Day 2

§ Hecke category

Unlike Hecke algebra, Hecke category depends on more than Coxeter system
(W, S) Coxeter system, $\Bbbk$ field (for simplicity)

Def [Elias – Williamson]

A realization of (W, S) over $\Bbbk$ is a triple
$$\mathfrak{h} = (V, \{\alpha_s^\vee\}_{s\in S} \subset V, \{\alpha_s\}_{s\in S} \subset V^*),$$
where V is finite-dim'l $\Bbbk$-vector space, $V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$, equipped with simple coroots $\alpha_s^\vee$ and simple roots α_s, satisfying:

(1) $\langle \alpha_s^\vee, \alpha_s \rangle = 2 \quad \forall s \in S$

(2) $s \longmapsto (v \longmapsto v - \alpha_s(v)\alpha_s^\vee)$ defines representation of W on V

(3) technical condition (involving 2-quantum numbers $\forall s, t \in S$, $s \neq t$, $m_{st} < \infty$)

More technical conditions: assume realization is balanced and satisfies Demazure surjectivity.

Rmk These technical conditions will be satisfied for realizations we care about, assuming $\mathrm{char}\,\Bbbk > h$ (= Coxeter number)

Let $(\mathfrak{h}, W)$ be a realization over $\Bbbk$ (i.e. $\mathfrak{h}$ is a realization over $\Bbbk$ of Coxeter system (W, S)). The associated Hecke category is a certain $\Bbbk$-linear additive monoidal category with grading shift that categorifies $H(W)$.

Two incarnations:

(1) (geometric) For "Cartan realizations" of crystallographic Coxeter systems (i.e. $m_{st} \in \{2, 3, 4, 6, \infty\}$ $\forall s, t \in S$, $s \neq t$), e.g. realizations arising as root datum of reductive group, can consider Borel-equivariant $\Bbbk$-parity complexes on (Kac–Moody) flag variety (see Day 3)

(2) (diagrammatic) Elias – Williamson diagrammatic Hecke category (later today, continued on Day 3)

§ Soergel bimodules

Concrete algebraic model of Hecke category, but not always "correct" (doesn't always categorify $H(W)$). Only as motivation.

Let $(\mathfrak{h}, W)$ be realization over $\Bbbk$.

$R = \mathrm{Sym}(V^*)$ viewed as ($\mathbb{Z}$-)graded algebra with $\deg V^* = 2$.

$R\text{-gmod-}R$ = category of ($\mathbb{Z}$-)graded R-bimodules and graded (degree 0) R-bimodule homomorphisms

additive monoidal ($\otimes_R$) $\Bbbk$-linear with grading shift (1), $(M(1))_i = M_{i+1}$ for $M = \bigoplus_{i\in\mathbb{Z}} M_i$, so can categorify a $\mathbb{Z}[v, v^{-1}]$-algebra.

$W \curvearrowright V \rightsquigarrow W \curvearrowright R$, $R^s :=$ s-invariants in R, $s \in S$

define graded R-bimodule $B_s := R \otimes_{R^s} R(1)$, $s \in S$.

<u>Def</u> For expression $\underline{w} = (s_1, \dots, s_k)$, <u>Bott-Samelson bimodule</u>

$$BS(\underline{w}) := B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \simeq R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_k}} R(k).$$

($BS(\phi) = R$, regular bimodule, ϕ = empty expression)

<u>Def</u> $\mathrm{SBim}_{BS}(\mathfrak{h}, W)$ = full subcategory of <u>ungraded</u> R-bimodules consisting of Bott-Samelson bimodules (monoidal, not additive)

The category of <u>Soergel bimodules</u> is

$$\mathrm{SBim}(\mathfrak{h}, W) := \text{graded Karoubi envelope of (additive envelope of) } \mathrm{SBim}_{BS}(\mathfrak{h}, W)$$

$$= \langle B_s \mid s \in S \rangle_{\otimes_R, \oplus, (1), \overset{\oplus}{\subset}, \simeq} \subset R\text{-gmod-}R$$

i.e. smallest strictly full subcategory R-gmod-R and closed under finite $\otimes_R$, finite $\oplus$, (1), and direct summand $\overset{\oplus}{\subset}$.

<u>Ex</u> (Cartan realization of SL_2) $W = S_2 = \{id, s\}$

$$\mathfrak{h}^{\Bbbk}_{SL_2} = (V = \Bbbk\alpha_s^\vee, \{\alpha_s^\vee\}, \{\alpha_s\}), \quad \mathrm{char}\,\Bbbk \neq 2$$

(for Demazure surjectivity)

$R = \Bbbk[\alpha_s]$, $\deg(\alpha_s) = 2$. $\quad s(\alpha_s) = -\alpha_s$

$R^s = \Bbbk[\alpha_s^2]$, $\quad R = R^s \oplus R^s \cdot \alpha_s$ as R^s-bimodule.

