Midterm 1
Introduction to Higher Math
Fall 2018
Instructor: Linh Truong

Name: Solutions

Instructions:

- Print your name in the space above.
- Show your reasoning. Write complete proofs.
- You have 75 minutes.
- No notes, books, calculators or any other electronic devices are allowed.
- As usual, \( \mathbb{Q} \) denotes the set of rational numbers and \( \mathbb{N} \) is the set of natural numbers.

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1. (6 points) Let \( f : \mathbb{Q} \rightarrow \mathbb{N} \) be any function. For each \( i \in \mathbb{N} \), consider the set \( M_i = \{ x \in \mathbb{Q} \mid f(x) = i \} \). In other words, \( M_i = f^{-1}(i) \). What is the following set? Prove your answer.

\[
\bigcap_{i \in \mathbb{N}} M_i
\]

Let \( x \in \bigcap_{i \in \mathbb{N}} M_i \). Then \( x \in M_i \) for all \( i \in \mathbb{N} \).

Then \( f(x) = i \) for all \( i \in \mathbb{N} \).

This is a contradiction to the fact that \( f \) is a function. So \( \bigcap_{i \in \mathbb{N}} M_i = \emptyset \).
2. (a) (2 points) Let $W$ be a set. Give the definition of what it means for a function

$$g : W \times W \rightarrow W \times W$$

to be injective. Be precise.

Let $(a, b) \in W \times W$ and $(c, d) \in W \times W$. If $g$ is injective, if

$$g(a, b) = g(c, d)$$

then $(a, b) = (c, d)$.  


(b) (6 points) Let $f : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q} \times \mathbb{Q}$ be a function defined as follows. If $(x, y) \in \mathbb{Q} \times \mathbb{Q}$, then

$$f(x, y) = (x + y, x - y).$$

Is the function $f$ injective?

Let $(a, b) \in \mathbb{Q} \times \mathbb{Q}$ and $(c, d) \in \mathbb{Q} \times \mathbb{Q}$ such that $f(a, b) = f(c, d)$.

Then $(a+b, a-b) = (c+d, c-d)$.

$$\Rightarrow a+b = c+d \quad \text{and} \quad a-b = c-d$$

Adding the left-hand sides together and setting equal to the sum of the right-hand sides:

$$a+b + a-b = c+d + c-d$$

$$2a = 2c$$

$$\Rightarrow a = c$$

$$\Rightarrow b = d$$

So $(a, b) = (c, d)$. So $f$ is injective.
3. (8 points) Let \( x = 7^5 + 3^5 \). Compute the last digit of \( x \).

\[
\begin{align*}
7^1 &\equiv 7 \mod 10 \\
7^2 &\equiv 9 \mod 10 \\
7^3 &\equiv 3 \mod 10 \\
7^4 &\equiv 1 \mod 10
\end{align*}
\]

\[
\begin{align*}
5^1 &\equiv 1 \mod 4 \\
5^3 &\equiv 1 \mod 4
\end{align*}
\]

Since \( 5^3 \equiv 1 \mod 4 \),
\[
7^5 \equiv 7^1 \mod 10.
\]

\[
x \equiv 7 + 3 \mod 10 \\
\equiv 0 \mod 10.
\]

The last digit of \( x \) is 0.
4. Let $R$ be a relation on $\mathbb{Q}$ defined as follows: If $x, y \in \mathbb{Q}$, then $x R y$ if and only if there exists $c \in \mathbb{Z}$ such that $2^c x = y$.

(a) (6 points) Is the relation $R$ reflexive?

$R$ is reflexive if for every $x \in \mathbb{Q}$, $x R x$.

For every $x \in \mathbb{Q}$, $2^0 \cdot x = x$. So $x R x$.

So $R$ is reflexive.
(b) (6 points) Is the relation \( R \) antisymmetric?

\[ R \text{ is antisymmetric if for every } x, y \in \mathbb{Q} \text{ s.t. } xRy \text{ and } yRx, \text{ we have } x = y. \]

\[ R \text{ is not antisymmetric: } \]

1. \( R \) since \( 2 \cdot 1 = 2 \)
2. \( R \) since \( 2^{-1} \cdot 2 = 1 \)

But \( 1 \neq 2 \).
(c) (6 points) Is the relation $R$ symmetric?

R is symmetric if for every $x, y \in \mathbb{Q}$ such that $x R y$, we have $y R x$.

If $x R y$ then $\exists c \in \mathbb{Z} \text{ s.t. } 2^c x = y$.

Then $x = 2^{-c} y$, or $y R x$.

So $R$ is symmetric.