Unlike Hecke algebra, Hecke category depends on more than Coxeter system (W, S) Coxeter system, \( k \) field (for simplicity)

**Def [Elias-Williamson]**

A realization of \((W, S)\) over \( k \) is a triple

\[ h = (V, \{\alpha_s\}_{s \in S} \subseteq V, \{\tilde{\alpha}_s\}_{s \in S} \subseteq V^*) \]

where \( V \) is finite-diml \( k \)-vector space, \( V^* = \text{Hom}_k(V, k) \), equipped with simple roots \( \alpha_s \) and simple coroots \( \alpha^*_s \), satisfying:

1. \( \langle \alpha_s, \alpha^*_s \rangle = 2 \quad \forall s \in S \)
2. \( s \mapsto (v \mapsto v - \alpha_s(v) \alpha^*_s) \) defines representation of \( W \) on \( V \)
3. technical condition (involving 2-quantum numbers \( \forall s, t \in S, s \neq t, m_{st} < \infty \))

More technical conditions: assume realization is balanced and satisfies Demazure surjectivity.

**Rmk** These technical conditions will be satisfied for realizations we care about, assuming char \( k \) > \( h \) (Coxeter number)

Let \((h, W)\) be a realization over \( k \) (i.e. \( h \) is a realization over \( k \) of Coxeter system \((W, S)\)). The associated Hecke category is a certain \( k \)-linear additive monoidal category with grading shift that categorifies \( H(W) \).

Two incarnations:

1. **geometric** For "Cartan realizations" of crystallographic Coxeter systems (i.e. \( m_{st} \in \{2, 3, 4, 6, \infty\} \) \( \forall s, t \in S, s \neq t \)), e.g. realizations arising as root datum of reductive group, can consider Borel-equivariant \( k \)-parity complexes on (Kac-Moody) flag variety

(see Day 3)

(2) **diagrammatic** Elias-Williamson diagrammatic Hecke category

(later today, continued on Day 3)
Soergel bimodules

Concrete algebraic model of Hecke category, but not always "correct" (doesn't always categorify \( H(W) \)). Only as motivation.

Let \((\mathfrak{h}, W)\) be realization over \(k\).

\[ R = \text{Sym} \left( V^* \right) \text{ viewed as } (\mathbb{Z})\text{-graded algebra with deg } V^* = 2. \]

\( R\text{-gmod-}R = \text{category of } (\mathbb{Z})\text{-graded } R\text{-bimodules and graded (degree 0) } R\text{-bimodule homomorphisms.} \]

Additive monoid \( (\otimes_R) \) \( \mathbb{K}\)-linear with grading shift \((1)\),

\[(M(1))_i = M_{i+1} \text{ for } M = \bigoplus_{i \in \mathbb{Z}} M_i, \text{ so can categorify a } \mathbb{Z}[v, v^{-1}]\text{-algebra.} \]

\[ W \circ V \rightarrow W \circ R, \quad R^s := s\text{-invariants in } R, \quad s \in S \]
define graded \( R\)-bimodule \( B_s := R \otimes_R (1), \quad s \in S \).

**Def.** For expression \( w = (s_1, \ldots, s_k) \), \underline{Bott--Samelson bimodule}

\[ BS(w) := B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \rightarrow R \otimes_R \cdots \otimes_R R \text{ (k).} \]

\[ (BS(\emptyset) = R, \text{ regular bimodule, } \emptyset = \text{empty expression}) \]

**Def.** \( SBim_{BS}(\mathfrak{h}, W) = \text{full subcategory of ungraded } R\text{-bimodules} \]

consisting of Bott--Samelson bimodules

(monoidal not additive)

The category of \underline{Soergel bimodules} is

\[ SBim(\mathfrak{h}, W) := \text{graded Karoubi envelope of } SBim_{BS}(\mathfrak{h}, W) \]

\[ = \langle B_s \mid s \in S \rangle \otimes_R, \oplus, (1), \ominus, = \subset R\text{-gmod-}R \]

i.e. smallest strictly full subcategory \( R\text{-gmod-}R \) and closed under

finite direct \( R, \otimes, (1), \) and direct summands \( \oplus \).

**Ex.** (Cartan realization of \( SL_2 \)) \( W = S_2 = \langle \text{sid, } s \rangle \)

\[ \mathfrak{h}_{SL_2} = (V = \mathfrak{h} \alpha_s, \{ \alpha_s \}, \{ \alpha_s \}), \quad \text{char } k \neq 2. \]

(for Demazure surjectivity)
\[ R = \text{lk}[\alpha_s], \quad \deg(\alpha_s) = 2. \quad s(\alpha_s) = -\alpha_s \]
\[ R^s = \text{lk}[\alpha_s^2], \quad R = R^s \oplus R^s \alpha_t \quad \alpha_s \quad R^s \text{-bimodule}. \]