Key computation: $\quad$ ($\deg \alpha_s = 2$)

$$B_s \underset{R}{\otimes} B_s = R \underset{R^s}{\otimes} R \underset{R^s}{\otimes} R(2) \simeq R \underset{R^s}{\otimes} (R^s \oplus R^s(-2)) \underset{R^s}{\otimes} R(2) \simeq B_s(-1) \oplus B_s(1).$$

$$\Longrightarrow \quad \{\text{indecomp. Soergel bimodules}\}/(1), \simeq \quad \xleftrightarrow{1:1} \quad S_2$$

$$B_{id} := R, \quad B_s$$

and $[B_s]^2 = (v + v^{-1})[B_s]$ in split Grothendieck group... quadratic rel.

$$\Longrightarrow \quad [\mathrm{SBim}(\mathfrak{h}^{\Bbbk}_{sl_2}, S_2)]_{\oplus} \simeq H(S_2) \quad (\text{as } \mathbb{Z}[v, v^{-1}]\text{-algebra})$$

$$[R] \longleftrightarrow b_{id}$$

$$[B_s] \longleftrightarrow b_s,$$

i.e. $\mathrm{SBim}(\mathfrak{h}^{\Bbbk}_{sl_2}, S_2)$ categorifies $H(S_2)$.

<u>Ex</u> (Cartan realization of SL_3)

$W = S_3$, $\quad S = \{s, t\}$

$B_{st} = B_s \underset{R}{\otimes} B_t$ indecomp.

$B_{st} \underset{R}{\otimes} B_s = B_{w_0} \oplus B_s \qquad B_{ts} \underset{R}{\otimes} B_t = B_{w_0} \oplus B_t$

new indecomposable direct summand, isomorphic

This categorifies computation of KL basis from Day 1.

§ KL basis and p-canonical basis

<u>Def</u> Let $\Bbbk$ be infinite field, char $\Bbbk \neq 2$. A realization $(\mathfrak{h}, W)$ over $\Bbbk$ is <u>reflection faithful</u> if for all $w \in W$,

w is a reflection in W $\Longleftrightarrow$ w acts as a reflection on $\mathfrak{h}$,
(i.e. conjugate of simple reflection) $\qquad$ (i.e. $\mathrm{codim}(V^w \subset V) = 1$.)

and is faithful as a W-representation.

For any Coxeter system, Soergel defined a reflection faithful "Kac-Moody" realization over $\mathbb{R}$. ("geometric representation" for finite Coxeter)

Thm (Soergel categorification theorem)
If $(\mathfrak{h}, W)$ is reflection faithful, then $\mathrm{SBim}(\mathfrak{h}, W)$ categorifies $H(W)$.

i.e. $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism

$$H(W) \xrightarrow{\sim} [\mathrm{SBim}(\mathfrak{h}, W)]_{\oplus}$$
$$b_s \longmapsto [B_s], \quad s \in S.$$

Let ch be inverse isom.

Thm ("Soergel's conjecture" Soergel for Weyl groups, Elias–Williamson for arbitrary Coxeter system)
For the Kac-Moody realization, $ch(B_w) = b_w \quad \forall w \in W$.

$\Longrightarrow$ positivity of KL polynomials and structure constants for KL basis.

"Def" Consider the Hecke category associated to a Cartan realization over a field of characteristic $p \geq 0$. Then

$$\{{}^p b_w := ch(B_w)\}_{w \in W}$$

is called the p-canonical basis (or p-KL basis). It only depends on the root system and p. Then p-KL polynomials ${}^p h_{x,w}$ (in general only Laurent polynomials) are defined by

$${}^p b_w = \sum_{x \in W} {}^p h_{x,w} \delta_x.$$

Rmk

(1) Soergel's conjecture $\Longrightarrow$ ${}^0 b_w = b_w \quad \forall w \in W$.

(2) Whereas b_w only depends on W, ${}^p b_w$ can differ in type B_n vs. C_n.

(3) Using geometric Hecke category, p-KL polynomial is Poincaré polynomial of stalk of indecomposable $\Bbbk$-parity complex, char $\Bbbk = p$. (Day 3)

§ Antispherical module

For characters of tilting modules, we need antispherical p-KL polynomials.

Let $G/\Bbbk$ as in Simon's Day 1, char $\Bbbk = p > h$

$\rightsquigarrow$ Coxeter systems $(W_{\mathrm{aff}} = \mathbb{Z}R \rtimes W, S_{\mathrm{aff}})$, (W, S)

Hecke algebras $H_{aff} \supset H$

Let ${}^f W_{aff} := \{ w \in W_{aff} \mid w \text{ minimal in } Ww \}$.

