One-color diagrammatics ([Elias-Khovanov] did $GL_n$)

Recall Cartan realization $V_{\mathfrak{sl}_2}$ for $\mathfrak{sl}_2$, char $k \neq 2$.

$W = S_2 = \{ \text{id}, s \}$, $R = k[\alpha_5]$, $s(\alpha_5) = -\alpha_5$.

The diagrammatic Bott-Samelson category $D_{BS}(V_{\mathfrak{sl}_2}, S_2)$ is the strict $k$-linear monoidal category defined as follows:

- **Generating object**: $S$
- **Generating morphisms** (read bottom to top)

  

  \[
  \begin{array}{cccc}
  \phi & s & s & (s, s) & \phi \\
  \uparrow & \downarrow & \downarrow & \uparrow & \uparrow \\
  s & \neq & (s, s) & s & \neq \\
  \downarrow & (s, s) & \downarrow & \phi & (\text{for homogeneous})
  \end{array}
  \]

- **Degree**: $1$ $1$ $-1$ $-1$ $-1$ $degf$

i.e. a morphism $\xrightarrow{w} \xrightarrow{x}$ is an $k$-linear combination of isotopy classes of diagrams in $R \times [0, 1]$ with bottom boundary $\xrightarrow{w}$, top boundary $\xrightarrow{x}$, and made up of local pieces as above.

**Ex**

\[
\begin{array}{c}
(\hat{s}, s, s, s) \\
\text{Shorthands:} \\
\wedge := \hat{s} \\
\wedge := Y
\end{array}
\]

**Rank** Rectilinear isotopy holds in any monoidal category (exchange law):

\[
\begin{array}{c}
\star \uparrow \star = \star \downarrow \star \\
\text{Isotopy in particular includes "cyclicity" under fixed biadjunctions:} \\
\wedge = \hat{s} = Y \\
\wedge = \hat{s} = Y, \ w = Y = \wedge
\end{array}
\]
Monoidal structure is horizontal concatenation.

Composition is vertical concatenation.

These morphisms are subject to the one-color relations
(see [Jensen–Williamson, §2.3.1])

\[ \begin{align*}
Ex. \quad \text{Equality in } \text{End}(s, s) : \quad & 1 = 1 \rightarrow 1 = \frac{1}{2} s_1 \rightarrow \frac{1}{2} s_{-1} + \frac{1}{2} s_1 \rightarrow \frac{1}{2} s_{-1}
\end{align*} \]

\[ \Rightarrow B_s(s, s) \cong B_s(1) \oplus B_s(-1) \quad \text{in additive envelope } D_{BS}^\oplus(\mathfrak{h}^k, S_2)
\]

because \( B_s(1) \quad \overset{p_1}{\rightarrow} \quad B_s(-1) \) satisfying
\[
\begin{align*}
\text{id}_{B_s(s, s)} &= i_1 \circ p_1 + i_2 \circ p_2 \\
p_1 \circ i_1 &= \text{id} \quad \overset{\text{check!}}{=} \quad p_2 \circ i_2 = \text{id}
\end{align*}
\]

Relations are homogeneous, so Hom spaces are (\( \mathbb{Z} \)-)graded.

Functor to bimodules

\[ F: D_{BS}(\mathfrak{h}^k, S_2) \rightarrow SBim_{BS}(\mathfrak{h}^k, S_2) \]

defined by generators and relations:

\[ s \quad \rightarrow \quad B_s \]

\[ \begin{array}{cc}
\uparrow & \downarrow \\
\rightarrow & \rightarrow \\
B_s & B_s
\end{array} \quad \text{mult. } \frac{1}{2} s_{-1} + \frac{1}{2} s_1

\[ \begin{array}{cc}
\uparrow & \downarrow \\
\rightarrow & \rightarrow \\
B_s & B_s
\end{array} \quad \text{comult.} \quad B_s \rightarrow B_s \oplus B_s
\[ \begin{array}{cc}
\uparrow & \downarrow \\
\rightarrow & \rightarrow \\
\rightarrow & \rightarrow \\
\rightarrow & \rightarrow \\
\rightarrow & \rightarrow \\
B_s \oplus B_s & B_s
\end{array} \quad \text{mult. by } \frac{1}{2}
\]

"divided difference" \( \partial_s: R \rightarrow R(-2) \), \( \partial_s(f) = \frac{f - s(f)}{s_1} \).
Must check that these images satisfy the relations (including isotopy).

\[ F(\bigvee) = F(\bigwedge) = \text{id}_{B_s} \]

\[ \begin{array}{c}
\text{Ex}
\end{array} \]

\[ \begin{array}{c}
F \\
\text{F}
\end{array} \]

Can calculate graded dimension of Hom spaces to show F is equivalence.

§ Elias-Williamson diagrammatic category

Let \((V, W)\) be a realization. [Elias-Williamson], building on [Elias-Khovanov, Elias]

Def \(D_{B_S}(V, W)\) is the strict \(k\)-linear monoidal category defined by the following diagrammatic presentation.

