.
2. True or false questions. Determine whether each of the following statements is true or false.

(I) (2 points) The ODE \( y''(t) + y'(t) + y(t) = 0 \) has exactly one solution which satisfies \( y(0) = 0 \).

(II) (2 points) The initial value problem
\[
\begin{align*}
y''(t) + \sin(t)y'(t) + e^{2t}y(t) &= 0 \\
y(0) &= 1 \\
y'(0) &= 2
\end{align*}
\]
has at least two distinct solutions, \( y_1(t), y_2(t) \), each defined for all \( t \in (-\infty, \infty) \).

(III) (2 points) Suppose that \( f(t) \) and \( g(t) \) are continuous functions defined for all \( t \in (-\infty, \infty) \) which satisfy
\[
f(0) = 1, \quad f'(0) = 2, \quad g(0) = 3, \quad g'(0) = 4.
\]
Then \( f(t)g'(t) - f'(t)g(t) \) is nonzero for all \( t \).

(IV) (2 points) Suppose that \( f(t) \) and \( g(t) \) are functions defined for all \( t \in (-\infty, \infty) \) which satisfy
\[
f'''(t) + \sin(t)f'(t) + e^t f(t) = 1
\]
and
\[
g''(t) + \sin(t)g'(t) + e^t g(t) = 1,
\]
and also
\[
f(0) = 1, \quad f'(0) = 2, \quad g(0) = 3, \quad g'(0) = 4.
\]
Then \( f(t)g'(t) - f'(t)g(t) \) is nonzero for all \( t \).

(V) (2 points) For any fixed constants \( a, b, c, d \in \mathbb{R} \), the general solution of the ODE
\[
ay''''(t) + by'''(t) + cy''(t) + dy'(t) = 0
\]
can be written in the form
\[
C_1e^{r_1t} + C_2e^{r_2t} + C_3e^{r_3t} + C_4e^{r_4t},
\]
where \( r_1, r_2, r_3, r_4 \in \mathbb{R} \) are real numbers (not necessarily distinct) and \( C_1, C_2, C_3, C_4 \) are arbitrary constants.
(VI) (2 points) The rational function $\frac{x^3 - 1}{x^2 + 1}$ can be written as a power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ...,$$

and this power series has infinite radius of convergence.

(VII) (2 points) Consider the system

\[
\begin{aligned}
  x_1'(t) &= ax_1(t) + bx_2(t) \\
  x_2'(t) &= cx_1(t) + dx_2(t).
\end{aligned}
\]

For any fixed constants $a, b, c, d \in \mathbb{R}$, this system has exactly one equilibrium solution.

(VIII) (2 points) The initial value problem

\[
\begin{aligned}
  y'(t) &= y(t)^2 \\
  y(0) &= 0
\end{aligned}
\]

has a unique solution, defined at least for $|t|$ sufficiently small.

(IX) (2 points) The initial value problem

\[
\begin{aligned}
  y(t) &= y'(t)^2 \\
  y(0) &= 0
\end{aligned}
\]

has a unique solution, defined at least for $|t|$ sufficiently small.

(X) (2 points) For any fixed constants $\alpha, \beta \in \mathbb{R}$, the Euler equation

$$t^2y''(t) + \alpha ty'(t) + \beta y(t) = 0$$

has a nonzero solution which is a polynomial $P(t)$. 
3. This question is about first order ordinary differential questions.

(I) (6 points) Consider the initial value problem

\[
\begin{align*}
\frac{dy}{dt} &= 3y^4 \\
y(0) &= 1.
\end{align*}
\]

Find the largest interval containing \( t = 0 \) on which the solution is defined, i.e. the smallest \( a < 0 \) and largest \( b > 0 \) such that there is a solution to the IVP defined all for \( t \in (a, b) \). Note that \( a \) could be \(-\infty\) and \( b \) could be \(+\infty\).

