Day 1

§ Coxeter systems

Def A <u>Coxeter system</u> (W, S) is a group W and finite set $S \subset W$ satisfying:

$\exists\ m_{st},\quad s,t \in S$

- $m_{ss} = 1 \quad \forall\ s \in S$
- $m_{st} = m_{ts} \in \{2, 3, \dots\} \cup \{\infty\} \quad \forall\ s,t \in S,\quad s \neq t,$

such that W has presentation

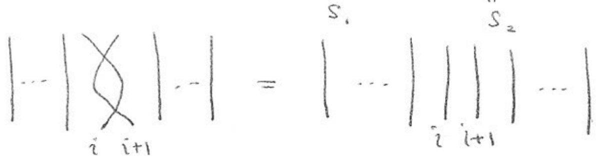
$$W = \langle s \in S \mid (st)^{m_{st}} = id \quad \forall\ s,t \in S,\quad m_{st} < \infty \rangle,$$

i.e. (<u>quadratic rels</u>) $\qquad s^2 = id \quad \forall\ s \in S$

(<u>braid rels</u>) $\qquad \underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{ts}} \quad \forall\ s,t \in S,\quad m_{st} < \infty.$

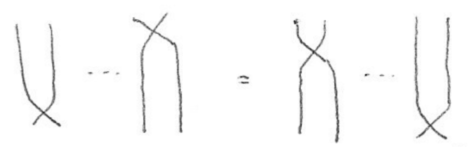
Elements of S are called <u>simple reflections</u>.

Ex (1) $W = S_n,\quad S = \{ \underset{s_1}{\overset{\shortparallel}{(1,2)}}, \underset{s_2}{\overset{\shortparallel}{(2,3)}}, \dots, \underset{s_{n-1}}{\overset{\shortparallel}{(n-1,n)}} \}$

relations: 

$$s_i^2 = id$$

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$



$s_i s_j = s_j s_i \quad$ if $|i-j| > 1$.

(2) $W = I_2(m) =$ dihedral group of order $2m$.

$S = \{s, t\},\quad m_{st} = m \in \{2, 3, \dots\} \cup \{\infty\}.$

If $w \in W$, can write $w = s_1 \cdots s_k,\quad s_1, \dots, s_k \in S$. Then $\underline{w} = (s_1, \dots, s_k)$ is an <u>expression</u> for w of length k.

Def $\ell : W \longrightarrow \mathbb{Z}_{\geq 0},\quad \ell(w) =$ <u>length</u> of w is the smallest k for which w has an expression of length k. Any expression for w with this minimal length is called a <u>reduced expression</u> for w.

Facts:
- $\ell(w) = 0 \iff w = id$
- If W is finite, $\exists$ unique element w_0 of greatest length (<u>longest element</u>)

<u>Def</u> <u>Bruhat order</u> is partial order $\leq$ on W generated by

$x \leq y$ if $l(x) < l(y)$ and $xt = y$, some $t \in T :=$ {conjugates of S}.

Given $y \in W$, fix reduced expression $\underline{y}$ for y. Then

$x \leq y \iff x$ is obtained as some subexpression of $\underline{y}$.

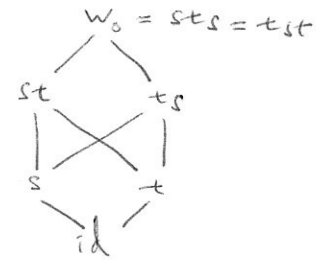
<u>Ex</u> ($W = S_3$)

$(1,3)$

$(1,2,3)$ $(1,3,2)$

$s := (1,2)$ $t := (2,3)$

id



Hasse diagram for $\leq$.

$T = \{s, t, w_0\}$.

