Sample Final
Math UN1101: Calculus I, Section 10
Instructor: Linh Truong
Fall 2017

Name: ________________________________
UNI: _________________________________

Instructions:

- Print your name in the space above.
- Show your reasoning and intermediate computations.
- You have 2 hours and 50 minutes.
- No notes, books, calculators or any other electronic devices are allowed.
- Write answers in the space provided. If you need extra space, use the backs of pages and clearly label your work.

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1. (10 points) Mark True or False (1 point each). No justification is needed.

(a) \[\boxed{T}\] If \( f(x) \) is increasing at \( x = 3 \), then \( f''(3) > 0 \).

(b) \[\boxed{T}\] The function \( f(x) = \tan(x) \) is continuous at \( x = \pi \).

(c) \[\boxed{F}\] The derivative of \( f(x) = e^{e^x} \) is \( f'(x) = e^{e^x} \).

(d) \[\boxed{F}\] The derivative of \( f(x) = 1/(1 + x^2) \) is \( f'(x) = \arctan(x) \).

(e) \[\boxed{F}\] \( \int_{\frac{1}{2}}^1 \ln x \, dx \) is positive.

(f) \[\boxed{F}\] \( \int_a^b f(x)g(x) \, dx = (\int_a^b f(x) \, dx) \cdot (\int_a^b g(x) \, dx) \)

(g) \[\boxed{F}\] If \( \int_0^2 f(x) \, dx = 2 \) and \( \int_3^5 f(x) \, dx = 6 \), then \( \int_0^5 f(x) \, dx = 8 \).

(h) \[\boxed{T}\] The function \( f(x) = \int_0^x (1 - t)e^{t^2} \) is decreasing when \( x > 1 \).

(i) \[\boxed{F}\] The fundamental theorem of calculus ensures that \[
\frac{d}{dx} \int_2^3 f'(t) \, dt = f(3) - f(2).\]

(j) \[\boxed{F}\] The substitution rule ensures that \( \int f(u) \, du = \int f(x^2) \, dx \) if \( u = x^2 \).
2. (4 points) Compute \( \lim_{x \to 4} \frac{x - 4}{x^2 - 9x + 20} \).

\[
= \lim_{x \to 4} \frac{x - 4}{(x - 4)(x - 5)}
\]

\[
= \lim_{x \to 4} \frac{1}{x - 5}
\]

\[
= \frac{1}{4 - 5} = \frac{1}{-1} = -1
\]
3. (3 points) \[ \lim_{x \to -\infty} \frac{x^5 + 5x + 2}{2x^4 - 3} \]

\[
= \lim_{x \to -\infty} \frac{x^5 + 5x + 2}{2x^4 - 3} \cdot \frac{1/x^4}{1/x^4}
\]

\[
= \lim_{x \to -\infty} \frac{x + 5/x^3 + 2/x^4}{2 - 3/x^4}
\]

Since \( x + 5/x^3 + 2/x^4 \to -\infty \) as \( x \to -\infty \)

and \( 2 - 3/x^4 \to 2 \) as \( x \to -\infty \)

\[ \lim_{x \to -\infty} \frac{x^5 + 5x + 2}{2x^4 - 3} = -\infty \]
4. (4 points) Compute \( \lim_{x \to 0} \left( \frac{1}{e^x - 1} - \frac{1}{x} \right) \).

\[
= \lim_{x \to 0} \left( \frac{x}{(e^x - 1) x} - \frac{e^x - 1}{x (e^x - 1)} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{x - e^x + 1}{x (e^x - 1)} \right) \quad \text{type } \frac{0}{0}
\]

\[
= \lim_{x \to 0} \left( \frac{1 - e^x}{e^x - 1 + xe^x} \right) \quad \text{by L'Hôpital's Rule}
\]

\[
= \lim_{x \to 0} \left( \frac{-e^x}{e^x + e^x + xe^x} \right) \quad \text{by L'Hôpital's Rule}
\]

\[
= \frac{-e^0}{e^0 + e^0 + 0e^0} = \frac{-1}{2}
\]
5. (4 points) Is the function \( f(x) \) continuous at \( x = 0 \)? Explain.

\[
f(x) = \begin{cases} 
x^4 \sin \left( \frac{1}{x} \right) & x > 0 \\
e^{-x} - 1 & x \leq 0
\end{cases}
\]

- To be continuous at \( x = 0 \) means
  \[
  \lim_{x \to 0} f(x) \text{ exists}
  \]
  and
  \[
  f(0) = \lim_{x \to 0} f(x).
  \]

- \( f(0) = e^0 - 1 = 0 \).