(II) (6 points) Consider the autonomous first order ODE

\[
y'(t) = y(t)(y(t) - 1)(y(t) - 3)(y(t) - 7).
\]

Suppose \( \phi(t) \) is a solution to this ODE which satisfies the initial condition \( \phi(0) = \phi_0 \), for some \( \phi_0 \in \mathbb{R} \). Does this exist some \( t > 0 \) such that \( \phi(t) = 2\phi_0 \)? Note that your answer might depend on \( \phi_0 \).
(III) (6 points) Find the general solution to the following ODE:

\[ y'(t) + 2y(t) = te^{-2t}. \]

(IV) (6 points) Find the solution to the initial value problem

\[
\begin{cases}
2ty(t)^2 + 2y(t) + (2t^2y(t) + 2t)y'(t) = 0 \\
y(0) = 1.
\end{cases}
\]

Note: your answer should involve an arbitrary constant. You may write your answer as an equation implicitly defining \( y(t) \).
4. This question is about ordinary differential equations with constant coefficients.

(I) (4 points) Find the general solution to the ODE

\[ y''(t) - 2y'(t) + y(t) = 0. \]

(II) (6 points) Find the general solution to the ODE

\[ y''(t) - 2y'(t) + y(t) = \sin(t). \]
(III) (6 points) Find the general solution to the ODE

$$y'''(t) - 5y(t) = 0.$$
5. This question is about second linear ordinary differential equations. Recall that, for an ODE of the form
\[ P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0, \]
a point \( t_0 \) is ordinary if \( \frac{Q(t)}{P(t)} \) and \( \frac{R(t)}{P(t)} \) are analytic at \( t_0 \), it is called regular singular if \((t - t_0)\frac{Q(t)}{P(t)} \) and \((t - t_0)^2\frac{Q(t)}{P(t)} \) are analytic at \( t_0 \), and otherwise it is called irregular singular.

(I) (6 points) Consider the ODE
\[ t^2(1 - t^2)y''(t) + 2y'(t) + 4y(t) = 0. \]
Find the ordinary, regular singular, and irregular singular points.

(II) (6 points) Find the general solution to the ODE
\[ t^2y''(t) + 5ty'(t) + 4y(t) = 0, \quad t > 0. \]
(III) (10 points) Consider the initial value problem

\[
\begin{aligned}
y''(t) &= ty(t) \\
y(0) &= 1 \\
y'(0) &= 0.
\end{aligned}
\]

Find the solution, written as a power series \( \sum_{n=0}^{\infty} a_n t^n \). You should write out the nth term \( a_n \) explicitly, although it does not need to be in “closed-form”. 
(IV) (8 points) Consider the ODE

\[ t^2 y''(t) - t(t + 2)y'(t) + (t + 2)y(t) = 0, \quad t > 0. \]

Given that \( y_1(t) = t \) is a solution, find the general solution. \textit{Hint: use the method of reduction of order.}
6. *This question is about systems of first order linear ordinary differential equations.*

(I) (8 points) Consider the system of first order linear ODEs

\[
\begin{align*}
    x_1'(t) &= x_2(t) \\
    x_2'(t) &= -x_1(t) + 3x_2(t).
\end{align*}
\]

Find the general solution. Note: it should involve two arbitrary constants.
(II) (8 points) Consider the system of first order linear ODEs

\[
\begin{cases}
  x_1'(t) = x_2(t) \\
  x_2'(t) = -2x_1(t) + \sqrt{8}x_2(t).
\end{cases}
\]

Find the general solution. Note: it should involve two arbitrary constants.
7. **Bonus questions.** The following are bonus questions. They more challenging and worth proportionally fewer points, so you should only attempt them after you are confident with your answers to the main problems. You can safely skip the bonus questions without any serious harm.

(I) (4 points (bonus)) Consider the ODE

\[ t^2 y''(t) + 3ty'(t) + (1 + t)y(t) = 0, \quad t > 0. \]

Find one nonzero solution, written as a Frobenius series \( \sum_{n=0}^{\infty} a_n t^{n+r} \). You should write out the nth term \( a_n \) explicitly, although it does not need to be in “closed-form”.

(II) (4 points (bonus)) Find the general solution to the ODE

\[ y(t) + 2ty(t)y'(t) - e^{-2y(t)}y'(t) = 0. \]

*Hint: look for an integrating factor of the form \( \mu(y) \).*
(III) (4 points (bonus)) Find the solution to the initial value problem

\[
\begin{cases}
y''(t) = y(t)^5 \\
y(0) = 1 \\
y'(0) = \frac{1}{\sqrt{3}}.
\end{cases}
\]

*Hint: try converting it to a system of two linear ODEs. It might help to notice that \( z'(t)/y'(t) = dz/dy. \)