Day 3

§ One-color diagrammatics

([Elias–Khovanov] did GL_n)

Recall Cartan realization $\mathfrak{h}^{\Bbbk}_{SL_2}$ for SL_2, $\operatorname{char} \Bbbk \neq 2$.

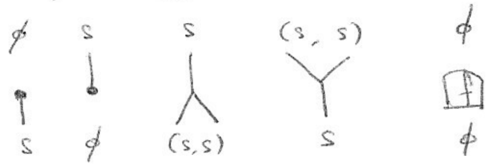
$W = S_2 = \{id, s\}$, $R = \Bbbk[\alpha_s]$, $s(\alpha_s) = -\alpha_s$.

The diagrammatic Bott-Samelson category $\mathcal{D}_{BS}(\mathfrak{h}^{\Bbbk}_{SL_2}, S_2)$ is the strict $\Bbbk$-linear monoidal category defined as follows:

generating object : s

(objects : expressions, i.e. ϕ, s, (s,s), (s,s,s), ...)

generating morphisms (read bottom to top)

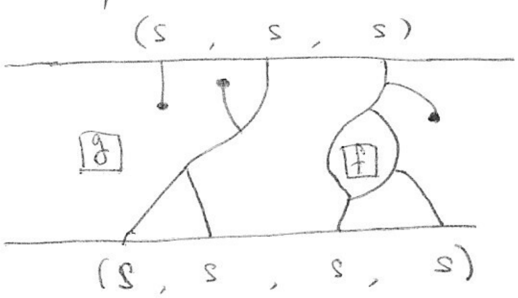


($f \in R$ homogeneous)

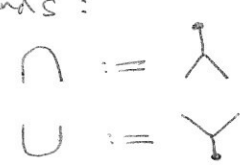
degree 1 1 −1 −1 $\deg f$

i.e. a morphism $\underline{w} \longrightarrow \underline{v}$ is an $\Bbbk$-linear combination of isotopy classes of diagrams in $\mathbb{R} \times [0,1]$ with bottom boundary $\underline{w}$, top boundary $\underline{v}$, and made up of local pieces as above.

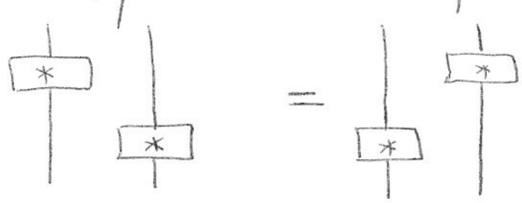
Ex



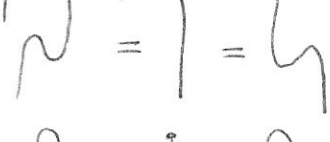
Shorthands :



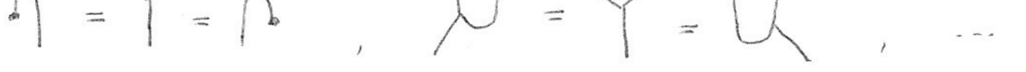
Rmk Rectilinear isotopy holds in any monoidal category (exchange law):



Isotopy in particular includes "cyclicity" under fixed biadjunctions :



(s is biadjoint to itself)

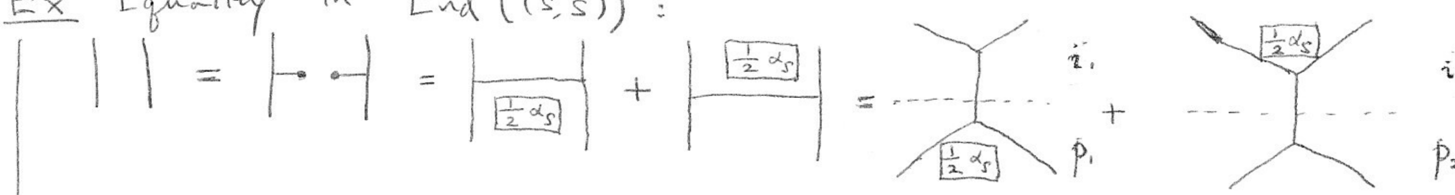


Monoidal structure is horizontal concatenation.

