Instructions:

- Print your name in the space above.
- Show your reasoning. Write complete proofs.
- You have 75 minutes.
- No notes, books, calculators or any other electronic devices are allowed.
- As usual, \( \mathbb{Q} \) denotes the set of rational numbers and \( \mathbb{N} \) is the set of natural numbers.

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1. Consider the following statements about real numbers $x$ and $y$. If true, prove it. If false, state its negation and prove that statement.

(a) (5 points) $\forall x > 0, \exists y > 0$ such that $xy < 1$.

(b) (5 points) $\exists y > 0$ such that $\forall x > 0, xy < 1$.

**Solution:**

(a) Let $x > 0$ and $y = \frac{1}{2x}$. Then $y > 0$ and $xy = \frac{1}{2} < 1$. So this statement is true.

(b) I will prove the negation of the statement:

$$\forall y > 0, \exists x > 0 \text{ such that } xy \geq 1.$$ 

Given $y > 0$, choose $x = 3/y$. Then $x > 0$ and $xy = 3 \geq 1$. Therefore, the statement (b) is false.
2. (a) (2 points) State the Principle of Mathematical Induction.

(b) (8 points) Give a careful proof by mathematical induction that

\[ n! < n^{n-1} \]

for \( n \geq 3 \).

Solution: (b) Given \( n \geq 3 \), let \( P(n) \) be the statement

\[ n! < n^{n-1}. \]

Base Case: \( P(3) \) says:

\[ 3! < 3^2 \]

which is true since \( 6 < 9 \).

Induction Step: Now suppose \( P(n) \) holds and consider \( P(n + 1) \).

\[ (n + 1)! < (n + 1)n^{n-1} \quad \text{by Induction Hypothesis} \]

\[ < (n + 1)(n + 1)^{n-1} \]

\[ = (n + 1)^n. \]

where we used the fact that \( n^{n-1} < (n + 1)^{n-1} \) in the second inequality.
3. (10 points) Prove by induction on $n$ that when $x > 0$:

$$(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$$

for all positive integers $n$.

**Solution:** Given $n \geq 1$, let $P(n)$ be the statement

$$(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2, \quad \forall x > 0.$$

Base Case: $P(1)$ says: for every $x > 0$,

$$1 + x \geq 1 + x + 0x^2 = 1 + x,$$

which is true.

Induction Step: Now suppose that $P(n)$ holds and consider $P(n + 1)$.

We have

$$(1 + x)^{n+1} \geq (1 + x)(1 + nx + \frac{n(n-1)}{2}x^2) \quad \text{by Induction Hypothesis}$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + x + nx^2 + \frac{n(n-1)}{2}x^3$$

$$\geq 1 + (n + 1)x + \left(\frac{n(n-1)}{2} + n\right)x^2 \quad \text{since} \quad \frac{n(n-1)}{2}x^3 > 0$$

$$= 1 + (n + 1)x + \frac{n(n+1)}{2}x^2$$
4. Consider the statement:

\[ \forall x \in \mathbb{R}, \ (x \geq 3) \implies (x^2 > 5). \]

(a) (5 points) Write the contrapositive.
(b) (5 points) Write the negation.

\[ \neg (x \in \mathbb{R}, \ (x \geq 3) \implies (x^2 > 5)) \]

\[ \exists x \in \mathbb{R} \] such that \[ x \geq 3 \] and \[ x^2 \leq 5 \]

\[ \neg \forall x \in \mathbb{R}, \ (x \geq 3) \implies (x^2 > 5) \]
5. (10 points) Show that $\sqrt{3}$ is an irrational number.

**Solution:**

We prove by way of contradiction. Assume $\sqrt{3}$ is rational. Then there exists relatively prime nonzero integers $p$ and $q$ such that $\sqrt{3} = \frac{p}{q}$. Therefore

$$3q^2 = p^2.$$ 

So $3$ divides $p^2$ and therefore, $3$ divides $p$. This implies $9$ divides $p^2$. So $9$ divides the left hand side of the equation. Therefore, $3$ divides $q^2$, or $3$ divides $q$. This contradicts the fact that $p$ and $q$ were relatively prime. So $\sqrt{3}$ is an irrational number.
6. (10 points) Let $a, b \in \mathbb{N}$. Show that $\gcd(a, b)$ is the smallest positive element in the set $\{ma + nb \mid m, n \in \mathbb{Z}\}$. You may use without proof the following two facts:

1. If positive integers $a$ and $b$ are relatively prime, then there are integers $s$ and $t$ such that $sa + nt = 1$.
2. If $a > b$, then $\gcd(a - b, b) = \gcd(a, b)$.

Prove all other results that you use.

**Solution:** Let $c = \gcd(a, b)$. Then there exists integers $i$ and $j$ such that

$$ci = a \quad \text{and} \quad cj = b.$$ 

We claim that $i$ and $j$ are relatively prime. Indeed, let $r = \gcd(i, j)$. Then there are $u$ and $v$ such that $ru = i$ and $rv = j$. Therefore,

$$cru = a \quad \text{and} \quad crv = b.$$ 

Hence $cr|a$ and $cf|b$. Hence,

$$\gcd(a, b) \geq rc \geq c.$$ 

Since $\gcd(a, b) = c$, $r = 1$. Hence $i$ and $j$ are relatively prime.

By the first fact above, there exist integers $s$ and $t$ such that $si + tj = 1$. Multiplying by $c$, we see

$$sic + tjc = c.$$ 

Therefore, $c$ is an element of the set $\{ma + nb \mid m, n \in \mathbb{Z}\}$.

To see that $c$ is the smallest positive element in the above set, let $m, n \in \mathbb{Z}$ and consider

$$ma + nb = mci + ncj = (mi + nj)c.$$ 

If $ma + nb > 0$ is a positive element of the above set, then $(mi + nj) \geq 1$ since $c$ is positive. Therefore if $ma + nb > 0$

$$ma + nb \geq c.$$