Name: ________________________________

Instructions:

- Print your name in the space above.
- Show your reasoning and intermediate computations.
- You have 75 minutes.
- No notes, books, calculators or any other electronic devices are allowed.
- Write answers in the space provided. If you need extra space, use the backs of pages and clearly label your work.

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1. (5 points) Find $y'$ if

$$y = xe^{-7x}.$$ 

**Solution:** Using the product rule,

$$y' = 1 \cdot e^{-7x} + x \cdot \frac{d}{dx} e^{-7x}$$

Letting $u = -7x$, the chain rule says

$$\frac{d}{dx} e^{-7x} = \frac{d}{du} e^u \cdot \frac{d}{dx} u = e^u \cdot (-7) = -7e^{-7x}$$

So

$$y' = e^{-7x} - 7xe^{-7x}$$

or

$$y' = (1 - 7x)e^{-7x}$$
2. (5 points) Find $y'$ if

$$y = \frac{4}{\ln(x)}.$$

Solution:

$$y' = \frac{0 - 4 \cdot \frac{d}{dx} \ln x}{(\ln x)^2}$$

$$= \frac{- \ln x}{(\ln x)^2}$$

$$= \frac{-1}{x(\ln x)^2}$$

$$= -\frac{1}{x\ln x}$$
3. (5 points) Find $y'$ if $y = x^{\sin(x)}$.

\begin{center}
\begin{tcolorbox}
Solution: Use logarithmic differentiation.

\begin{align*}
\ln y &= \sin(x) \ln x \\
\frac{d}{dx} \ln y &= \frac{d}{dx} (\sin(x) \ln x) \\
y' &= \cos(x) \ln x + \sin(x) \cdot \frac{1}{x} \\
y' &= x^{\sin(x)} (\cos(x) \ln x + \frac{\sin(x)}{x})
\end{align*}
\end{tcolorbox}
\end{center}
4. (5 points) Compute
\[
\lim_{x \to 1^+} \frac{1}{\ln(x)} - \frac{x}{x - 1}.
\]

Solution:
\[
\lim_{x \to 1^+} \frac{1}{\ln(x)} - \frac{x}{x - 1} = \lim_{x \to 1^+} \frac{x - 1 - x \ln x}{(x - 1) \ln x} \quad \text{(type 0/0)}
\]
\[
= \lim_{x \to 1^+} \frac{1 - \ln x - x \cdot \frac{1}{x}}{1 \cdot \ln x + (x - 1) \cdot \frac{1}{x}} \quad \text{(by L’Hôpital’s rule)}
\]
\[
= \lim_{x \to 1^+} -\frac{\ln x}{\ln x + 1 - \frac{1}{x}} \quad \text{(type 0/0)}
\]
\[
= \lim_{x \to 1^+} \frac{-\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} \quad \text{(by L’Hôpital’s rule)}
\]
\[
= \frac{-\frac{1}{1}}{\frac{1}{1} + \frac{1}{1^2}} = -\frac{1}{2}
\]
5. (5 points) A cylindrical tank with radius 5 m is being filled with water at a rate of 4 m³/min. How fast is the height of the water increasing?

**Solution:** The volume of the water in the cylindrical tank is given by $V = \pi r^2 h$, where $h$ is the height of the water, and $r = 5$ m. By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}.$$ 

By assumption,

$$\frac{dV}{dt} = 4$$

and

$$\frac{dV}{dh} = \pi r^2 = 25\pi$$

Therefore, the height of the water is increasing at the rate (in m/min):

$$\frac{dh}{dt} = \frac{4}{25\pi}.$$
6. Consider the function \( f(x) = x^3 + 3x^2 - 24x + 8 \).

(a) (3 points) Find the critical points of \( f \). Find the intervals on which \( f \) is increasing or decreasing.

**Solution:**

\[
f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x + 4)(x - 2).
\]

The critical points of \( f(x) \) occur at \( x = -4 \) and \( x = 2 \).

- On \((-\infty, -4)\), \( f'(x) > 0 \), so \( f \) is increasing.
- On \((-4, 2)\), \( f'(x) < 0 \), so \( f \) is decreasing.
- On \((2, \infty)\), \( f'(x) > 0 \), so \( f \) is increasing.

(b) (2 points) Find the inflection points of \( f \). Find the intervals on which \( f \) is concave up or concave down.

**Solution:**

\[
f''(x) = 6x + 6 = 6(x + 1)
\]

So \( f''(x) = 0 \) at \( x = -1 \).

- On \((-\infty, -1)\), \( f''(x) < 0 \), so \( f \) is concave down.
- On \((-1, \infty)\), \( f''(x) > 0 \), so \( f \) is concave up.

Since \( f \) changes concavity at \( x = -1 \), \( x = -1 \) is an inflection point.
(c) (2 points) Use the above information to sketch the graph of $y = f(x)$.

(d) (3 points) Find the absolute maximum and absolute minimum of $y = f(x)$ on the interval $[0, 4]$.

**Solution:** We compute the values $f(x)$ at the critical points and endpoints, then pick out the max/min.
- There is only one critical point in $[0, 4]$: $x = 2$, which gives $f(2) = -20$.
- The two endpoints give $f(0) = 8$ and $f(4) = 24$.
So the absolute maximum is $f(4) = 24$ and the absolute minimum is $f(2) = -20$ on the interval $[0, 4]$. 
7. (5 points) Find the point on the parabola $y^2 = 4x$ that is closest to the point (2, 8).

**Solution:** The distance between a point $(x, y)$ on the parabola $y^2 = 4x$ and the point (2, 8) is

$$d = \sqrt{(x - 2)^2 + (y - 8)^2}.$$

We would like to minimize $d$, or equivalently, minimize $d^2$.

$$d^2 = (x - 2)^2 + (y - 8)^2.$$

We use $x = y^2/4$ to get $d^2$ in terms of $y$ only.

$$f(y) = d^2 = \left(\frac{y^2}{4} - 2\right)^2 + (y - 8)^2.$$

To minimize $f$, find the critical points:

$$f'(y) = 2\left(\frac{y^2}{4} - 2\right)(\frac{2y}{4}) + 2(y - 8) = \frac{y^3}{4} - 16$$

The critical point of $f$ occurs when $y^3 = 64$, or $y = 4$.

- Since $f'(y) < 0$ for all $y < 4$, and $f'(y) > 0$ for all $y > 4$, $f$ has an absolute minimum at $y = 4$ by the First Derivative Test for Extreme Values.
- When $y = 4$, $x = y^2/4 = 4$.

So the point $(4, 4)$ is the closest point on the parabola to the point (2, 8).