- \( \lim_{x \to 0^{-}} f(x) = e^0 - 1 = 0 \).

- \( \lim_{x \to 0^{+}} f(x) = \lim_{x \to 0^{+}} x^4 \sin \left( \frac{1}{x} \right) \)

- \( -1 \leq \sin \left( \frac{1}{x} \right) \leq 1 \)

- \( -x^4 \leq x^4 \sin \left( \frac{1}{x} \right) \leq x^4 \)

and \( \lim_{x \to 0^{-}} (-x^4) = \lim_{x \to 0^{+}} x^4 = 0 \).

By the Squeeze Theorem, \( \lim_{x \to 0^{+}} x^4 \sin \left( \frac{1}{x} \right) = 0 \).

So \( f(x) \) is continuous at \( x = 0 \).

Since \( f(0) = \lim_{x \to 0} f(x) \).
6. (4 points) Find $f'(x)$ if

$$f(x) = (\cos x)(\ln(6x + 1)).$$

By the Product Rule,

$$f'(x) = (-\sin x)(\ln(6x + 1)) + \cos x \cdot \frac{d}{dx} \ln(6x + 1).$$

To find $\frac{d}{dx} \ln(6x + 1)$, use the chain rule with $u = 6x + 1$.

$$\frac{d}{dx} \ln(6x + 1) = \frac{d}{du} (\ln u) \cdot \frac{du}{dx} = \frac{1}{u} \cdot 6 = \frac{6}{6x + 1}$$

Finally,

$$f'(x) = (-\sin x)(\ln(6x + 1)) + \frac{6 \cos x}{6x + 1}.$$
7. (5 points) Find the tangent line of \( y = \frac{e^x}{x^{1/3} + 1} \) at \( x = 1 \).

Need to find \( \frac{dy}{dx} \) at \( x = 1 \). By the Quotient Rule,

\[
\frac{dy}{dx} = \frac{e^x \left( x^{1/3} + 1 \right) - e^x \left( \frac{1}{3} x^{-2/3} \right)}{(x^{1/3} + 1)^2}
\]

\[
\left. \frac{dy}{dx} \right|_{x=1} = \frac{e^1 (1+1) - e^1 \left( \frac{1}{3} \cdot 1 \right)}{(1+1)^2}
\]

\[
= \frac{2e - \frac{1}{3}e}{4} = \frac{5}{12}e
\]

So the slope of the tangent line is \( m = \frac{5}{12}e \).

The equation of the tangent line is

\[ y = \frac{5}{12}e + b \]

We need to find \( b \):

At \( x = 1 \), \( y = \frac{e}{2} \). So,

\[
\frac{e}{2} = \frac{5}{12}e + b \Rightarrow b = \frac{1}{12}e
\]

\[ y = \frac{5}{12}e \cdot x + \frac{1}{12}e \]
8. (5 points) Let \( f(x) = x - 2 \arctan x \). Find the absolute maximum and absolute minimum of \( f(x) \) on the interval \([0, 4]\).

Hint: \( \arctan(4) \) is about equal to 1.3.

\[
f'(x) = 1 - \frac{2}{1 + x^2}
\]

The critical points of \( f(x) \) occur at
\[
f'(x) = 0 \quad \text{(and where } f'(x) \text{ DNE, but } f'(x) \text{ exists everywhere)}
\]

\[
1 - \frac{2}{1 + x^2} = 0
\]

\[
1 + x^2 = 2
\]

\[
x^2 = 1
\]

\[
x = 1 \text{ or } x = -1.
\]

So the only critical point in the interval \([0, 4]\) is \( x = 1 \).