Composition is vertical concatenation.

These morphisms are subject to the <u>one-color relations</u> (see [Jensen–Williamson, §2.3.1])

<u>Ex</u> Equality in $\mathrm{End}((s,s))$:



$\Rightarrow B_{(s,s)} \simeq B_s(1) \oplus B_s(-1)$ in additive envelope $\mathcal{D}_{BS}^{\oplus}(\mathfrak{h}_{SL_2}^{\Bbbk}, S_2)$

because

$$\begin{tikzcd} B_s(1) & & B_s(-1) \\ & B_{(s,s)} \arrow[ul, "p_1"] \arrow[ur, "p_2"'] & \\ B_s(1) \arrow[ur, "i_1"] & & B_s(-1) \arrow[ul, "i_2"'] \end{tikzcd}$$

satisfying
$$\left\{\begin{array}{l} \mathrm{id}_{B_{(s,s)}} = i_1 \circ p_1 + i_2 \circ p_2 \\ p_1 \circ i_1 = \mathrm{id}, \quad p_2 \circ i_2 = \mathrm{id} \\ p_2 \circ i_1 = 0, \quad p_1 \circ i_2 = 0 \end{array}\right. \Bigg) \text{check!}$$

Relations are homogeneous, so Hom spaces are ($\mathbb{Z}$-)graded.

Functor to bimodules
$$F : \mathcal{D}_{BS}(\mathfrak{h}_{SL_2}^{\Bbbk}, S_2) \longrightarrow \mathrm{SBim}_{BS}(\mathfrak{h}_{SL_2}^{\Bbbk}, S_2)$$
defined by generators and relations:

$s \longmapsto B_s$

 $\longmapsto \begin{array}{c} R \\ \uparrow \text{mult.} \\ B_s \end{array}$ $\quad \begin{array}{c} fg \\ \uparrow \\ f \otimes g \end{array}$

 $\longmapsto \begin{array}{c} B_s \\ \uparrow \text{comult.} \\ R \end{array}$ $\quad \begin{array}{c} \frac{1}{2}\alpha_s \otimes 1 + 1 \otimes \frac{1}{2}\alpha_s \\ \uparrow \\ 1 \end{array}$

 $\longmapsto \begin{array}{c} B_s \\ \uparrow \\ B_s \otimes_R B_s \end{array}$ $\quad \begin{array}{c} \partial_s(f) \otimes 1 = 1 \otimes \partial_s(f) \\ \uparrow \\ 1 \otimes f \otimes 1 \end{array}$

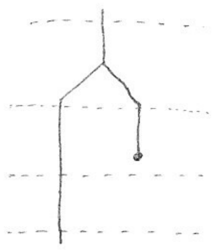
 $\longmapsto \begin{array}{c} B_s \otimes_R B_s \\ \uparrow \\ B_s \end{array}$ $\quad \begin{array}{c} 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \end{array}$

$\boxed{f} \longmapsto \begin{array}{c} R \\ \uparrow \\ R \end{array}$ mult. by f.

"divided difference" $\partial_s : R \longrightarrow R(-2)$, $\quad \partial_s(f) = \dfrac{f - s(f)}{\alpha_s}$.

Must check that these images satisfy the relations (including isotopy).

Ex $F(\lambda) \overset{?}{=} F(|) = \mathrm{id}_{B_s}$



$$\xrightarrow{F} \quad \begin{array}{c} B_s \\ \uparrow \\ B_s \otimes_R B_s \\ \uparrow \quad \text{id} \otimes \text{comult.} \\ B_s \otimes_R R \\ \wr\!| \\ B_s \end{array} \qquad\qquad \begin{array}{c} 1\otimes 1 \\ \uparrow \\ 1\otimes \tfrac{1}{2}\alpha_s \otimes 1 + 1\otimes 1\otimes \tfrac{1}{2}\alpha_s \\ \uparrow \\ 1\otimes 1\otimes 1 \\ \uparrow \\ 1\otimes 1 \end{array}$$

(use $\partial_s(\frac{1}{2}\alpha_s) = 1$)

Can calculate graded dimension of Hom spaces to show F is equivalence.

