(1) Cromwell Exercise 1.2. (You don’t have to prove your answer to the last subquestion.)
(2) Cromwell Exercise 1.3.
(3) Let \( f: S^1 \to \mathbb{R}^3 \) be a continuous, injective map. Prove that the image \( f(S^1) \) of \( f \) is homeomorphic to \( S^1 \). (Hint: this is easy.)
(4) Describe explicitly a smooth knot (image of a \( C^1 \) function with nonvanishing derivative) \( K \) in \( \mathbb{R}^3 \) so that the projection of \( K \) to \( \mathbb{R}^2 = \{(x,y,0)\} \subset \mathbb{R}^3 \) has infinitely many transverse double-points (crossings at which the crossing strands are not tangent).

In the next few exercises, we give the beginning of another proof that any smooth knot has a knot diagram. The proof can be completed using similar ideas and a little more differential geometry. (We will use one key theorem, Sard’s theorem, without proving it.)

(5) The Baire Category Theorem states: A (nonempty) complete metric space is not a countable union of nowhere-dense subsets. Prove this.

(Hint: Suppose \( X \) is a complete metric space and \( X = \bigcup_{i=1}^{\infty} C_i \) where \( C_i \) is nowhere dense. We may assume \( C_i \) is closed (why?). Choose a point \( p_1 \in X \) and an \( \epsilon_1 > 0 \) so that the ball \( B(p_1, \epsilon_1) \) is contained in \( X \setminus C_1 \). Choose a point \( p_2 \in B(p_1, \epsilon_1/2) \) and an \( \epsilon_2 < \epsilon_1/2 \) so that \( B(p_2, \epsilon_2) \) is contained in \( X \setminus (C_1 \cup C_2) \). Repeat.)

(6) Let \( F: \mathbb{R}^n \to \mathbb{R}^m \) be a \( C^1 \) function. A point \( p \in \mathbb{R}^n \) is a critical point of \( F \) if the total derivative \( dF(p) \) of \( F \) is not surjective at \( p \), i.e., if the matrix

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix},
\]

where \( f = (f_1, \ldots, f_m) \), has rank less than \( m \). A point \( q \in \mathbb{R}^m \) is a critical value of \( F \) if \( q = F(p) \) for some critical point \( p \) of \( F \).

One version of Sard’s theorem states: The set of critical values of a \( C^1 \) function \( F \) is a countable union of nowhere-dense sets. (Another version states that the set of critical values of \( F \) has measure 0.)

Use Sard’s theorem to prove that there is no surjective, \( C^1 \) function \( F: \mathbb{R}^1 \to \mathbb{R}^2 \). (Hint: this is easy.)

(7) Use Sard’s theorem to prove that there is no surjective, \( C^1 \) function \( F: S^1 \to S^2 \).

(A function \( F: S^1 \to S^2 \) is differentiable if \((i \circ F \circ p): \mathbb{R} \to \mathbb{R}^3 \) is differentiable,
where \( p: \mathbb{R} \to S^1 \) is the map \( \theta \mapsto e^{i\theta} \) and \( i: S^2 \to \mathbb{R}^3 \) is the usual inclusion of the unit sphere in \( \mathbb{R}^3 \).

(8) Suppose \( \gamma: S^1 \to \mathbb{R}^3 \) is a smooth embedding, i.e., a \( C^\infty \) map with non-vanishing derivative. Prove that there is a plane \( P \) so that for the orthogonal projection map \( \pi_P: \mathbb{R}^3 \to P, \pi_P \circ \gamma: S^1 \to P \) has non-vanishing derivative. (Hint: use the previous problem.)

Remark 1. Problem 3 shows, in particular, that the subspace of \( \mathbb{R}^3 \) illustrated in Cromwell’s Figure 1.15 is homeomorphic to the standard circle. (This might seem surprising.)

Remark 2. In contrast to the result in Problem 6, there are continuous, surjective maps \( \mathbb{R} \to \mathbb{R}^2 \); such maps are called space-filling curves.

Remark 3. In the language of differential geometry, in Problem 8 you find a plane \( P \) so that \( \pi_P \circ \gamma \) is an immersion.

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