To find the absolute max and min values of \( f(x) \), evaluate \( f(x) \) at \( x = 1 \) and \( x = 0 \) and \( x = 4 \).

\[
x = 0: \quad f(0) = 0 - 2 \arctan(0) = 0
\]

\[
\leftarrow \text{MIN}
\]

\[
x = 1: \quad f(1) = 1 - 2 \arctan(1) = 1 - 2 \cdot \frac{\pi}{4} = 1 - \frac{\pi}{2} < 0
\]

\[
x = 4: \quad f(4) = 4 - 2 \arctan(4) \approx 4 - 2(1.3)
\]

\[
= 4 - 2.6 = 1.4 > 0. \quad \text{MAX}
\]
9. (4 points) A particle moves along the curve \(3x^2 + y^2 = 13\). At the point \((2, 1)\), the \(x\)-coordinate is increasing at a rate of 5 in/sec; in other words \(\frac{dx}{dt} = 5\) in/sec. Find the rate of change of the \(y\)-coordinate at the point \((2, 1)\).

Take the derivative with respect to \(t\) of the equation:

\[
\frac{d}{dt} (3x^2 + y^2) = \frac{d}{dt} (13)
\]

\[
6x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

At \((2, 1)\), \(x = 2\), \(y = 1\), and \(\frac{dx}{dt} = 5\).

So:

\[
6(2) \cdot 5 + 2(1) \cdot \frac{dy}{dt} = 0
\]

\[
60 + 2 \frac{dy}{dt} = 0
\]

\[
\frac{dy}{dt} = -30 \text{ in/sec}
\]
10. (4 points) Evaluate the limit:

\[ \lim_{n \to \infty} \sum_{i=1}^{n} 4e^{-2x_i} \Delta x, \]

where \( x_i = 1 + i \Delta x \) and \( \Delta x = \frac{3}{n} \).

Recall \( \int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \)

where \( x_i = a + i \Delta x \) and \( \Delta x = \frac{b-a}{n} \).

Here, \( a = 1 \), and \( b-a = 3 \Rightarrow b = 4 \).

\[ \lim_{n \to \infty} \sum_{i=1}^{n} 4e^{-2x_i} \Delta x = \int_{1}^{4} 4e^{-2x} \, dx \]

Using a substitution \( u = -2x \), \( du = -2 \, dx \),

\[ \int_{1}^{4} 4e^{-2x} \, dx = \int_{-2}^{-8} 4e^{u} \cdot \left( -\frac{1}{2} \right) \, du \]

\[ = -\frac{1}{2} \cdot 4 \left. e^{u} \right|_{-2}^{-8} \]

\[ = -2(e^{-8} - e^{-2}) \]

\[ = 2(e^{-2} - e^{-8}) \]
11. (4 points) \[ \int_0^1 \sqrt{1+7x} \, dx. \]

Use a substitution.

\[ u = 1 + 7x \quad \text{with} \quad \begin{align*}
  x = 0 & \Rightarrow u = 1 \\
  x = 1 & \Rightarrow u = 8
\end{align*} \]

\[
\int_0^1 \sqrt{1+7x} \, dx = \int_1^8 u^{1/3} \frac{du}{7}
\]

\[
= \frac{1}{7} \cdot \frac{3}{4} u^{4/3} \bigg|_1^8
\]

\[
= \frac{3}{28} \left( 8^{4/3} - 1^{4/3} \right)
\]

\[
= \frac{3}{28} \left( 2^4 - 1 \right)
\]

\[
= \frac{45}{28}
\]
12. (4 points) \( \int_{-3}^{0} (1 + \sqrt{9 - x^2}) \, dx \). Hint: Interpret the integral as the area of a region.

\[
y = 1 + \sqrt{9 - x^2}
\]
is the equation for the top half of a circle of radius 3 with center (0,1).

\[
\int_{-3}^{0} 1 + \sqrt{9 - x^2} \, dx = \frac{9}{4} \pi + 3
\]

The area of the region that is \( \frac{1}{4} \) of a circle is \( \frac{1}{4} \pi \cdot 3^2 = \frac{9}{4} \pi \).

The area of the rectangle is \( 3 \cdot 1 = 3 \).

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13. (5 points) Find $f''(\sqrt{\pi})$ if $f(x) = \int_0^{x^2} \cos t \, dt$

Let $u = x^2$.

$$f'(x) = \frac{d}{dx} \int_0^u \cos t \, dt$$

$$= \left( \frac{d}{du} \int_0^u \cos t \, dt \right) \frac{du}{dx}$$

(by the chain rule)

$$= \cos(u) \cdot 2x$$

(by FTC I)

$$= \cos(x^2) \cdot 2x$$

Plug in $x = \sqrt{\pi}$ to get:

$$f'(\sqrt{\pi}) = \cos(\pi) \cdot 2\sqrt{\pi}$$

$$= -2\sqrt{\pi}$$
14. (5 points) Find the indefinite integral \( \int \sec^2 x \tan^4 x \, dx \).