§ Elias–Williamson diagrammatic category

Let $(\mathfrak{h}, W)$ be a realization. [Elias–Williamson], building on [Elias–Khovanov, Elias]

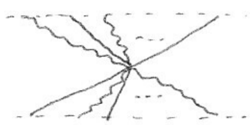
Def $\mathcal{D}_{BS}(\mathfrak{h}, W)$ is the strict $\Bbbk$-linear monoidal category defined by the following diagrammatic presentation.

generating objects: B_s, $s \in S$

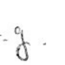
($\Rightarrow$ objects: $B_{\underline{w}}$, $\underline{w}$ expression)

generating morphisms:

- polynomial boxes $\boxed{f}$, $f \in R = \mathrm{Sym}(V^*)$ homogeneous
- one-color generating morphisms $\forall\ s \in S$
- $\forall\ s,t \in S$, $s \neq t$, $m_{st} < \infty$, $2m_{st}$-valent vertex



$$: B_{\underbrace{(s,t,\dots)}_{m_{st}}} \longrightarrow B_{\underbrace{(t,s,\dots)}_{m_{st}}}, \quad \text{degree } 0.$$

(e.g.  if $m_{st} = 2$.)

Rmk Let $w = \underbrace{st\cdots}_{m_{st}} = \underbrace{ts\cdots}_{m_{st}}$. The $2m_{st}$-valent vertex corresponds to R-bimodule map

$$B_{(s,t,\dots)} = B_w \oplus (\text{lower}) \twoheadrightarrow B_w \hookrightarrow B_w \oplus (\text{lower}) = B_{(t,s,\dots)}.$$

The morphisms are subject to the following (local, homogeneous) relations:

- one-color relations $\forall\ s \in S$
- two-color relations $\forall\ s,t \in S$, $s \neq t$, $m_{st} < \infty$. (see [Jensen–Williamson, §2.3.2])
- three-color relations $\forall$ finitary $\{s,t,u\} \subset S$ of order 3 ([JW])

  Zamolodchikov relation for every parabolic A_3, B_3, $A_1 \times I_2(m)$, H_3

<u>Def</u> $\mathcal{D}(\mathfrak{h}, W) := \mathrm{Kar}(\mathcal{D}_{BS}^{\oplus}(\mathfrak{h}, W))$.

These categories are called <u>Elias–Williamson diagrammatic category</u> or simply <u>diagrammatic Hecke category</u> associated to $(\mathfrak{h}, W)$.

<u>Thm</u> (Elias–Williamson)

(1) For each $w \in W$, $\exists$ indecomposable $B_w \in \mathcal{D}(\mathfrak{h}, W)$ characterized as follows. For any reduced expression $\underline{w}$ for w,
$$B_w \overset{\oplus}{\subset} B_{\underline{w}},$$
and B_w does not appear as direct summand of $B_{\underline{v}}$ for any expression $\underline{v}$ with $\ell(\underline{v}) \le \ell(w)$.

(2) $\mathcal{D}(\mathfrak{h}, W)$ categorifies $H(W)$

(3) If $(\mathfrak{h}, W)$ is reflection faithful, then $\exists$ monoidal equivalence
$$\mathcal{D}_{BS}(\mathfrak{h}, W) \xrightarrow{\sim} \mathrm{SBim}_{BS}(\mathfrak{h}, W)$$
and
$$\mathcal{D}(\mathfrak{h}, W) \longrightarrow \mathrm{SBim}(\mathfrak{h}, W).$$

§ Parity sheaves [Juteau–Mautner–Williamson]

X = complex algebraic variety (classical topology), $\Bbbk$ = field (for simplicity)

$X = \bigsqcup_{\lambda \in \mathcal{S}} X_\lambda$ stratification $\mathcal{S}$ into affine spaces