Key computation:
\[
\begin{align*}
B_s \otimes_R B_t &= R_R \otimes_R (R \oplus R(-2)) \otimes_R R \cong B_s \oplus B_t \oplus B_s \oplus B_t \oplus B_s \oplus B_t \\
&\cong \{ \text{indecomp. Soergel bimodules} \} / (1) \xrightarrow{1:1} S_2
\end{align*}
\]

and \([B_s]^2 = (v + v^{-1})[B_s]\) in split Grothendieck group... quadratic rel.

\[
[S\text{Bim}(\mathfrak{h}_{s,t}, S_2)] \xrightarrow{\phi} H(S_2) \quad (\text{as } \mathbb{Z}[v, v^{-1}]\text{-algebra})
\]

\[
\begin{array}{c}
[R] \leftrightarrow b_{ik} \\
[B_s] \leftrightarrow b_s
\end{array}
\]

i.e. \(S\text{Bim}(\mathfrak{h}_{s,t}, S_2)\) categorifies \(H(S_2)\).

Ex: (Cartan realization of \(SL_3\))
\[
W = S_3, \quad S = \{s, t\}
\]
\[
B_{st} = B_s \otimes_R B_t \text{ indecomp.}
\]
\[
B_{st} \otimes_R B_s = B_{w_0} \oplus B_s, \quad B_{ts} \otimes_R B_t = B_{w_0} \oplus B_t
\]

\(\text{new indecomposable direct summand, isomorphic}\)

This categorifies computation of KL basis from Day 1.

\textbf{§ KL basis and p-canonical basis}

Def Let \(k\) be infinite field, \(\text{char } k \neq 2\). A realization \((\mathfrak{g}, W)\) over \(k\) is reflection faithful if for all \(w \in W\),
\[
\begin{align*}
w &\text{ is a reflection in } W \iff w \text{ acts as a reflection on } \mathfrak{h}, \\
&\text{(i.e. conjugate of simple reflection)} \quad \text{(i.e. codim}(V^w \subset V) = 1). \\
\end{align*}
\]

and \(w\) is faithful as a \(W\)-representation.

For any Coxeter system, Soergel defined a reflection faithful "Kac-Moody" realization over \(R\) ("geometric representation" for finite Coxeter).
Thm (Soergel categorification theorem)
If \((\mathfrak{g}, W)\) is reflection faithful, then \(SBim(\mathfrak{g}, W)\) categorifies \(H(W)\).

\[
H(W) \cong [SBim(\mathfrak{g}, W)]_{\otimes}
\]

\[
b_{s} \mapsto [B_{s}], \quad s \in S.
\]

Let \(ch\) be inverse isom.

Thm ("Soergel's conjecture," Soergel for Weyl groups, Elias-Williamson for arbitrary Coxeter system)
For the Kac-Moody realization,
\[
ch(B_{w}) = b_{w} \quad \forall \ w \in W.
\]

\[\Rightarrow\] positivity of KL polynomials and structure constants for KL basis.

Def Consider the Hecke category associated to a Cartan realization
over a field of characteristic \(p \geq 0\). Then
\[
\{p_{b_{w}} := ch(B_{w})\}_{w \in W}
\]
is called the \(p\)-canonical basis (or \(p\)-KL basis). It only depends
on the root system and \(p\). Then \(p\)-KL polynomials \(p_{b_{w}}\) (in general
only Laurent polynomials) are defined by
\[
p_{b_{w}} = \sum_{x \in W} p_{h_{x \cdot w}} s_{x}.
\]

Rem
(1) Soergel's conjecture \(\Rightarrow\) \(p_{b_{w}} = b_{w} \quad \forall \ w \in W\).
(2) Whereas \(b_{w}\) only depends on \(W\), \(p_{b_{w}}\) can differ in
    type \(B_n\) vs. \(C_n\).
(3) Using geometric Hecke category, \(p\)-KL polynomial is Poincaré
    polynomial of stalk of indecomposable \(k\)-parity complex, \(char k = p\).
(Day 3)

\[\text{\textcolor{red}{\textbf{8. Antispherical module}}}
\]
For characters of tilting modules, we need antispherical \(p\)-KL polynomials.

Let \(G/\mathfrak{m}\) as in Simon's Day 1, \(char k = p > h\)
\[\Rightarrow\] Coxeter systems \((W_{aff} = \mathbb{Z} R \rtimes W, S_{aff})\), \((W, S)\).
Hecke algebras $H_{\text{aff}} \supset H$

Let $W_{\text{aff}} := \{ w \in W_{\text{aff}} | w \text{ minimal in } W_{\text{aff}} \}$.

**Definition:** The antispherical (right) module of $H_{\text{aff}}$ is

$M_{\text{aff}} := \text{sign} \otimes H_{\text{aff}}$

where $\text{sign} = \mathbb{Z}[v, v^{-1}]$ is sign representation ($S_\infty$ acts by $-v \forall s \in S$).