Let \( u = \tan x \).
\[
du = \sec^2 x \, dx.
\]

\[
\int \sec^2 x \tan^4 x \, dx = \int u^4 \sec^2 x \, dx
\]
\[
= \int u^4 \, du
\]
\[
= \frac{u^5}{5} + C
\]
\[
= \left( \tan x \right)^5 + C
\]
15. Consider the function
\[ f(x) = \int_1^x (t^2 - 4t - 5) \, dt \]

(a) (3 points) Find the intervals on which \( f(x) \) is increasing or decreasing.

(b) (2 points) Find the intervals on which \( f(x) \) is concave up or concave down.

(a) \( f'(x) = x^2 - 4x - 5 \) by FTC I.

\[ f'(x) = 0 \text{ when } 0 = x^2 - 4x - 5 \]
\[ 0 = (x - 5)(x + 1) \]

critical points: \( x = 5 \) or \( x = -1 \).

\[
\begin{array}{c|c|c}
(-\infty, -1) & (-1, 5) & (5, \infty) \\
\text{sign } f''(x) & + & - & + \\
f''(x) & \text{increase} & \text{decrease} & \text{increase} \\
\end{array}
\]

(b) \( f''(x) = 2x - 4 \)

\[ f''(x) = 0 \text{ when } 2x - 4 = 0 \]
\[ 2x = 4 \]
\[ x = 2 \]

\[
\begin{array}{c|c|c}
(-\infty, 2) & (2, \infty) \\
\text{sign } f''(x) & - & + \\
f''(x) & \text{concave down} & \text{concave up} \\
\end{array}
\]
16. A function \( f(t) \) satisfies \( f''(t) = t + 5 \) and \( f'(0) = 4 \).

(a) (2 points) Find \( f'(t) \).

(b) (3 points) Find \( \int_0^8 f'(t) \, dt \).

(a) The most general antiderivative of \( f''(t) \) is

\[
\int f''(t) \, dt = \int (t + 5) \, dt = \frac{t^2}{2} + 5t + C.
\]

So \( f'(t) = \frac{t^2}{2} + 5t + C \)

\[
f'(0) = \frac{0^2}{2} + 5 \cdot 0 + C = 4 \quad \Rightarrow \quad C = 4.
\]

\[
f'(t) = \frac{t^2}{2} + 5t + 4
\]

(b) \( \int_0^8 \frac{t^2}{2} + 5t + 4 \, dt \)

\[
= \frac{t^3}{6} + \frac{5t^2}{2} + 4t \bigg|_0^8
\]

\[
= \frac{8^3}{6} + \frac{5 \cdot 8^2}{2} + 4 \cdot 8 - 0
\]

\[
= \frac{512}{6} + \frac{320}{2} + 32 - 0
\]

\[
= \frac{832}{3} \quad \text{simplified}
\]

Unsimplified answer okay here.
17. (5 points) Find the area of the region enclosed by the curves $4x + y^2 = 12$ and $x = y$.

This is a parabola that opens sideways with vertex $(3, 0)$.

Points of Intersection:

\[4x + x^2 = 12\]
\[x^2 + 4x - 12 = 0\]
\[(x + 6)(x - 2) = 0\]
\[x = -6 \quad \text{or} \quad x = 2\]

\[y = -6 \quad \text{or} \quad y = 2\]

Area = \[
\int_{-6}^{2} \left(3 - \frac{y^2}{4} - y\right) dy
\]

\[= 3y - \frac{y^3}{12} - \frac{y^2}{2}\bigg|_{-6}^{2}
\]

\[= \left(3 \cdot 2 - \frac{2^3}{12} - \frac{2^2}{2}\right) - \left(3 \cdot (-6) - \frac{(-6)^3}{12} - \frac{(-6)^2}{2}\right)
\]

\[= \left(6 - \frac{8}{3} - 2\right) + (18 - 18 + 18) = 22 - \frac{2}{3} = \frac{64}{3}
\]