<u>Def</u> The category $D^b_{\mathcal{S}}(X, \Bbbk)$ of <u>$\mathcal{S}$-constructible complexes of $\Bbbk$-vector spaces</u> is the full subcategory of $D^b(X, \Bbbk) := D^b\mathrm{Sh}(X, \Bbbk)$ consisting of complexes $\mathcal{F}$ s.t. $\mathcal{H}^i(\mathcal{F})$ is $\mathcal{S}$-constructible for all $i \in \mathbb{Z}$, i.e.
$$\mathcal{H}^i(\mathcal{F})|_{X_\lambda} \text{ is a local system } \forall \lambda \in \mathcal{S}.$$

Let $j_\lambda : X_\lambda \hookrightarrow X$ be inclusion.

<u>Def</u> ([JMW]) $\mathcal{F} \in D^b_{\mathcal{S}}(X, \Bbbk)$ is <u>even</u> if $\mathcal{H}^i(j_\lambda^* \mathcal{F}) = \mathcal{H}^i(j_\lambda^! \mathcal{F}) = 0$ for i odd, all $\lambda \in \mathcal{S}$. It is <u>odd</u> if $\mathcal{F}[1]$ is even, <u>parity</u> if it is of form even$\oplus$odd.

<u>Thm</u> ([JMW]) Let $\mathcal{F}$ be an indecomposable parity complex. Then
$$\mathrm{supp}\,\mathcal{F} = \overline{X_\lambda}, \text{ some } \lambda \in \mathcal{S}, \quad \mathcal{F}|_{X_\lambda} \simeq \underline{\Bbbk}_{X_\lambda}[n], \text{ some } n \in \mathbb{Z}.$$
Any two indecomposable parity complexes with same support are isomorphic up to shift $\Rightarrow$ unique Verdier self-dual $\mathcal{E}_\lambda$, called <u>parity sheaf</u>.

May not exist in general; not unique up to unique isomorphism like IC.

<u>Def</u> Let $X = \bigsqcup_{\lambda \in \Lambda_X} X_\lambda$, $Y = \bigsqcup_{\mu \in \Lambda_Y} Y_\mu$ be stratified varieties.

A morphism $\pi : X \longrightarrow Y$ is <u>stratified</u> if
(1) for all $\mu \in \Lambda_Y$, $\pi^{-1}(Y_\mu)$ is a union of strata.
(2) for each $X_\lambda \subset \pi^{-1}(Y_\mu)$, the induced morphism $\pi|_{X_\lambda} : X_\lambda \longrightarrow Y_\mu$ is a submersion with smooth fibers.

A stratified morphism is <u>even</u> if cohomology of fiber is concentrated in even degrees.

<u>Fact</u> Direct image of a parity complex under a proper even morphism is parity.
This easy fact is a substitute for the decomposition theorem in this setting.

<u>Cor</u> If $\overline{X_\lambda}$ has an <u>even resolution</u> $\pi_\lambda : \widetilde{X}_\lambda \longrightarrow \overline{X_\lambda}$, then $\mathcal{E}_\lambda$ exists, and $\mathcal{E}_\lambda \overset{\oplus}{\subset} \pi_{\lambda *} \Bbbk_{\widetilde{X}_\lambda} [\dim \widetilde{X}_\lambda]$.

Analogous statements hold in the equivariant derived category of Bernstein–Lunts. The definition of parity remains the same, but assumption on stratification is replaced by equivariant analogue.

<u>Ex</u> $X = \mathbb{P}^1 = \underset{(X_s)}{\mathbb{A}^1} \sqcup \underset{(X_{id})}{pt}$, $\mathcal{E}_s = \Bbbk_X[1]$, $\mathcal{E}_{id} = \Bbbk_{pt}$.

§ Parity complexes on flag varieties

Let $A = (a_{st})_{s,t \in S}$ be a generalized Cartan matrix, $|S| < \infty$, i.e.
$a_{ss} = 2 \ \forall s \in S$, $a_{st} \leq 0$ if $s \neq t$, $a_{st} = 0 \Leftrightarrow a_{ts} = 0$.