- \text{generating objects} : \(B_s, s \in S\)
  
  \[ \Rightarrow \text{objects} : B_w, w \text{ expression} \]

- \text{generating morphisms}:
  - polynomial boxes
  - one-color generating morphisms \(A \in S\)

\[ \forall s, t \in S, s \neq t, m_{st} < \infty, \text{ } 2m_{st}\text{-valent vertex} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{X} \\
\text{X}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
: B(s, t, \ldots) \rightarrow B(t, s, \ldots), \text{ degree } 0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
(\text{e.g., if } m_{st} = 2)
\end{array}
\end{array} \]

Rmk Let \(w = \frac{st}{m_{st}} = \frac{ts}{m_{st}}\). The \(2m_{st}\text{-valent vertex corresponds to} \]

\[ \begin{array}{c}
\begin{array}{c}
R\text{-module map}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
B(s, t, \ldots) = B_w \oplus (\text{lower}) \rightarrow B_w \rightarrow B_w \oplus (\text{lower}) = B(t, s, \ldots)
\end{array}
\end{array} \]

The morphisms are subject to the following (local, homogeneous) relations:

- one-color relations \(A \in S\)
- two-color relations \(\forall s, t, e \in S, s \neq t, m_{st} < \infty \) (see [JeuRn-Williamson, §2.3.2])
- three-color relations \(\forall \text{ finite } \{s, t, e\} \subseteq S \text{ of order } 3 \) (\text{[JW]})

Zamolodchikov relation for every parabolic \(A_3, B_3, A, \times I_2(m), H_3\)
Def \( \mathcal{D}(\mathfrak{g}, W) := \text{Ker}(\mathcal{D}^{\text{BS}}_{\mathfrak{g}}(\mathfrak{g}, W)). \)

These categories are called \textit{Elias-Williamson diagrammatic category} or simply \textit{diagrammatic Hecke category} associated to \((\mathfrak{g}, W)\).

**Thm (Elias-Williamson)**

1. For each \( w \in W \), \( \exists \) indecomposable \( B_w \in \mathcal{D}(\mathfrak{g}, W) \) characterized as follows. For any reduced expression \( w \) for \( w \),

\[ B_w \subseteq B_x \]

and \( B_w \) does not appear as direct summand of \( B_x \) for any expression \( x \) with \( l(x) \leq l(w) \).

2. \( \mathcal{D}(\mathfrak{g}, W) \) categorifies \( H(W) \).

3. If \((\mathfrak{g}, W)\) is reflection faithful, then \( \exists \) monoidal equivalence

\[ \mathcal{D}^{\text{BS}}_{\mathfrak{g}}(\mathfrak{g}, W) \overset{\sim}{\longrightarrow} \text{SBim}^{\text{BS}}_{\mathfrak{g}}(\mathfrak{g}, W) \]

and

\[ \mathcal{D}(\mathfrak{g}, W) \overset{\sim}{\longrightarrow} \text{SBim}(\mathfrak{g}, W). \]

\\

\section{Parity sheaves [Juteau-Mantena-Williamson]}

\( X = \) complex algebraic variety (classical topology), \( k = \) field (for simplicity), \( X = \bigsqcup_{\lambda \in \mathcal{A}} X_{\lambda} \) stratification \( \mathcal{A} \) into affine spaces.

**Def** The category \( \mathcal{D}^{\text{b}}_\mathcal{A}(X, k) \) of \( \mathcal{A} \)-constructible complexes of \( k \)-vector spaces is the full subcategory of \( \mathcal{D}^{\text{b}}(X, k) := \mathcal{D}^{\text{Sh}}(X, k) \) consisting of complexes \( F \) s.t. \( H^i(F) \) is \( \mathcal{A} \)-constructible for all \( i \in \mathbb{Z} \), i.e.

\[ H^i(F)|_{X_{\lambda}} \text{ is a local system } \forall \lambda \in \mathcal{A}. \]

Let \( j_\lambda : X_{\lambda} \longrightarrow X \) be inclusion.

**Def (IJMW)** \( F \in \mathcal{D}^{\text{b}}_\mathcal{A}(X, k) \) is \textit{even} if \( H^i(j_\lambda^*F) = H^i(j_\lambda^!F) = 0 \) for \( i \) odd, \( \forall \lambda \in \mathcal{A} \). It is \textit{odd} if \( F|_{\mathcal{I}} \) is even, \textit{parity} if it is of form even@odd.

**Thm (IJMW)** Let \( F \) be an indecomposable parity complex. Then

\[ \text{supp } F = X_{\lambda}, \text{ some } \lambda \in \mathcal{A}, \]

\[ F|_{X_{\lambda}} \overset{\sim}{\longrightarrow} \mathbb{C}^{\mathcal{A}}_{\lambda} \text{, some } n \in \mathbb{Z}. \]

Any two indecomposable parity complexes with same support are isomorphic up to shift \( \Rightarrow \) unique Verdier self-dual \( E_{\lambda} \), called \textit{parity cheat}.

