MATH V2020 PROBLEM SET 8
DUE NOVEMBER 18, 2008.

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Note: problem 1 corrected to work out more nicely.

(1) Consider the system of differential equations
\[
\begin{align*}
y_1' &= -3y_1 + 2y_2 \\
y_2' &= -2y_1 + 2y_2
\end{align*}
\]
subject to the initial conditions \(y_1(0) = 1, \ y_2(0) = 3\).
(a) Solve the system by decoupling it (the first method we used in class).
(b) Check that your solution is, indeed, a solution.
(c) Solve the system using the matrix exponential.
(d) Check that your two solutions agree.

(2) Write the differential equation \(y''' = 5y'' - y' + 5y\) as a system of first-order differential equations.

(3) Exponentiating JNF matrices, I. Let \(A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\).
(a) Compute \(A^2, A^3, A^4\).
(b) Give a formula for \(A^n\). Prove your answer by induction.
(c) What is \(e^A\)?
(d) What is \(e^{tA}\)? (Be careful.)
(e) Consider the system of differential equations \(y' = Ay\). Use matrix exponentiation to find the solution subject to the initial conditions \(y(0) = (1, 3)^T\). Verify that what you found is, in fact, a solution.

(4) Exponentiating JNF matrices, II. Let \(A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\).
(a) Compute \(A^2, A^3, A^4\).
(b) Give a formula for \(A^n\). You don’t have to prove your answer this time.
(c) What is \(e^A\)?
(d) What is \(e^{tA}\)? (Be careful.)

(5) Exponentiating JNF matrices, III. Let
\[
A = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}
\]
be a Jordan block. What is \(e^A\)? \(e^{tA}\)? Justify your answers.
(6) Consider the system of differential equations

\[ y'_1(t) = y_1(t) + y_2(t) \]
\[ y'_2(t) = -y_1(t) + 3y_2(t). \]

Use matrix exponentiation to find the solution satisfying \( y_1(0) = 1, y_2(0) = 2 \). (Hint: put the corresponding matrix in JNF.)

(7) Lengths and angles.
(a) On \( \mathbb{R}^3 \) with its usual dot product, compute the lengths of \((1,1,1)^T\) and \((1,2,3)^T\), and the angle between them. (Not cooked to come out nicely.)
(b) On \( \mathcal{C}^\infty[0,1] \) with inner product \( \langle f,g \rangle = \int_0^1 f(x)g(x)\,dx \), compute the lengths of \( f(x) = x \) and \( g(x) = \sin(2\pi x) \), and the angle between them. (Also not cooked.)

(8) Let

\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}. \]

Use the Gram-Schmidt process to find an orthonormal basis for the image (column space) of \( A \), with respect to the standard dot product on \( \mathbb{R}^3 \).

(9) Define an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{P}_{\leq 2} \) by \( \langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)\,dx \).
(a) Prove that \( \langle \cdot, \cdot \rangle \) does, in fact, give an inner product.
(b) Apply the Gram-Schmidt process to the basis \([2, x, x^2]\) to obtain an orthonormal basis for \( \mathcal{P}_{\leq 2} \) (with respect to this inner product).

(10) Let \( S = \{v_1, \ldots, v_k\} \subset V \) be a set of vectors in an inner product space \( V \). Prove: if the vectors in \( S \) are orthonormal then \( S \) is linearly independent.

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