Consider an <u>associated Kac–Moody root datum</u>

$$(\Lambda, \{\alpha_s^\vee\}_{s \in S}, \{\alpha_s\}_{s \in S}),$$

i.e. Λ is a free $\mathbb{Z}$-module of finite rank, $\alpha_s^\vee \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$, $\alpha_s \in \Lambda$, s.t. $a_{st} = \langle \alpha_s^\vee, \alpha_t \rangle \ \forall s,t \in S$.

$\rightsquigarrow$ Kac–Moody group $\mathcal{G}$, an ind group scheme$/\mathbb{C}$, with canonical Borel $\mathcal{B} \supset$ max torus T; Weyl group (W, S).

<u>Ex</u> A complex reductive group $G \supset$ Borel $B \supset$ max torus T $\rightsquigarrow$ root datum.
Corresponding Kac–Moody group recovers $G \supset B \supset T$.

KM flag variety has Bruhat decomposition:

$$X := \mathcal{G}/\mathcal{B} = \bigsqcup_{w \in W} X_w, \qquad X_w := \mathcal{B}w\mathcal{B}/\mathcal{B} \simeq \mathbb{C}^{\ell(w)}.$$

Consider $D^b_{\mathcal{B}}(X, \Bbbk)$, $\Bbbk$ field, monoidal under convolution product:

$$\mathcal{G} \times X \xrightarrow{q} \mathcal{G} \overset{\mathcal{B}}{\times} X \xrightarrow{\text{mult}} X$$

Given $\mathcal{E}, \mathcal{F} \in D^b_{\mathcal{B}}(X, \Bbbk)$,

$$\mathcal{E} * \mathcal{F} := \text{mult}_*(\mathcal{E} \,\tilde{\boxtimes}\, \mathcal{F}), \quad \text{where } \mathcal{E} \,\tilde{\boxtimes}\, \mathcal{F} \text{ is unique with } q^*(\mathcal{E} \,\tilde{\boxtimes}\, \mathcal{F}) \simeq \mathcal{E} \boxtimes \mathcal{F}.$$

For $s \in S$,

$$\mathcal{E}_s := \underline{\Bbbk}_{\mathcal{B}s\mathcal{B}/\mathcal{B}}[1] \in \mathrm{Parity}_{\mathcal{B}}(X, \Bbbk).$$

For expression $\underline{w} = (s_1, \dots, s_k)$, <u>Bott–Samelson parity complex</u>

$$\mathcal{E}_{\underline{w}} := \mathcal{E}_{s_1} * \cdots * \mathcal{E}_{s_k} \in \mathrm{Parity}_{\mathcal{B}}(X, \Bbbk)$$

(parity because $(-) * \mathcal{E}_s \simeq q_s^* q_{s*}[1]$, $q_s : X \to \mathcal{G}/\mathcal{P}^s$ even)

<u>Def</u> $\mathrm{Parity}^{BS}_{\mathcal{B}}(X, \Bbbk)$ is the full subcategory of $\mathrm{Parity}_{\mathcal{B}}(X, \Bbbk)$ consisting of $\mathcal{E}_{\underline{w}}[n]$, $\underline{w}$ expression, $n \in \mathbb{Z}$.

For every $w \in W$, parity sheaf $\mathcal{E}_w$ arises as direct summand of $\mathcal{E}_{\underline{w}}$, $\underline{w}$ reduced expression for w:

$$\text{supp}\, \mathcal{E}_{\underline{w}} = \overline{\mathcal{B}w\mathcal{B}/\mathcal{B}} = \overline{X_w}, \qquad \mathcal{E}_{\underline{w}}|_{X_w} \simeq \underline{\Bbbk}_{X_w}[\ell(w)],$$

so $\mathcal{E}_w$ is unique direct summand with full support. Hence

$$\mathrm{Kar}(\mathrm{Parity}^{BS,\oplus}_{\mathcal{B}}(X, \Bbbk)) \simeq \mathrm{Parity}_{\mathcal{B}}(X, \Bbbk).$$