(1) Let \((V, \langle \cdot, \cdot \rangle)\) be a finite-dimensional inner product space and \(S \subset V\) a subspace. Prove that \((S^\perp)^\perp = S\).

**Hint.** Prove that \(S \subset (S^\perp)^\perp\). Then use the fact that \(\dim(V) = \dim(S) + \dim(S^\perp)\) and \(\dim(V) = \dim(S^\perp) + \dim((S^\perp)^\perp)\).

(2) Let
\[
A = \begin{pmatrix}
25 & -4 & -4 \\
-4 & 10 & 1 \\
-4 & 1 & 10
\end{pmatrix}.
\]

Find an orthogonal matrix \(P\) such that \(P^T AP\) is diagonal. What is \(P^T AP\)?

(Note: some of the numbers get pretty big, but the problem is cooked.)

(3) An \(n \times n\) matrix \(A\) is called **skew-Hermitian** if \(A^H = -A\). If \((V, \langle \cdot, \cdot \rangle)\) is a Hermitian inner product space then a linear transformation \(F: V \to V\) is called **skew-adjoint** if for any \(v, w \in V\), \(\langle F(v), w \rangle = -\langle v, F(w) \rangle\).

(a) Give an example of a skew-Hermitian matrix.

(b) Prove that if \(F: V \to V\) is skew-adjoint and \(B = [e_1, \ldots, e_n]\) is an orthonormal basis for \(V\) then the matrix for \(F\) with respect to \(B\) is skew-Hermitian.

(c) Prove that if \(F\) is skew-adjoint then the eigenvalues of \(F\) are all purely imaginary (i.e., of the form \(ir\) for some \(r \in \mathbb{R}\)).

(d) Prove that if \(F\) is skew-adjoint and \(v, w\) are eigenvectors of \(F\) of eigenvalues \(\lambda\) and \(\mu\) respectively, with \(\lambda \neq \mu\) then \(v\) and \(w\) are orthogonal.

(e) Prove that if \(F\) is skew-adjoint, \(v\) is an eigenvector of \(F\) and \(w\) is any vector so that \(w \perp v\) then \(F(w) \perp v\).

(f) Prove by induction that if \(F\) is skew-adjoint then there is an orthonormal basis \(B\) for \(V\) so that \([F]_B\) is diagonal.

**Hint for all parts.** All of the proofs in this problem are very similar to proofs we did in class, for analogous properties of self-adjoint transformations.

(4) An \(n \times n\) matrix with real entries is called **skew-symmetric** if \(A^T = -A\). In this exercise, you will prove that if \(A\) is skew-symmetric then \(e^A\) is orthogonal.

(a) Find matrices \(A\) and \(B\) so that \(e^{A+B} \neq e^A e^B\).

(b) Explain why if \(AB = BA\) then \(e^{A+B} = e^A e^B\). (Hint: each side gives you an infinite sum. Figure out what the coefficient of \(A^m B^n\) is in each sum.)

(c) Explain why if \(A\) is skew-symmetric then \(AA^T = A^T A\) (very short).

(d) Prove that if \(A\) is skew-symmetric then \(e^A\) is orthogonal. (Hint: this is not hard from parts (4b) and (4c). Compute \((e^A)(e^{A^T})\).)

**Remark.** In very fancy terminology, this statement is the fact that the **Lie algebra** of the orthogonal group is the space of skew-symmetric matrices.

**Remark.** It’s also true that if \(A\) is skew-Hermitian then \(e^A\) is unitary. The proof is essentially the same.