Updated: I’ve made the first problem easier by adding a hypothesis, dropped Cramer’s rule from this problem set (deleting last problem and modifying second to last), added an instruction to compute one of the determinants in Problem 7 by row-reduction.

(1) Let \( F, G : V \to V \) be linear maps, and assume that \( G \) is invertible. Using the definition in terms of volume functions, prove that \( \det(F \circ G) = \det(F) \det(G) \). (Hint: your proof should start “Let \( \text{Vol} \) be a volume form on \( V \) and \( \mathcal{B} = [v_1, \ldots, v_n] \) a basis for \( V \).” The proof should not be more than three or four mathematical “sentences.”)

(2) Let \( F : V \to V \) be a linear transformation and \( \lambda \in \mathbb{F} \). Let \( n = \dim(V) \). Prove that \( \det(\lambda F) = \lambda^n \det(F) \). (Here, \( \lambda F \) denotes the linear transformation \( (\lambda F)(v) = \lambda(F(v)) \).) (Hint: start the same way as the previous one. Your proof should again be short.)

(3) Suppose that \( A \) and \( B \) are \( n \times n \) matrices, and \( AB = I \). We proved in class that this implies \( A \) is invertible. Prove this again, using the determinant. (Hint: the proof should again be very short.)

(4) Prove that if \( A \) and \( B \) are similar \( n \times n \) matrices then \( \det(A) = \det(B) \). (Hint: short again. Remind yourself the definition of “similar matrices.”)

(5) Compute the determinant of

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
0 & 2 & 2 & \pi & e \\
0 & 0 & 3 & 7 & 17 \\
0 & 0 & 4 & -8 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

by expansion by minors. Explain why your answer makes sense from the point of view of volume functions. (You might like to use an analogous \( 2 \times 2 \) or \( 3 \times 3 \) matrix to make your point.)

(6) An \( n \times n \) matrix \( A \) is invertible if and only if \( A^T \) is invertible.
   (a) Prove this using determinants (one sentence).
   (b) Prove this directly, using the fact that \( (AB)^T = B^T A^T \). (Roughly three sentences.)

(7) Compute the following determinants:
   (a) \[
   \begin{pmatrix}
   1 & 2 & 3 \\
   4 & 5 & 6 \\
   -1 & -2 & -1 \\
   \end{pmatrix}
   \]
   Do this one both by expansion by minors and by row reduction.
(b) 
\[
\begin{pmatrix}
0 & 5 & 0 & 0 & 0 \\
12 & 11 & 2 & 0 & 10 \\
3 & 13 & 0 & 0 & 14 \\
9 & 8 & 6 & 1 & 7 \\
0 & 15 & 0 & 0 & 4
\end{pmatrix}
\]

(c) 
\[
\begin{pmatrix}
a & b & 0 & 0 & 0 \\
c & d & 0 & 0 & 0 \\
0 & 0 & e & f & 0 \\
0 & 0 & g & h & 0 \\
0 & 0 & 0 & i & j \\
0 & 0 & 0 & 0 & k & l
\end{pmatrix}
\]

(8) Is it usually faster to compute a determinant of an \(n \times n\) (for \(n\) large) matrix by expanding by minors immediately or by row reducing first? Roughly how many arithmetic operations does each take? (You don’t have to “prove” your answers; just explain them. But try to be precise and complete; this might take a couple of drafts.)

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