Day 1

\section*{Coxeter Systems}

**Def.** A Coxeter system \((W, S)\) is a group \(W\) and finite set \(S \subseteq W\) satisfying:

1. \(m_{st}, s, t \in S\)
2. \(m_{ss} = 1\) \quad \forall \ s \in S\)
3. \(m_{st} = m_{ts} \in \{2, 3, \ldots \} \cup \{\infty\}\) \quad \forall \ s, t \in S, \ s \neq t,

such that \(W\) has presentation

\[ W = \langle S \mid (st)^{m_{st}} = id \quad \forall \ s, t \in S, \ m_{st} < \infty \rangle \]

i.e.,

- (quadratic rels) \(s^2 = id\) \quad \forall s \in S
- (braid rels) \(s_{it} s_{jt} s_{it} \cdots = s_{jt} s_{it} s_{jt} \cdots \quad \forall s, t \in S, \ m_{st} < \infty\)

Elements of \(S\) are called simple reflections.

**Ex.**

1. \(W = S_n, \ S = \{ (1, 2), (2, 3), \ldots, (n-1, n) \}\)

   relations: \[\ldots s_{i+1} s_i \cdots = s_i s_{i+1} \cdots \]

   \[ s_i^2 = id \]

   \[ s_i s_{i+n}, s_i = s_{i+n} s_i \]

   \[ s_i s_j = s_j s_i \quad \text{if } |i-j| > 1 \]

2. \(W = I_a(m)\) = dihedral group of order \(2m\)

   \[ S = \{ s, t \}, \ m_{st} = m \in \{2, 3, \ldots \} \cup \{\infty\}. \]

If \(w \in W\), we can write \(w = s_i \cdots s_k, \ s_i, \ldots, s_k \in S\). Then \(w = (s_i, \ldots, s_k)\) is an expression for \(w\) of length \(k\).

**Def.** \(l : W \to \mathbb{Z}_{\geq 0}, \ l(w) = \text{length of } w\) is the smallest \(k\) for which \(w\) has an expression of length \(k\). Any expression for \(w\) with this minimal length is called a reduced expression for \(w\).

**Facts:**

- \(l(w) = 0 \iff w = id\)
- If \(W\) is finite, \(\exists \) unique element \(w_0\) of greatest length

\[ (\text{longest element})] \]
Def Bruhat order is partial order ≤ on \( W \) generated by
\[ x \preceq y \text{ if } \lambda(x) < \lambda(y) \text{ and } xt = y, \text{ some } t \in T := \{ \text{conjugates of } s \}. \]

Given \( y \in W \), fix reduced expression \( y \) for \( y \). Then
\[ x \preceq y \iff x \text{ is obtained as some subexpression of } y. \]

Ex (\( W = S_3 \))
\[
(1, 3) \\
(1, 2, 3) \\
(1, 3, 2) \\
S := (1, 2) \\
t := (2, 3) \\
id
\]

Affine reflection groups
\[ V = \text{Euclidean space } (-\mathbb{R}^n), \text{ equipped with a collection of affine hyperplanes } \]
\[ W := \text{group generated by reflections in these hyperplanes } \subset \text{Aff}(V) \]

Ex Affine Weyl groups
\[ (\tilde{A}_1) \]
\[ ... s \circ t \circ s \circ t \circ s \circ t \circ s \circ t ... \]
\[ \tilde{A}_2 \]

\[ W = I_2(\infty) \]

\[ \Delta := \{ \text{connected components of } V \setminus U \text{ hyperplanes} \}, \text{ Fix fundamental alcove } \Delta \in \mathcal{A} \]

\[ S := \text{reflections in walls of } \Delta. \]

Lem \( W \cap \Delta \) simply transitively:
\[ W \rightarrow \Delta \]

Rank left mate by \( s \in S \): reflection
Rank right mate by \( s \in S \): cross a hyperplane \((\Delta, s\Delta) \text{ are adjacent, so } x\Delta, xs\Delta \text{ are...} \)
Thm. \((W, S)\) is a Coxeter system.

Express \(w = (s_1, \ldots, s_k)\) determines a scroll, a sequence of adjacent alcoves
\[
A(w) = (\Delta, s_1 \Delta, s_2 \Delta, \ldots, s_k \Delta).
\]

**Lemma**

1. \(w\) is reduced \(\iff A(w)\) never crosses the same hyperplane twice.
2. \(l(w) = \#\) hyperplanes separating \(\Delta\) and \(w\Delta\).

**Example**

![Not reduced vs. reduced](image)

reduced, length 6

### Hecke algebra

Fix Coxeter system \((W, S)\).

**Definition** Hecke algebra \(H = \mathcal{H}(W)\) is (finite associative) \(\mathbb{Z}[v, v^{-1}]\)-algebra
generated by \(\{s_s\}_{s \in S}\), subject to relations
- (quadratic rels) \(s_s^2 = (v^{-1} - v)s_s + 1\) \(\forall s \in S\) \((\iff (s_s + v)(s_s - v^{-1}) = 0)\)
- (braid rels) \(s_t s_s s_t \cdots \overset{m_{st}}{=} s_t s_t s_s \cdots \overset{m_{st}}{=} \) \(\forall s, t \in S, s \neq t, m_{st} < \infty\).

