Midterm 1 Sample
Math UN1101: Calculus III, Section 2
Spring 2019
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Name: ________________________________

Instructions:

• Print your name in the space above.
• Show your reasoning and intermediate computations.
• You have 75 minutes.
• No notes, books, calculators or any other electronic devices are allowed.
• Write answers in the space provided. If you need extra space, use the backs of pages.
1. Let \( \vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}, \vec{b} = -\vec{i} + 2\vec{j} + 2\vec{k} \) and \( \vec{c} = \vec{i} + \vec{k} \). Compute the followings:

(a) \( \vec{a} + \vec{b} - \vec{c} = \)

\[
\text{Solution: } \vec{a} + \vec{b} - \vec{c} = -\vec{i} + 4\vec{j} + 4\vec{k}
\]

(b) \( \vec{a} \times \vec{c} = \)

\[
\text{Solution: } \vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 2\vec{i} + 2\vec{j} - 2\vec{k}
\]

(c) Volume of the parallelepiped determined by \( \vec{a}, \vec{b} \) and \( \vec{c} \).

\[
\text{Solution: } |\vec{b} \cdot (\vec{a} \times \vec{c})| = | -2 | = 2
\]

(d) \( \text{Proj}_{\vec{b}} \vec{c} = \)

\[
\text{Solution: } \text{Proj}_{\vec{b}} \vec{c} = \frac{\vec{c} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = -\frac{1}{9}\vec{i} + \frac{2}{9}\vec{j} + \frac{2}{9}\vec{k}
\]
2. (a) Identify the surface in $\mathbb{R}^3$ described in spherical coordinates by $\phi = 3\pi/4$.

**Solution:** Downward cone with angle $\pi/4$.

(b) Find an equation for the surface in rectangular coordinates.

**Solution:**

\[
\tan(\phi) = \frac{r}{z} = \frac{\sqrt{x^2 + y^2}}{z}
\]

Thus, \(\sqrt{x^2 + y^2} = -z\).
3. Let $P_1$ be the plane $x + y + z = 1$, and $P_2$ be the plane $x - y + z = 1$.

(a) Find cosine of the angle between $P_1$ and $P_2$.

Solution:

\[ \vec{n}_1 = \langle 1, 1, 1 \rangle \quad \text{and} \quad \vec{n}_2 = \langle 1, -1, 1 \rangle \]

Thus, \( \cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1}{3} \).

(b) Find symmetric equation for the line of intersection of $P_1$ and $P_2$.

Solution:

Point on the line of intersection: $(0, 0, 1)$

\[ \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{k} \]

Equation of the line: $x = -(z - 1)$ and $y = 0$

(c) The plane $P : 2y - 2x - 2z = 3$ is parallel to either $P_1$ or $P_2$. Which one?

Solution: $P_2$. (Explain your answer).

(d) Compute the distance between the parallel plane to $P$, from part (c), and $P$.

Solution: Point on $P_2$: $p = (0, 0, 1)$

\[ D = \frac{|2(0) - 2(0) - 2(1) - 3|}{\sqrt{4 + 4 + 4}} = \frac{5}{2\sqrt{3}} \]
4. Let \( L_1 \) and \( L_2 \) be the lines:

\[
\begin{align*}
L_1 : & \quad x = 1 + t, \quad y = 1 + 6t, \quad z = 2t \\
L_2 : & \quad x = 1 + 2s, \quad y = 5 + 15s, \quad z = 6s - 2.
\end{align*}
\]

(a) The line \( x - 2 = -\frac{1 - y}{6} = \frac{z - 2}{2} \) is parallel to either \( L_1 \) or \( L_2 \). Which one?

**Solution:** \( L_1 \). (Explain your answer).

(b) The plane \( 2x + 12y + 4z = 5 \) is orthogonal to either \( L_1 \) or \( L_2 \). Which one?

**Solution:**

\( L_1 \). (Explain your answer).

(c) Show that \( L_1 \) and \( L_2 \) are skew lines.

**Solution:** \( L_1 \) and \( L_2 \) are not parallel, because \( \langle 1, 6, 2 \rangle \) is not parallel to \( \langle 2, 15, 6 \rangle \).

\[
\begin{align*}
1 + t &= 1 + 2s \rightarrow t = 2s \\
1 + 6t &= 5 + 15s \\
2t &= 6s - 2 \rightarrow 4s = 6s - 2 \rightarrow 2s = 2 \rightarrow s = 1
\end{align*}
\]

It follows from 1st and 3rd equations that \( s = 1 \) and \( t = 2 \). But these values don’t satisfy in 2nd equation, so the lines don’t intersect.

(d) Find an equation for a plane containing \( L_1 \) and which doesn’t intersect \( L_2 \).

**Solution:**

\[
\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 6 & 2 \\ 2 & 15 & 6 \end{vmatrix} = 6i - 2j + 3k
\]

Point on \( L_1 \): \((1, 1, 0)\)

Equation of plane: \( 6(x - 1) - 2(y - 1) + 3z = 0 \) \( \rightarrow \) \( 6x - 2y + 3z = 4. \)
5. (a) Describe the trace of the quadric surface
\[ z = x^2 + 2y^2 \]
in the plane \( z = 6 \).

**Solution:** The \( z = 6 \) trace is an ellipse \( 1 = \frac{x^2}{6} + \frac{y^2}{3} \).

(b) Sketch the trace of the quadric surface
\[ z = x^2 + 2y^2 \]
in the \( yz \)-plane.

**Solution:** The \( x = 0 \) trace is the curve \( z = 2y^2 \), a parabola.
(c) Use the previous parts to sketch the quadric surface

\[ z = x^2 + 2y^2. \]

**Solution:** The surface is an elliptic paraboloid which opens upward.