Limits

Evaluate the following limits. If the limit does not exist, write "DNE" and explain why. Show the intermediate steps in your calculations, and name the important rules that you are using.

1. (4 points) \( \lim_{x \to -4} \frac{1}{(x+3)(x+4)} + \frac{1}{x+4} \)

\[
= \lim_{x \to -4} \frac{1}{(x+3)(x+4)} + \frac{1}{x+4} \cdot \frac{(x+3)}{(x+3)}
\]

\[
= \lim_{x \to -4} \frac{1 + x + 3}{(x+3)(x+4)}
\]

\[
= \lim_{x \to -4} \frac{x + 4}{(x+3)(x+4)}
\]

\[
= \lim_{x \to -4} \frac{1}{x + 3}
\]

\[
= \frac{1}{-4 + 3} = -1 \quad \text{by Direct Substitution}
\]
2. (4 points) \[ \lim_{x \to -6} \frac{2x + 12}{|x + 6|} \]

\[ |x + 6| = \begin{cases} x + 6 & \text{if } x \geq -6 \\ -(x + 6) & \text{if } x < -6 \end{cases} \]

Left limit:

\[ \lim_{x \to -6^{-}} 2 \cdot \frac{(x + 6)}{|x + 6|} = \lim_{x \to -6^{-}} 2 \cdot \frac{(x + 6)}{-(x + 6)} = -2 \]

Right limit:

\[ \lim_{x \to -6^{+}} 2 \cdot \frac{(x + 6)}{|x + 6|} = \lim_{x \to -6^{+}} 2 \cdot \frac{(x + 6)}{(x + 6)} = 2 \]

Since the left-sided limit does not equal the right sided limit, the limit does not exist.
3. (4 points) \[ \lim_{x \to \infty} \frac{1-3x}{\sqrt{x^2 + 1}} \]

\[ \lim_{x \to \infty} \frac{1-3x}{\sqrt{x^2 + 1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \]

\[ \lim_{x \to \infty} \frac{\frac{1}{x} - 3}{\sqrt{1 + \frac{1}{x^2}}} = -3 \]

As \( x \to \infty \), \( \frac{1}{x} - 3 \to -3 \)

and \( \sqrt{1 + \frac{1}{x^2}} \to 1 \)
4. Continuity

(a) (2 points) What does it mean for a function \( f(x) \) to be continuous at \( x = a \)?

\[
\lim_{x \to a} f(x) \text{ exists and equals } f(a).
\]

(b) (5 points) Consider the function

\[
f(x) = \begin{cases} 
  x^2 \cos^3\left(\frac{1}{x^2}\right) & \text{if } x < 0 \\
  b & \text{if } x = 0 \\
  2^{\sin(x)} - 1 & \text{if } x > 0 
\end{cases}
\]

Does there exist a value of \( b \) so that \( f(x) \) is continuous at \( x = 0 \)? If so, what is \( b \)? Explain.

- As \( x \to 0^- \), \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 \cos^3\left(\frac{1}{x^2}\right) \leq 1 \)

\[
-1 \leq \cos^3\left(\frac{1}{x^2}\right) \leq 1
\]

\[
-x^2 \leq x^2 \cos^3\left(\frac{1}{x^2}\right) \leq x^2
\]

Since \( \lim_{x \to 0^-} -x^2 = 0 \) and \( \lim_{x \to 0^-} x^2 = 0 \), by Squeeze Theorem, \( \lim_{x \to 0^-} x^2 \cos^3\left(\frac{1}{x^2}\right) = 0 \).

- As \( x \to 0^+ \), \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2^{\sin(x)} - 1 = 2^0 - 1 = 0 \)

- So \( 0 = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0 \).

To be continuous, need \( f(0) = \lim_{x \to 0} f(x) \), or \( b = 0 \).
Derivative definition

The definition of the derivative of \( f(x) \) at the point \( x = a \) is:

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

5. (5 points) \( f(x) = \sqrt{x} + 2 \). Use the definition of the derivative to compute \( f'(4) \).

\[
f'(4) = \lim_{h \to 0} \frac{\sqrt{4+h} + 2 - (\sqrt{4} + 2)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \cdot \frac{(\sqrt{4+h} + \sqrt{4})}{(\sqrt{4+h} + \sqrt{4})}
\]

\[
= \lim_{h \to 0} \frac{4+h-4}{h(\sqrt{4+h} + \sqrt{4})}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + \sqrt{4}} = \frac{1}{4}
\]
Derivative Rules

6. (5 points) \( g(x) = 5x \cos(x) + e^x + 4 \). Using any method, compute \( g'(x) \) and \( g'(0) \).

\[
g'(x) = 5 \left[ \frac{d}{dx} (x) \cos(x) + x \cdot \frac{d}{dx} (\cos(x)) \right] + \frac{d}{dx} e^x \\
= 5 \left( \cos(x) + x (-\sin(x)) \right) + e^x \\
= 5 \cos(x) - 5x \sin(x) + e^x
\]

\[
g'(0) = 5 \cos(0) - 5 \cdot 0 \cdot \sin(0) + e^0 \\
= 5 + 1 = 6
\]
7. (5 points) Using any method, compute \( f'(x) \) and \( f'(1) \) for the function

\[
f(x) = \frac{x^{1/3}}{2 + x^4}
\]

\[
f'(x) = \frac{\frac{d}{dx} \left(x^{1/3}\right) \left(2 + x^4\right) - \frac{d}{dx} \left(2 + x^4\right) \cdot x^{1/3}}{(2 + x^4)^2}
\]

\[
= \frac{\frac{1}{3} x^{-2/3} \left(2 + x^4\right) - 4x^3 x^{1/3}}{(2 + x^4)^2}
\]

\[
f'(1) = \frac{\frac{1}{3} \cdot 1^{2/3} \cdot (2 + 1^4) - 4 \cdot 1^3 \cdot 1^{1/3}}{(2 + 1^4)^2} = \frac{-2}{3^2} = -\frac{1}{3}
\]
8. (6 points) Match the following functions to their properties. Write your answers ((I), (II), (III), (IV), (V) and (VI)) in the box below.

(a) $f(x) = \cos(x)$
(b) $f(x) = 2x$
(c) $f(x) = \ln(x)$
(d) $f(x) = \tan(x)$
(e) $f(x) = \frac{x - x^2}{x^2}$
(f) $f(x) = \cos\left(\frac{1}{x-3}\right)$

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(I) The graph of $f(x)$ has a horizontal asymptote at $y = -1$.

(II) $\lim_{x \to 1} f(x)$ does not exist.

(III) $f(x)$ is an even function.

(IV) The inverse of $f(x)$ is $g(x) = \frac{1}{2}x$.

(V) The graph of $f(x)$ has a vertical asymptote at $x = \pi/2$.

(VI) The graph of $f(x)$ is obtained by reflecting the graph of $y = e^x$ across the line $y = x$. 