MIDTERM 2 PRACTICE PROBLEMS  
MATH 2030 ODE, FALL 2018

Instructions: Solve as many problems or subproblems as you can in the given time. Except for the first problem, you must justify your answers for full credit. I recommend first working on the problems you know how to solve before spending time on the trickier ones. Please check the back side of this page for possible additional problems. Good luck!

Problem 1. Determine whether the following statements are true or false. Please write out “true” or “false”. No justification is needed.

(1) (2.5 pts / 100) For any real constants \( a, b, c, d \in \mathbb{R} \), the ODE
\[
y'''(t) + by''(t) + cy'(t) + dy(t) = \sin(t)
\]
has three linearly independent solutions on \((-\infty, \infty)\).

(2) (2.5 pts / 100) The rational function \( \frac{x^2 - 2x + 1}{x^2 + 1} \) can be written as a convergent power series \( \sum_{n=0}^{\infty} a_n(x - \pi)^n \) with infinite radius of convergence.

(3) (2.5 pts / 100) The functions \( \sinh(t) \) and \( \cosh(t) \) are linearly independent on \((-\infty, \infty)\).

(4) (2.5 pts / 100) The ODE
\[
y'''(t) - y(t) = e^t
\]
has a solution of the form \( y(t) = Ce^t \) for some \( C \in \mathbb{R} \).

(5) (2.5 pts / 100) The functions \( \sin(2t), \cos(2t) \), and \( \sin(4t) \) are linearly independent on \((-\pi, \pi)\).

(6) (2.5 pts / 100) If \( t \) and \( t^2 \) are solutions to an ODE of the form \( P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0 \), with \( P, Q, R : \mathbb{R} \rightarrow \mathbb{R} \) analytic functions, then \( t = 0 \) is an ordinary point.

(7) (2.5 pts / 100) Suppose \( y(t) \) is any analytic function defined for all \( t > 0 \) which satisfies
\[
t^2y''(t) + ty'(t) - y(t) = 0.
\]
Then \( \lim_{t \rightarrow 0^+} y(t) \) is finite.
(8) (2.5 pts / 100) If \( y_1(t) \) and \( y_2(t) \) are solutions to the ODE
\[
y'''(t) + 2y''(t) + 2y'(t) + 2y(t) = e^t,
\]
then \( 2y_1(t) - y_2(t) \) is also a solution.

(9) (2.5 pts / 100) If \( p(t) \) and \( q(t) \) are functions which are analytic for all \( t \in \mathbb{R} \), then \( p(t)/q(t) \) is analytic for all \( t \in \mathbb{R} \).

(10) (2.5 pts / 100) The functions \( y_1(t) = e^t \), \( y_2(t) = \sin(t) \), and \( y_3(t) = \cos(2t) \) all solve an ODE of the form
\[
y^{(4)}(t) + ay'''(t) + by''(t) + cy'(t) + dy(t) = 0
\]
for some choice of \( a, b, c, d \in \mathbb{R} \).

Problem 2.  (1) (6 pts / 100) Find a nonzero solution to the ODE
\[
y^{(4)}(t) - y'''(t) - 2y'(t) + 2y(t) = 0.
\]
*Hint: it might help to look for rational roots.*

(2) (6 pts / 100) Find a solution to the ODE
\[
y^{(4)}(t) - y'''(t) - 2y'(t) + 2y(t) = \sin(t).
\]

(3) (6 pts / 100) Find the general solution to the ODE
\[
y^{(4)}(t) - y'''(t) - 2y'(t) + 2y(t) = 0.
\]
Your answer should be a real-valued function involving some arbitrary real constants. *Hint: you might encounter expressions of the form \( \exp(ae^{ib}) \). Recall from Euler’s formula that \( e^{a+bi} = e^a(\cos(b) + i\sin(b)) \) for \( a, b \in \mathbb{R} \).*

Problem 3. (12 pts / 100) Let \( \phi(t) \) be a solution to the initial value problem
\[
(t^2 - 2t - 3)y''(t) + ty'(t) + 4y(t) = 0 \quad y(1) = a \quad y'(1) = b,
\]
for some real constants \( a, b \in \mathbb{R} \). Find \( \phi''(1) \) and \( \phi^{(3)}(1) \) in terms of \( a \) and \( b \).

Problem 4. Consider the ODE
\[
(t - 3)^3(t^4 - t^2)y''(t) + \sin(t) \cos(t - 3)y'(t) + \left( \frac{e^t}{(t-1)^2} \right) y(t) = 3\sin(t)y'(t).
\]

(1) (7.5 pts / 100) Find the ordinary, regular singular, and irregular singular points.

(2) (7.5 pts / 100) Pick one of the regular singular points and find the exponents of the singularity (i.e. the roots of the indicial equation).

Problem 5. (15 pts / 100) Given that \( e^t \) is a solution, find the general solution to the ODE
\[
(t - 1)y''(t) - ty'(t) + y(t) = 0,
\]
valid at least for $t > 1$. Hint: use the method of reduction of order to find a second solution. It may be helpful to know that $(a - b) \ln(|b + x|) + x$ is an antiderivative for $\frac{x + a}{x + b}$ and that $(x - 1)e^x$ is an antiderivative for $xe^x$.

**Problem 6.** (15 pts / 100) Find a nonzero solution to the ODE

$$t^2 y''(t) + ty'(t) + (t^2 - 1/4)y(t) = 0,$$

defined as least for $t > 0$. If possible, write the answer in terms of elementary functions (as opposed to say a power series).