Def The antispherical (right) module of H_{aff} is
$$M^{asph} := \mathrm{sgn} \underset{H}{\otimes} H_{aff},$$
where $\mathrm{sgn} = \mathbb{Z}[v, v^{-1}]$ is sign representation (δ_s acts by $-v$ $\forall s \in S$)
(Recall that $(\delta_s + v)(\delta_s - v^{-1}) = 0$)

M^{asph} again has two bases:

$\{ n_w := 1 \otimes \delta_w \}_{w \in {}^f W_{aff}}$ standard basis

$\{ \underline{n}_w \}_{w \in {}^f W_{aff}}$ KL basis (again defined using an involution)

Fact Under H_{aff}-module map
$$\varphi : H_{aff} \twoheadrightarrow M^{asph} : h \longmapsto 1 \otimes h,$$
we have
$$\varphi(b_w) = \begin{cases} \underline{n}_w & \text{if } w \in {}^f W_{aff}, \\ 0 & \text{otherwise}. \end{cases}$$

M^{asph} admits categorification, as follows.
First, let
$$\mathfrak{h} := (V, \{\alpha_s\}_{s \in S_{aff}}, \{\alpha_s^\vee\}_{s \in S_{aff}}),$$
where

- $V := \mathbb{k} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}R$,
- for $s \in S$, α_s and $\alpha_s^\vee$ are (images of) roots and coroots,
- if $s \in S_{aff} \setminus S$, let $\gamma \in R^+$ be the unique positive root s.t.
$$W_{aff} = W \ltimes \mathbb{Z}R \longrightarrow W$$
$$s \longmapsto s_\gamma,$$
and set α_s to be image of $-\gamma^\vee$, $\alpha_s^\vee$ to be image of $-\gamma$.

Then $\mathfrak{h}$ is a realization of W_{aff} over $\mathbb{k}$. (This uses $p > h$, see [Riche-Williamson, §4.2])

Let $\mathcal{D} = \mathcal{D}(\mathfrak{h}, W_{aff})$, Hecke category associated to $(\mathfrak{h}, W_{aff})$.

Rmk $\mathfrak{h}$ is a realization related to the Langlands dual of G.

Geometrically, $\mathcal{D}$ is Iwahori-equivariant $\Bbbk$-parity complexes on affine flag variety of the complex reductive group $G^\vee$ (semisimple simply-connected) Langlands dual to G.

We can imitate earlier Fact to define categorification of M^{asph}:

Def Define the right $\mathcal{D}$-module category

$$\mathcal{D}^{asph} := \mathcal{D} \Big/ \langle B_w(n) : w \notin {}^fW_{aff},\ n \in \mathbb{Z} \rangle_{\oplus}$$

additive quotient.

Fact $\mathcal{D}^{asph}$ categorifies M^{asph}, i.e.

$$\{\text{indecomp. objects in } \mathcal{D}^{asph}\}/\simeq \xleftrightarrow{1:1} {}^fW_{aff} \times \mathbb{Z}$$

$$\overline{B_w}(n) \longleftrightarrow (w, n),$$

where $\overline{B_w}$ = Image of B_w in $\mathcal{D}^{asph}$

$\Rightarrow$ $ch: [\mathcal{D}^{asph}]_{\oplus} \simeq M^{asph}$ as right $[\mathcal{D}]_{\oplus} \simeq H_{aff}$-modules.

Def Antispherical p-canonical basis

$$\{ {}^p\underline{n}_w := ch(\overline{B_w}) = 1 \otimes {}^p b_w \}_{w \in {}^fW_{aff}},$$

antispherical p-KL polynomial ${}^p n_{y,w}$ defined by

$${}^p\underline{n}_w = \sum_{y \in {}^fW_{aff}} {}^p n_{y,w}\, n_y$$

$$\Rightarrow {}^p n_{y,w} = \sum_{x \in W} (-v)^{l(x)}\, n_{xy,w}.$$

Rmk geometrically, stalk of Iwahori-Whittaker parity sheaves on affine flag variety.

§ diagrammatic Hecke category : intro

Presentation for an algebra:

$\mathbb{Z}[v,v^{-1}]\langle b_s\rangle$ = unital associative algebra freely generated under multiplication by
- element b_s

$$\begin{tikzcd}
\mathbb{Z}[v,v^{-1}]\langle b_s\rangle \arrow[d, two heads] \arrow[dr, "\text{surjection}"] & \\
\mathbb{Z}[v,v^{-1}]\langle b_s\rangle \big/ \substack{\text{relations} \\ (b_s^2 = (v+v^{-1})\, b_s)} \arrow[r, "\sim"] & H(S_2)
\end{tikzcd}$$

Presentation for a monoidal category:

$D^{\mathrm{free}}_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2)$ = $\Bbbk$-linear monoidal category freely generated under composition and $\otimes$ by
- object B_s
- morphisms (NEW)

$$\begin{tikzcd}
D^{\mathrm{free}}_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2) \arrow[d] \arrow[dr, "\substack{\text{essentially surjective} \\ +\ \text{full}}"] & \\
D_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2) := D^{\mathrm{free}}_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2) \big/ \substack{\text{relations} \\ \text{on morphisms}} \arrow[r, "\sim"] & SBim_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2)
\end{tikzcd}$$

($\Rightarrow$ relations on objects, e.g. $B_s \otimes_R B_s \simeq B_s(1) \oplus B_s(-1)$)

Then define

$$D(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2) := Kar\left(D^{\oplus}_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2)\right)$$

(graded Karoubi envelope of additive envelope)

Advantages of (diagrammatic) presentation:

(1) simplifies computations

(2) can define monoidal functor

$$D_{BS}(\mathfrak{h}, W) \longrightarrow \mathcal{C} \text{ monoidal}$$

by generators and relations: specify images of generating objects and morphisms, and check that these images satisfy the relations.