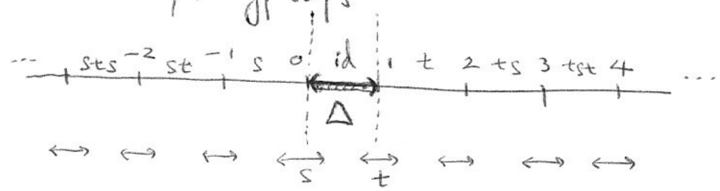
§ Affine reflection groups

V = Euclidean space ($= \mathbb{R}^n$), equipped with a collection of affine hyperplanes

$W :=$ group generated by reflections in these hyperplanes $\subset \mathrm{Aff}(V)$

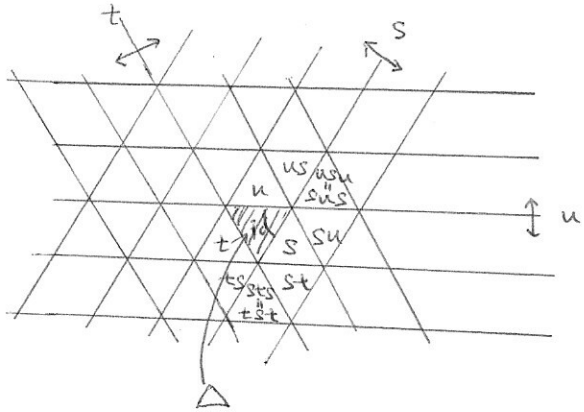
<u>Ex</u> Affine Weyl groups

$(\tilde{A}_1)$



$W = I_2(\infty)$

$(\tilde{A}_2)$



$\mathcal{A} :=$ {connected components of $V \setminus \bigcup$ hyperplanes}, Fix <u>fundamental alcove</u> $\Delta \in \mathcal{A}$
(<u>alcoves</u>)

$S :=$ reflections in walls of Δ.

<u>Lem</u> $W \curvearrowright \mathcal{A}$ simply transitively : $W \xrightarrow{\sim} \mathcal{A}$

$w \longmapsto w.\Delta$

<u>Rmk</u> left mult. by $s \in S$: reflection

right mult. by $s \in S$: cross a hyperplane (Δ, $s\Delta$ are adjacent, so $x\Delta$, $xs\Delta$ are adjac

Thm (W, S) is a Coxeter system.

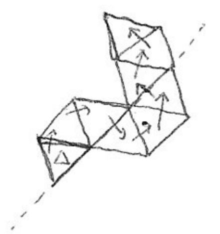
Expression $\underline{w} = (s_1, \dots, s_k)$ determines a <u>stroll</u>, a sequence of adjacent alcoves

$$\underline{A}(\underline{w}) := (\Delta, s_1\Delta, s_1s_2\Delta, \dots, s_1\cdots s_k\Delta).$$

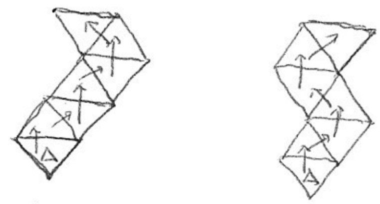
Lem (1) $\underline{w}$ is reduced $\iff$ $\underline{A}(\underline{w})$ never crosses the same hyperplane twice

(2) $\ell(w)$ = # hyperplanes separating Δ and $w\Delta$.

Ex



not reduced vs.



reduced, length 6.

Hecke algebra

Fix Coxeter system (W, S).

Def <u>Hecke algebra</u> $H = H(W)$ is (unital associative) $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\{\delta_s\}_{s\in S}$, subject to relations

- <u>(quadratic rels)</u> $\delta_s^2 = (v^{-1} - v)\delta_s + 1 \quad \forall s \in S \quad (\iff (\delta_s + v)(\delta_s - v^{-1}) = 0)$
- <u>(braid rels)</u> $\underbrace{\delta_s\delta_t\delta_s\cdots}_{m_{st}\text{ factors}} = \underbrace{\delta_t\delta_s\delta_t\cdots}_{m_{st}\text{ factors}} \quad \forall s, t \in S,\ s \neq t,\ m_{st} < \infty.$