For \(w \in W\), choose reduced \(x = (s_1, \ldots, s_m)\) for \(w\),
\[
s_w := s_1 \cdots s_m. \quad (s_i^2 = 1)
\]

**Fact**

\[ H = \bigoplus_{w \in W} \mathbb{Z}[v, v^{-1}] s_w, \quad \text{standard basis} \]

\[ H \otimes_{\mathbb{Z}[v, v^{-1}]} H \overset{\sim}{\longrightarrow} \mathbb{Z}[W], \quad s_w \mapsto e_w. \quad \text{(deformation of } \mathbb{Z}[W] \text{)}
\]

**Definition** Kazhdan-Lusztig involution is ring map
\[
H \rightarrow H \quad h \mapsto \overline{h}
\]
determined by
\[
\overline{s} = s^{-1} = s + (v - v^{-1}), \quad s \in S, \quad \overline{v} = v^{-1}.
\]

\[\]
Def KL basis \( \{ b_w \}_{w \in W} \subset H \) is determined by two conditions:

1. (self-duality) \( b_w = b_w \)
2. ("degree bound") \( b_w = \delta_w + \sum_{x \leq w} h_{x,w} \delta_x \), \( h_{x,w} \in \mathbb{R} \)

Unipotent basis because upper-triangular change of basis w.r.t. standard basis.

Uniqueness: exercise

Existence: \( \exists \) algorithm to construct it (next)

Induct on Bruhat order:

\[
\begin{align*}
\delta_{id} &= \delta_{id} = 1, \\
b_s &= \delta_s + v, \quad \forall s \in S \\
\delta_x b_s &= \begin{cases} \\
\delta_{xs} + v \delta_x & \text{if } x < s \\
\delta_{xs} + v^{-1} \delta_x & \text{if } x > s \\
\end{cases} \\
\end{align*}
\]

KL basis example

\( W = S_3, \quad S = \{ s, t \} \)

Draw elements of \( H \) on Coxeter complex:

\[
\text{Ex: } \quad v^2 + v^{-1} \delta_s + (1 + v) \delta_{st}
\]

\((*)\) becomes

\[
\begin{cases}
\delta_{id} \\
\delta_x \\
\delta_{xs} \\
\end{cases}
\]

Recursive construction of KL basis

\[
\begin{align*}
\text{st: } \quad b_s b_t &= \begin{array}{c}
\text{Diagram}
\end{array} \quad b_t &= \begin{array}{c}
\text{Diagram}
\end{array} \\
\text{satisfies defining conditions, so } \\
\text{by uniqueness, } \\
est: \quad b_{st} &= b_s b_t \\
\end{align*}
\]
\[ b_{st} b_s = \begin{array}{c} \text{self-dual but violates degree bound. Subtract smaller KL basis element to} \\
\text{eliminate constant term:} \\
\text{subtract smaller KL basis element to} \\
\text{eliminate constant term:} \\
b_{st} b_s - b_s = \begin{array}{c} \text{self-dual but violates degree bound. Subtract smaller KL basis element to} \\
\text{eliminate constant term:} \\
\text{subtract smaller KL basis element to} \\
\text{eliminate constant term:} \\
\end{array} b_{st} (= b_{st} b_t - b_t) \]

\[ \text{Thm (Kazhdan–Lusztig for Weyl groups, Egan–Williamson for any Coxeter)} \]

\[ (1) \quad b_x \in \mathcal{B}_x + \bigoplus_{y \leq x} v \mathbb{Z}_{>0} [v] b_y \]

\[ (2) \quad b_x b_y \in \bigoplus_{z \in W} \mathbb{Z}_{>0} [v, v^{-1}] b_z \]

This suggests categorification

\[ \text{§ Categorification of } H(W) \]

\[ A = \text{additive monoidal category with grading shift} \]

\[ \text{auto-equivalence (1): } A \rightarrow A \]

\[ \text{+ compatibility with monoidal structure} \]

graded Hom:
\[ \text{Hom}_A (X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A (X, Y(n)) \]

Def Split Grothendieck group of \( A \) is abelian group
\[ [A]_\oplus := \bigoplus_{X \in \text{Ob}(A)} \mathbb{Z} [X] / [X] = [Y] + [Z] \]

whenever \( X = Y \oplus Z \)

made into \( \mathbb{Z} [v, v^{-1}] \)-algebra via
\[ [X] \cdot [Y] := [X \otimes Y], \quad v \cdot [X] := [X(1)] \]

Def (ad hoc) \( A \) categorifies (or is a categorification of) \( H(W) \) if:

\[ (1) \quad A \text{ is Krull–Schmidt, and } B \text{ indecomposable objects } B_w, \text{ w } \in W, \text{ st} \]

\[ \text{[indecomposable objects } B_w, \text{ w } \in W, \text{ st} \]

\[ \text{[indecomposable objects } B_w, \text{ w } \in W, \text{ st} \]

\[ \begin{array}{c}
\text{determine a } \mathbb{Z} [v, v^{-1}] \text{-algebra isomorphism.}
\end{array} \]
(1) \[ \Rightarrow \text{every object in } A \text{ is isomorphic to } \bigoplus_{w \in W} B_w(n)^{w_0 n} \]

for uniquely determined multiplicities \( w_0 n \geq 0 \)

\[ [A]_\oplus = \bigoplus_{w \in W} Z[v, v^{-1}] \cdot [B_w] \]

i.e. isom to \( H \) as \( Z[v, v^{-1}] \)-module.

(2): multiplication also correspond

Let \( \text{ch}: [A]_\oplus \rightarrow H \) be inverse to isomorphism of (2).

Then \( \{ \text{ch}(B_w) \}_{w \in W} \) is \( Z[v, v^{-1}] \)-basis of \( H \) with structure constants in \( Z_{w_0} [v, v^{-1}] \).