MATH V2020 PROBLEM SET 3
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This is the first problem set where you'll write proofs of interesting results on your own. I've given sketches, but it still may be a bit of a struggle. Don't forget that help is available.

(1) Compute the following matrix arithmetic:

(a)

$$\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 6 & -2 \\ 18 & -6 \end{pmatrix}^2$$

(2) Some abstract computations with matrix multiplication. (The answers and a little justification is enough; you don't have to give proofs or detailed arguments.)

(a) Let $E_{i,j}$ denote the $n \times n$ matrix with a 1 in the $(i,j)^{th}$ position and 0's everywhere else. What is the product $E_{i,j}E_{k,l}$ in terms of i, j, k and l? (Suggestion: do a couple of 2×2 or 3×3 examples.)

(b) Let $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and let A be a 3×3 matrix. What is the effect of left multiplying A by M (i.e., MA)? What about right multiplying (i.e., AM)? (Suggestion: do an example.)

(c) Let $N = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and let A be a 3×3 matrix. What is the effect of left multiplying A by N? Right multiplying A by N?

(d) Let $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and let A be a 3×3 matrix. What is the effect of left multiplying A by P? Right multiplying A by P?

(e) What are the inverses of M, N and P? (Guess the answer by thinking about your answers from the first three parts, not by computing directly. Check that they're correct.)

(3) Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$. Find the inverse B of A. (Hint: B is a 2×2 matrix. Write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, the condition that B be the inverse of A boils down to a system of linear equations. Solve them.)

(You might already have learned some tricks for inverting matrices somewhere. Please don't use them unless you can explain why they work.)

(4) Prove that the matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$ has no inverse, two ways:
   (a) Write down the condition for B to be the inverse of A, in terms of linear equations, and check that they are not satisfied.
   (b) Consider the linear map induced by A, and show that it has no inverse. (Hint: is it bijective?)

   (Both parts should be quite short; if you're having trouble writing them, talk to someone (e.g., me, Evan).)
(5) Let $F\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation.
   (a) Prove that if $n < m$ then F is not surjective. (Hint: take a basis for $\mathbb{F}^n$, apply F to it. Explain why the resulting vectors can't span W. Explain why this implies F is not surjective.)
   (b) Let $F\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Prove that if $n > m$ then F is not injective. (Hint: take a basis for V, apply F to it. Explain why the resulting vectors can't be linearly independent. Explain why this implies F is not injective.)
   (c) Use the first two parts to prove that if A is an $m \times n$ matrix and $m \neq n$ then A is not invertible.
(6) Last week you found a matrix for the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line making an angle θ with the x-axis. Let R_θ denote this linear transformation. Find a matrix for the composition $R_\theta \circ R_\eta$ in two different ways:
   (a) Directly, by geometric arguments similar to those you used last week.
   (b) By multiplying the matrices for R_θ and R_η.

   Check that your answers agree.
(7) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y)^T = (x + y, x - y)^T$.
   (a) Find the matrix for F with respect to the standard basis $\mathcal{B}$ for $\mathbb{R}^2$.
   (b) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}' = [(1, 2)^T, (3, 7)^T]$. Find it's inverse (hint: you've already done that in this problem set).
   (c) Use the first two parts to compute the matrix for F with respect to $\mathcal{B}'$. (This is not rigged to have a particularly nice answer.)
(8) Let $F\colon \mathcal{P}_{\leq 1} \to \mathcal{P}_{\leq 1}$ be given by $F(p(x)) = xp(1) + p'(x)$.
   (a) Find the matrix for F with respect to the basis $\mathcal{B} = [1, x]$.
   (b) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}' = [1, 2 + x]$ and its inverse. (Hint: you figured out how to invert the relevant matrix in Problem 2.)
   (c) Use parts 8a and 8b to compute the matrix for F with respect to the basis $\mathcal{B}'$.
   (d) Check your answer by computing the matrix for F with respect to $\mathcal{B}'$ directly.
(9) Find two 2×2 matrices which are not similar. Prove that they're not similar. (This can be easy, if you choose the right matrices.)

Note: we'll do some more change of basis problems next week, but you still might like to find some more to practice on your own this week.

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