(1) Find all solutions to the following systems of equations by row reduction. Identify the pivot variables and free variables.
(a) 
\[
\begin{align*}
2x + 3y + z &= 0 \\
x + y + z &= 0 \\
3x + 4y + 2z &= 0 \\
y + z &= 0
\end{align*}
\]
(b) 
\[
\begin{align*}
3x + y - 3z &= 14 \\
2x + y - 3z &= 9 \\
-2x - y + 4z &= -8
\end{align*}
\]
(c) 
\[
\begin{align*}
x + y + 3z &= 5 \\
-2x - 2y - 6z &= -20
\end{align*}
\]
(2) Define \( F : \mathbb{R}^3 \to \mathbb{R}^2 \) by
\[
F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + 3z \\ -2x - 2y - 6z \end{pmatrix}.
\]
Find a basis for the kernel of \( F \). Find a basis for the image of \( F \).
(3) All of the entries in the row-reduced echelon forms of the matrices in problem (1) were integers (hopefully).
(a) Explain why this was not obvious \textit{a priori} (i.e., beforehand). (One or two sentences should be enough.)
(b) Explain how to cook up complicated-looking examples whose row-reduced echelon forms have all entries integers. Illustrate your algorithm with a couple of examples.
(4) Use the row-reduction technique from class to compute the inverse of the matrix
\[
\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.
\]
(5) Prove that a matrix \( A \) is invertible if and only if the row-reduced echelon form of \( A \) is the identity map. (There are two directions. The “only if” part is easier; for the other direction use elementary matrices in a similar way to what we did in class.)
(6) The matrix \[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] is invertible if and only if \( ad - bc \neq 0 \). Use problem (5) to prove this. (There are several cases here, depending on whether, say, \( a = 0 \), so the proof is a bit annoying.)

(7) Let \( P \) be the plane in \( \mathbb{R}^3 \) given by the equation \( y = x \). Let \( F: \mathbb{R}^3 \to P \) denote projection onto \( P \).
   (a) Find a basis \( B_P \) for \( P \).
   (b) Find a basis \( B' \) for \( \mathbb{R}^3 \) so that the matrix for \( F \) with respect to \( B', B_P \) is
       \[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \]
       (Hint: you found two useful basis vectors already.)
   (c) Find the change of basis matrix from \( B' \) to the standard basis \( B \) for \( \mathbb{R}^3 \). Find its inverse.
   (d) Find the matrix for \( F \) with respect to \( B \) and \( B_P \).

(8) Let \( F: \mathbb{R}^3 \to \mathbb{R}^2 \) be defined by
   \[ F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 2x + 4y + 6z \end{pmatrix} . \]
   (a) Find bases \( B_3 \) and \( B_2 \) for \( \mathbb{R}^3 \) and \( \mathbb{R}^2 \) so that the matrix for \( F \) with respect to \( B_3 \) and \( B_2 \) is
       \[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
       by examining the second (direct) proof of the rank theorem from class.
   (b) **Optional:** Do the same thing, but by using row and column operations instead, and keeping track of the elementary matrices you use. (This is more work, but should solidify some concepts.)

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