Please keep track of how many hours of undistracted work this takes you. I’ll ask you this on Tuesday, to help calibrate future assignments. (My target: roughly six hours per week.)

(This is really a foreign language assignment. And while there are a lot of problems, most of the solutions are quite short.)

[Problem 9 corrected in this version.]

1. Let $S = \{ x \in \mathbb{N} \mid x \text{ is prime} \}$, $T = \{ x \in \mathbb{N} \mid x \text{ is odd, } x < 10 \}$. List
   the elements of
   
   (a) $S \cap T$
   
   (b) $T \setminus S$
   
   (c) $(S \cap T) \times (T \setminus S)$.

2. Let $p(x)$ be an even-degree polynomial with real coefficients. View $p(x)$ as a map $\mathbb{R} \to \mathbb{R}$. Explain why the map $p(x)$ is not bijective.

   (Note on interpreting the problem: the phrasing means you don’t get to choose $p(x)$. You’re supposed to prove that no matter what even-degree polynomial I give you, the corresponding map is not bijective.)

3. Let $p(x) = x^3 + ax$. For which $a \in \mathbb{R}$ is $p(x)$
   
   (a) injective?
   
   (b) surjective?
   
   (c) bijective?

   (Justify, but don’t necessarily prove, your answer.)

4. Fill in the blanks in the proof on page 4.

5. Using the axioms of vector spaces, give a two-column proof of the following:

   **Lemma 1.** Let $V$ be a real vector space, and $x, y, z, w$ elements of $V$. Then

   $$((5(x + y)) + z) + w = (5x) + ((5y) + (z + w)).$$

   See Page 3 for an example of what I mean.

6. Let $\mathcal{P}$ denote the set of all polynomials in the variable $x$, with real coefficients. Let $+$ be the usual operation of addition and define scalar multiplication by

   $$\lambda(a_0 + \cdots + a_n x^n) = (\lambda a_0) + \cdots + (\lambda a_n) x^n.$$

   Recall that the degree of a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with $a_n \neq 0$ is $n$. (For instance, the degree of $0x^3 + 3x^2 + x$ is 2.)

   (a) $\mathcal{P}$ is a vector space. Why? Verify a couple of the axioms.
(b) Let $P_{\leq 5}$ be the subset of $P$ of polynomials of degree at most 5. Is $P_{\leq 5}$ a vector subspace of $P$? Why or why not?

(c) Let $P_{= 5}$ be the subset of $P$ of polynomials of degree exactly 5. Is $P_{= 5}$ a vector subspace of $P$? Why or why not?

(7) Prove that $S = \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$ is a vector subspace of $\mathbb{R}^2$ if and only if $b = 0$.

(Note: there are two parts. One starts “Suppose that $b = 0$.” The other starts either “Suppose that $S$ is a vector subspace” or “Suppose that $b \neq 0$.”)

(8) Let $P$ denote the vector space of polynomials in $x$.

(a) Prove that $P$ is not finite-dimensional, by finding an infinite number of linearly independent elements of $P$. (This might be tricky because it’s so easy.)

(b) Find a finite-dimensional subspace of $P$ and give its dimension. (Yes, lots of easy choices here.)

(9) (a) Find a basis for $V = \{(x, y) \in \mathbb{R}^2 \mid 2x + 4y = 0\}$. What is the dimension of $V$?

(b) Find a basis for $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 4z = 0\}$. (This takes a little work.) What is the dimension of $W$?

(c) The vector $(10, -1, -2)$ is an element of $W$. Write $(10, -1, -2)$ as a linear combination of your basis vectors.

(10) For $a \in \mathbb{R}$, let $V = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$. Let $W = \text{Span}\left\{\begin{pmatrix} a \\ a \\ b \end{pmatrix}\right\}$.

(a) For $a = b = 1$, what are $\dim(V)$, $\dim(W)$, $\dim(V \cap W)$ and $\dim(V + W)$? Check that your answer is consistent with Theorem 3, p. 49.

(b) For $a = 1$ and $b = 0$, what are $\dim(V)$, $\dim(W)$, $\dim(V \cap W)$ and $\dim(V + W)$? Check that your answer is consistent with Theorem 3, p. 49.

(c) For $a = b = 0$, what are $\dim(V)$, $\dim(W)$, $\dim(V \cap W)$ and $\dim(V + W)$? Check that your answer is consistent with Theorem 3, p. 49.
Sample two-column proof:

**Lemma 2.** Let $V$ be a real vector space and $x, y$ elements of $V$. Then $2(x + y) + 3(x + y) = (5x) + (5y)$.

**Proof.**

1. $2(x + y) = 2x + 2y$  
   Distributivity
2. $3(x + y) = 3x + 3y$  
   Distributivity
3. $2(x + y) + 3(x + y) = (2x + 2y) + (3x + 3y)$  
   Steps (1) and (2)
4. $(2x + 2y) + (3x + 3y) = (2x + (2y + 3x)) + 3y$  
   Associativity for vector addition
5. $(2x + (2y + 3x)) + 3y = (2x + (3x + 2y)) + 3y$  
   Commutativity for vector addition
6. $(2x + (3x + 2y)) + 3y = (2x + 3x) + (2y + 3y)$  
   Associativity for vector addition
7. $(2x + 3x) + (2y + 3y) = (2 + 3)x + (2 + 3)y$  
   Distributivity (twice)
8. $(2 + 3)x + (2 + 3)y = 5x + 5y$  
   (Arithmetic in $\mathbb{R}$; see below.)
9. $2(x + y) + 3(x + y) = 5x + 5y$  
   Steps (3), (4), (5), (6), (7) and (8)

and the transitive property of equality (5 times)  

□

(Yes, I know this is a pain: I had to typeset this whole monster. I won’t ask this of you after this problem set. The point is that, in principle, any proof can be reduced to a sequence of steps like this. For what it’s worth, I remember being surprised that the short list of axioms we have really were enough to do any manipulation like this.)

**Remark.** We haven’t discussed axioms for arithmetic in $\mathbb{Z}$ or $\mathbb{R}$, so let’s not make a big deal of it. In the usual way of axiomatizing arithmetic of $\mathbb{Z}$, the proof that $2 + 3 = 5$ would boil down to $(1+1)+((1+1)+1) = (((1+1)+1)+1)+1$, by a sequence of applications of the associative law. (Two is defined to be $1+1$, and 3 to be $((1+1)+1)$. But that’s for another course.)
Theorem 1. Let $X$, $Y$ and $Z$ be sets. If $f: X \to Y$ is injective and $g: Y \to Z$ is injective then $g \circ f: X \to Z$ is injective.

Proof. Suppose that $x_1, x_2 \in X$ are such that $g \circ f(x_1) = g \circ f(x_2)$. We will show that $\quad$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $g(y_1) = \quad$. So, since $g$ is injective, $\quad$. So, $f(x_1) = \quad$. So, since $\quad$, $x_1 = x_2$, as desired. $\square$

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