(Recall that $(S_\infty + v)(S_\infty - v^{-1}) = 0$)

$M_{\text{aff}}$ again has two bases:

- $\{ n_w := 1 \otimes s_w \}$ we $W_{\text{aff}}$
- $\{ n_w \}$ we $W_{\text{aff}}$

**Fact:** Under $H_{\text{aff}}$-module map

$q : H_{\text{aff}} \longrightarrow M_{\text{aff}} : h \longmapsto 1 \otimes h$,

we have

$q(b_w) = \begin{cases} n_w & \text{if } w \text{ we } W_{\text{aff}}, \\ 0 & \text{otherwise} \end{cases}$

$M_{\text{aff}}$ admits categorification, as follows.

First, let

$b := (V, \{ s \}_{s \in S_{\text{aff}}}, \{ s \}_{s \in S_{\text{aff}}})$

where

$V := \mathbb{A} \otimes \mathbb{Z} R$

for $s \in S$, $\alpha_s$ and $\check{\alpha}_s$ are (images of) roots and coroots,

If $s \in S_{\text{aff}} \setminus S$, let $Y \in \mathbb{R}^+$ be the unique positive root st.

$W_{\text{aff}} = W \times \mathbb{Z} R \longrightarrow W$

$s \longmapsto s_Y$

and set $\alpha_s$ to be image of $-Y$, $\check{\alpha}_s$ to be image of $-Y$.

Then $b$ is a realization of $W_{\text{aff}}$ over $\mathbb{A}$. (This uses $p > h$, see [Riche-Williamson, §4.2]).

Let $D = D(b, W_{\text{aff}})$, Hecke category associated to $(b, W_{\text{aff}})$. 
Rank 1 is a realization related to the Langlands dual of G.

Geometrically, \( D \) is Iwahori-equivariant \( k \)-parity complexes on affine flag variety of the complex reductive group \( G \).

(Semitsimple simply-connected) Langlands dual to \( G \).

We can initiate新一轮 to define categorification of \( \text{Morph} \).

**Def** Define the right \( D \)-module category

\[ \text{D}^{\text{Morph}} := D / \langle B_w(n) : n \in \text{W}^{\text{aff}}, n \in \mathbb{Z} \rangle. \]

**Fact** \( \text{D}^{\text{Morph}} \) categorifies \( \text{Morph} \), i.e.

\[ \text{Indecomposable objects in } \text{D}^{\text{Morph}} \]

\[ \overset{\sim}{\rightarrow} \text{W}^{\text{aff}} \times \mathbb{Z} \]

where \( B_w \) = image of \( B_w \) in \( \text{D}^{\text{Morph}} \)

\[ \Rightarrow \text{ch}[\text{D}^{\text{Morph}}] \rightarrow \text{Morph} \text{ as right } [D] \rightarrow \text{Haff-modules}. \]

**Def** Antispherical \( p \)-canonical basis

\[ \{ p^n_w := \text{ch}(B_w) = 1 \otimes p^n_w \}_{n \in \text{W}^{\text{aff}}}, \]

antispherical \( p \)-KL polynomial \( p^n_{y,w} \) defined by

\[ p^n_w = \sum_{y \in \text{W}^{\text{aff}}} p^n_{y,w} y. \]

\[ \Rightarrow p^n_{y,w} = \sum_{x \in \text{W}} (-1)^{y(x)} n_{xy,w}. \]

Rank geometrically, stalk of Iwahori-Whittaker parity sheaves on affine flag variety.
Diagrammatic Hecke category: intro

Presentation for an algebra:

\[ \mathbb{Z}[v, v^{-1}] \langle b_s \rangle = \text{unital associative algebra freely generated under multiplication by element } b_s \]

\[ \mathbb{Z}[v, v^{-1}] \langle b_s \rangle / \text{relations} \]

\[ b_s^2 = (v + v^{-1}) b_s \]

\[ \sim \rightarrow H(S_2) \]

Presentation for a monoidal category:

\[ D_{BS} \left( \mathfrak{h}^k_{SL_2}, S_2 \right) = \text{monoidal category freely generated h-linear under composition and } \otimes \text{ by object } B_s \text{ morphisms (NEW)} \]

\[ D_{BS}(\mathfrak{h}^k_{SL_2}, S_2) : = D_{BS}^{free}(\mathfrak{h}^k_{SL_2}, S_2) / \text{relations on morphisms} \]

\[ \Rightarrow \text{relations on objects, e.g. } B_s \otimes B_s \sim B_s (1) \otimes B_s (-1) \]

Then define

\[ D(\mathfrak{h}^k_{SL_2}, S_2) := \text{Kar}_{\otimes} (D_{BS}^{\otimes}(\mathfrak{h}^k_{SL_2}, S_2)) \]

graded Karoubi envelope of additive envelope

Advantages of (diagrammatic) presentation:

1. Simplifies computations
2. Can define monoidal functor

\[ D_{BS}(\mathfrak{h}, W) \rightarrow C \text{ monoidal by generators and relations: specify images of generating objects and morphisms, and check that these images satisfy the relations.} \]