For $w \in W$, choose reduced exp $\underline{w} = (s_1, \dots, s_m)$ for w,

$$\delta_w := \delta_{s_1}\cdots\delta_{s_m}. \qquad (\delta_{id} = 1)$$

Fact

$$H = \bigoplus_{w\in W} \mathbb{Z}[v, v^{-1}]\delta_w, \qquad \underline{\text{standard basis}}$$

$$\Rightarrow \mathbb{Z} \underset{\mathbb{Z}[v,v^{-1}]}{\otimes} H \xrightarrow{\sim} \mathbb{Z}[W], \quad \delta_w \mapsto e_w. \qquad \underline{(\text{deformation}} \text{ of } \mathbb{Z}[W$$

$1 \leftarrow v$

Def <u>Kazhdan-Lusztig involution</u> is ring map

$$H \longrightarrow H : h \longmapsto \overline{h}$$

determined by

$$\overline{\delta_s} = \delta_s^{-1} = \delta_s + (v - v^{-1}), \quad s \in S, \quad \overline{v} = v^{-1}.$$

<u>Def</u> <u>KL basis</u> $\{b_w\}_{w\in W} \subset H$ is determined by two conditions:

(self-duality) $\overline{b_w} = b_w$

("degree bound") $b_w = \delta_w + \sum_{x<w} h_{x,w}\, \delta_x$, $\quad h_{x,w} \in v\mathbb{Z}[v]$

$\uparrow$ <u>KL polynomial</u>

Basis because unipotent upper-triangular change of basis w.r.t. standard basis.

Uniqueness: exercise

Existence: $\exists$ algorithm to construct it (next)

Induct on Bruhat order.

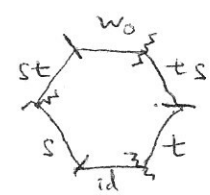
$b_{id} = \delta_{id} = 1$,

$b_s = \delta_s + v \quad \forall\ s \in S$

$$\delta_x b_s = \begin{cases} \delta_{xs} + v\,\delta_x & \text{if } x < xs \\ \delta_{xs} + v^{-1}\delta_x & \text{if } x > xs \end{cases} \qquad (*)$$

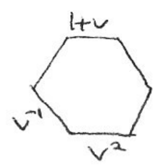
$\boxed{\text{§ KL basis example}}$

$W = S_3$, $S = \{s, t\}$



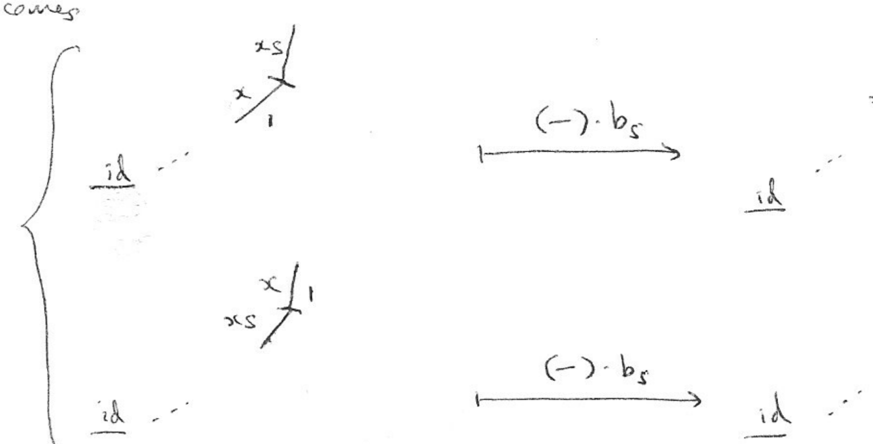
Draw elements of H on <u>Coxeter complex</u>:

<u>Ex</u>

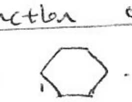
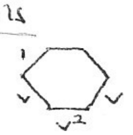


$= v^2 + v^{-1}\delta_s + (1+v)\,\delta_{sts}$.

