MATH V2020 PROBLEM SET 4
DUE SEPTEMBER 30, 2008.

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(1) Find all solutions to the following systems of equations by row reduction. Identify the pivot variables and free variables.

(a)

$$\begin{aligned} 2x + 3y + z &= 0 \\ x + y + z &= 0 \\ 3x + 4y + 2z &= 0 \\ y + z &= 0 \end{aligned}$$

.

(b)

$$\begin{aligned} 3x + y - 3z &= 14 \\ 2x + y - 3z &= 9 \\ -2x - y + 4z &= -8. \end{aligned}$$

(c)

$$\begin{aligned} x + y + 3z &= 5 \\ -2x - 2y - 6z &= -20. \end{aligned}$$

(2) Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + 3z \\ -2x - 2y - 6z \end{pmatrix}.$$

Find a basis for the kernel of F. Find a basis for the image of F.

(3) All of the entries in the row-reduced echelon forms of the matrices in problem (1) were integers (hopefully).

(a) Explain why this was not obvious *a priori* (i.e., beforehand). (One or two sentences should be enough.)

(b) Explain how to cook up complicated-looking examples whose row-reduced echelon forms have all entries integers. Illustrate your algorithm with a couple of examples.

(4) Use the row-reduction technique from class to compute the inverse of the matrix

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

(5) Prove that a matrix A is invertible if and only if the row-reduced echelon form of A is the identity map. (There are two directions. The "only if" part is easier; for the other direction use elementary matrices in a similar way to what we did in class.)

(6) The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$. Use problem (5) to prove this. (There are several cases here, depending on whether, say, $a = 0$, so the proof is a bit annoying.)

(7) Let P be the plane in $\mathbb{R}^3$ given by the equation $y = x$. Let $F\colon \mathbb{R}^3 \to P$ denote projection onto P.
  (a) Find a basis $\mathcal{B}_P$ for P.
  (b) Find a basis $\mathcal{B}'$ for $\mathbb{R}^3$ so that the matrix for F with respect to $\mathcal{B}', \mathcal{B}_P$ is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
(Hint: you found two useful basis vectors already.)
  (c) Find the change of basis matrix from $\mathcal{B}'$ to the standard basis $\mathcal{B}$ for $\mathbb{R}^3$. Find its inverse.
  (d) Find the matrix for F with respect to $\mathcal{B}$ and $\mathcal{B}_P$.

(8) Let $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 2x + 4y + 6z \end{pmatrix}.$$
  (a) Find bases $\mathcal{B}_3$ and $\mathcal{B}_2$ for $\mathbb{R}^3$ and $\mathbb{R}^2$ so that the matrix for F with respect to $\mathcal{B}_3$ and $\mathcal{B}_2$ is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
by examining the second (direct) proof of the rank theorem from class.
  (b) **Optional**: Do the same thing, but by using row and column operations instead, and keeping track of the elementary matrices you use. (This is more work, but should solidify some concepts.)

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