May not exist in general; not unique up to unique isomorphism like IC.
Def Let \( X = \bigsqcup_{\lambda \in \Lambda_X} X_\lambda \) and \( Y = \bigsqcup_{\mu \in \Lambda_Y} Y_\mu \) be stratified varieties.

A morphism \( \pi : X \to Y \) is stratified if

1. for all \( \mu \in \Lambda_Y \), \( \pi^{-1}(Y_\mu) \) is a union of strata.
2. for each \( X_\lambda \subset \pi^{-1}(Y_\mu) \), the induced morphism \( \pi|_{X_\lambda} : X_\lambda \to Y_\mu \) is a submersion with smooth fibers.

A stratified morphism is even if cohomology of fiber is concentrated in even degrees.

Fact Direct image of a parity complex under a proper even morphism is parity.

This easy fact is a substitute for the decomposition theorem in this setting.

Cor If \( X_\lambda \) has an even resolution \( X_\lambda \xrightarrow{\alpha} X_\lambda \xrightarrow{\pi} \), then \( \mathcal{E}_\lambda \) exists, and \( \mathcal{E}_\lambda \subset \pi_* \mathcal{F} \subset \mathcal{H}^\wedge \mathcal{F} \).

Analogous statements hold in the equivariant derived category of Bernstein-Lunts. The definition of parity remains the same, but assumption on stratification is replaced by equivariant analogue.

Ex \( X = \text{IP}^1 = \mathbb{P}^1 \cup \text{pt} \)

\[ (x_s)(x_{id}) \]

\[ E_s = \mathbb{A}^1_{X_s} \]

\[ E_{id} = \mathbb{A}^1_{X_{id}} \]

\[ 3 \text{ Parity complexes on flag varieties} \]

Let \( A = (a_{st})_{s,t \in S} \) be a generalized Cartan matrix, \( |S| < \infty \), i.e. \( a_{ss} = 2 \) \( \forall s \in S \), \( a_{st} \leq 0 \) if \( s \neq t \), \( a_{st} = 0 \) \( \iff a_{ts} = 0 \).

Consider an associated Kac-Moody root datum

\[ (\Lambda, \{\alpha_r\}_{\alpha \in S}, \{\alpha_s\}_{\alpha \in S}) \]

i.e. \( \Lambda \) is a free \( \mathbb{Z} \)-module of finite rank, \( \alpha_s \in \text{Hom}_{\mathbb{Z}} (\Lambda, \mathbb{Z}) \), \( \alpha_s \in \Lambda \), s.t. \( a_{st} = \langle \alpha_t^* , \alpha_s \rangle \) \( \forall s,t \in S \).

\( \Rightarrow \) Kac-Moody group \( G \), an ind group scheme over \( \mathbb{C} \), with canonical Borel \( B \supset \text{max torus } T \); Weyl group \( (W, S) \).

Ex A complex reductive group \( G \supset B \supset \text{max torus } T \) is root datum.

Corresponding Kac-Moody group recovers \( G \supset B \supset T \).
\[ X := G/B = \bigcup_{w \in W} X_w, \quad X_w := BwB/B \cong C^{(w)} \]

Consider \( D^b_B(X, k) \), \( k \) field, monoidal under convolution product:
\[ G \times X \xrightarrow{\cdot} G \times X \xrightarrow{\text{mult}} X \]

Given \( E, F \in D^b_B(X, k) \),
\[ E \ast F := \text{mult}^\ast(E \boxtimes F), \quad \text{where } E \boxtimes F \text{ is unique with } q^\ast(E \boxtimes F) = E \boxtimes F. \]

For \( s \in S \),
\[ E_s := [1]^{B_sB/B} \in \text{Parity}_B(X, k). \]

For expression \( w = (s_1, ..., s_k) \), the Bott-Samelson parity complex
\[ E_w := E_{s_1} \ast ... \ast E_{s_k} \in \text{Parity}_B(X, k) \]

(parity because \( (-) \ast E_s = q^\ast E_s \ast [1], \quad q_s : X \rightarrow G/B \) even)

\[ \text{Def } \text{Parity}^{BS}_B(X, k) \text{ is the full subcategory of } \text{Parity}_B(X, k) \]

consisting of \( E_w[n], \) \( w \) expression, \( n \in \mathbb{Z} \).

For every \( w \in W \), parity sheaf \( E_w \) arises as direct summand of \( E_w \), \( w \) reduced expression for \( w \):
\[ \text{supp } E_w = BwB/B = X_w, \quad E_w |_{X_w} \cong [k]^{X_{(w)}} \]

so \( E_w \) is unique direct summand with full support. Hence
\[ \text{Kan}(\text{Parity}^{BS}_B(X, k)) \cong \text{Parity}_B(X, k). \]