(*) becomes



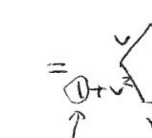
<u>Recursive construction of KL basis</u>

st: $b_s b_t =$  $\cdot\, b_t =$  satisfies defining conditions, so

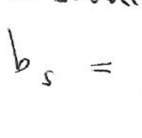
$b_{st} = b_s b_t$ by uniqueness.

ts: $b_{ts} = b_t b_s =$ 

sts : $b_{st} b_s =$ 

$\cdot\, b_s =$ 

self-dual but violates degree bound. Subtract smaller KL basis element to eliminate constant term:

$b_{st} b_s - b_s =$ 

$= b_{sts} \quad (= b_{ts} b_t - b_t)$

Miracle : when subtracting $b_s = \delta_s + v$, no minus sign appears.

Thm (Kazhdan-Lusztig for Weyl groups, Elias-Williamson for any Coxeter)

(1) $b_x \in \delta_x + \bigoplus_{y<x} v\mathbb{Z}_{\geq 0}[v]\,\delta_y$ (Nonnegativity of KL polyn.)

(2) $b_x b_y \in \bigoplus_{z \in W} \mathbb{Z}_{\geq 0}[v, v^{-1}]\, b_z$ (Nonnegativity of structure constants)

This suggests categorification.

§ Categorification of $H(W)$

$\mathcal{A}$ = $\underbrace{\text{additive}}_{\oplus}$ $\underbrace{\text{monoidal}}_{\otimes}$ category with grading shift

autoequivalence $(1) : \mathcal{A} \to \mathcal{A}$

(+ compatibility with monoidal structure)

graded Hom : $\quad \mathrm{Hom}^{\bullet}_{\mathcal{A}}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X, Y(n))$

Def Split Grothendieck group of $\mathcal{A}$ is abelian group

$$[\mathcal{A}]_{\oplus} := \bigoplus_{X \in \mathrm{Ob}(\mathcal{A})} \mathbb{Z}\cdot[X] \Big/ \begin{array}{c}[X] = [Y] + [Z] \\ \text{whenever } X \simeq Y \oplus Z\end{array}.$$

runs over isom classes

made into $\mathbb{Z}[v, v^{-1}]$-algebra via

$$[X]\cdot[Y] := [X \otimes Y], \qquad v \cdot [X] := [X(1)].$$

Def (ad hoc) $\mathcal{A}$ categorifies (or is a categorification of) $H(W)$ if:

(1) $\mathcal{A}$ is Krull-Schmidt, and $\exists$ indecomposable objects B_w, $w \in W$, s.t.

$$\{\text{indec. obj.}\}/\simeq \;\xleftrightarrow{1:1}\; W \times \mathbb{Z}$$
$$B_w(n) \longleftrightarrow (w, n).$$

(2)
$$H \longrightarrow [\mathcal{A}]_{\oplus} : b_s \longmapsto [B_s], \quad \forall s \in S,$$
determines a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism.

(1) $\Rightarrow$ every object in $\mathcal{A}$ is isomorphic to $\bigoplus_{w,n} B_w(n)^{\oplus m_{w,n}}$

for uniquely determined multiplicities $m_{w,n} \geq 0$

$$\Rightarrow \quad [\mathcal{A}]_\oplus = \bigoplus_{w \in W} \mathbb{Z}[v,v^{-1}] \cdot [B_w]$$

i.e. isom to H as $\mathbb{Z}[v,v^{-1}]$-module.

(2) : multiplication also correspond

Let $ch : [\mathcal{A}]_\oplus \xrightarrow{\sim} H$ be inverse to isomorphism of (2).
Then $\{ch(B_w)\}_{w \in W}$ is $\mathbb{Z}[v,v^{-1}]$-basis of H with structure constants in $\mathbb{Z}_{\geq 0}[v,v^{-1}]$.