

§ Beilinson-Ginzburg-Soergel Koszul duality

$G =$ connected reductive group / $\mathbb{C} \supset B \supset T$, Langlands dual $G^\vee \supset B^\vee \supset T^\vee$

Symplectic duality $T^*(G/B) \longleftrightarrow T^*(G^\vee/B^\vee)$, (principal block):

$\mathcal{O}_0 \simeq P(U \backslash G/B) = U$ -equiv (i.e. B -constructible) \mathbb{C} -perv sheaves on G/B

$G \supset B \supset T / \mathbb{F}_q \uparrow \text{For ("degrading functor")}$

$\mathcal{O}_0^{\text{mix}} \simeq P^{\text{mix}}(U \backslash G/B) =$ certain U -equiv mixed \mathbb{Q}_ℓ -perv sheaves on G/B

$\langle 1 \rangle = (-\frac{1}{2})$ Tate twist

$D^{\text{mix}}(U \backslash G/B) := D^b P^{\text{mix}}(U \backslash G/B)$

Thm (Koszul duality [BGS]) \exists triangulated equiv.

$\kappa : D^{\text{mix}}(U \backslash G/B) \xrightarrow{\sim} D^{\text{mix}}(G^\vee \backslash B^\vee / U^\vee)$

$\Delta_w^{\text{mix}} \longmapsto \Delta_w^{\vee, \text{mix}}$
 $IC_w^{\text{mix}} \longmapsto T_w^{\vee, \text{mix}}$
 \vdots

"standard sheaves"
 "indec. tilting"

$\forall w \in W =$ Weyl group

and intertwining

$[1] \longleftrightarrow [1]$
 $\langle 1 \rangle \longleftrightarrow [1] \langle -1 \rangle \quad (\Rightarrow \text{"mix" is essential})$

shuffling
 twisting

$\begin{cases} Sh_s \longleftrightarrow T_w^{\vee} \\ Tw_s \longleftrightarrow Sh_s^{\vee} \end{cases}$

$\forall s \in S = \{\text{simple reflections}\}$

\Rightarrow exchanging the commuting braid group actions

§ Monoidal Koszul duality, char 0 [Bezrukavnikov-Yun]

Grothendieck gp: as $\mathbb{Z}[v, v^{-1}]$ -modules

$[D^{\text{mix}}(U \backslash G/B)]_\Delta \xrightarrow{\sim} H =$ Hecke alg / $\mathbb{Z}[v, v^{-1}]$

$[\Delta_w^{\text{mix}}] \longleftrightarrow \delta_w$ "standard basis"

$[IC_w^{\text{mix}}] \longleftrightarrow$ Kazhdan-Lusztig basis

$\langle 1 \rangle \longleftrightarrow v$

κ induces involution

$\iota : H \longrightarrow H$
 $\delta_w \longmapsto \delta_w$
 $v \longmapsto -v^{-1}$

(\Rightarrow "mix" is essential ... can't set $v=1$)

In fact, ι is a ring involution!

Q Upgrade κ to monoidal equiv?

$D^{\text{mix}}(U \backslash G/B) \hookrightarrow (D^{\text{mix}}(B \backslash G/B), \ast_B)$ "Hecke category"

$H \hookrightarrow H$

right regular rep of H .

$$(D^{mix}(B \backslash G/B), \star_B) \Rightarrow \nabla_S^{mix}$$

$$Sh_S \subset D^{mix}(U \backslash G/B) \supset Tw_S \xrightarrow{(-)} \star_B \nabla_S^{mix}$$

$$(D^{mix}(B \backslash G/B), \star_B)$$

$$Tw_S \subset D^{mix}(B \backslash G/U) \supset Sh_S$$

Seek monoidal category in \square .

Thm (Monoidal Koszul duality [BY])

\exists triangulated monoidal cat $(\widehat{D^{mix}(U \backslash G/U)}, \star_U)$ of free-monodromic cx and monoidal equivalence

(completion involves "free unipotent monodromy" local systems)

$$(D^{mix}(B \backslash G/B), \star_B) \xrightarrow{\sim} (\widehat{D^{mix}(U \backslash G/U)}, \star_U)$$

such that BGS Koszul duality is recovered as quotient. (Even G Kac-Moody)

$\{$ Positive char coeffs: parthy cx [Juteau-Mautner-Williamson], mixed modular derived cat [Achar-Riche]

For application to modular rep thry, want BGS Koszul duality with positive char coeff (especially for affine flag var). Need suitable analogue of D^{mix} .

$G > B > T/\mathbb{C}$, (W, S) Weyl gp, k field.

$D^b(B \backslash G/B, k) = B$ -equiv derived cat of sheaves of k -v.s.

$D^b(U \backslash G/B, k) = B$ -const.

$[i]$ = homological shift (usually denoted by $[1]$)

$s \in S$, $\pi_s: G/B \rightarrow G/P^s$ minimal partial flag var

$$\rightsquigarrow D^b(G/B, k) \xrightleftharpoons[\pi_s^*]{\pi_{s*}} D^b(G/P^s, k) \quad (B\text{-equiv or } B\text{-const.})$$

Given $w \in W$, choose reduced expression $\underline{w} = (s_1 \dots s_\ell(w))$.

$$E_{\underline{w}} := \pi_{s_1}^* \pi_{s_2}^* \dots \pi_{s_\ell}^* \pi_{s_\ell} \underbrace{\mathbb{K}_{B \backslash B/B}}_{\text{skyscraper at point section}} \{l(w)\}$$

$$\text{supp } E_{\underline{w}} = \overline{BwB/B} = \text{Schubert var } \overline{X_w}, \quad E|_{X_w} \simeq \mathbb{K}_{X_w} \{l(w)\}$$

$\Rightarrow \exists!$ direct summand E_w (in $D^b(B \backslash G/B, k)$ or $D^b(U \backslash G/B, k)$) with full support.

- Facts
- indep of choice of w
 - if $\text{char } k = 0$, then $E_w \cong IC_w$ (\Rightarrow 0-canonical basis = KL basis)
 - in general, agrees with parity sheaf (characterized by parity vanishing of stalk/costalk cohomologies)
 - $(\text{Parity}(B \backslash G/B, \mathbb{Z}), \star_B)$ categorifies Hecke alg, class of $E_w =: \frac{p\text{-canonical basis}}{p = \text{char } k}$
 - in char 0, \oplus of $\{ \}$ of IC_w 's

$$K^b \text{SemiSimple}(U \backslash G/B) \xrightarrow{\sim} D^{\text{mix}}(U \backslash G/B)$$

$$[1] \longleftrightarrow [1]$$

$$[1] \{-1\} \longleftrightarrow \langle 1 \rangle$$

Def [Achar-Riche] For any field k , mixed nodular derived category

$$D^{\text{mix}}(B \backslash G/B, k) := K^b \text{Parity}(B \backslash G/B, k)$$

$$D^{\text{mix}}(U \backslash G/B, k) := K^b \text{Parity}(U \backslash G/B, k)$$

$$\text{Tate twist } \langle 1 \rangle := [1] \{-1\}$$

\exists recollement structure \rightsquigarrow perverse t -structure

all defined as $\Delta_w^{\text{mix}}, \nabla_w^{\text{mix}}, IC_w^{\text{mix}}, \dots \in P^{\text{mix}}(U \backslash G/B, k)$ and $P^{\text{mix}}(B \backslash G/B, k)$

complexes of parity complexes! $\rightarrow T_w^{\text{mix}} \in P^{\text{mix}}(U \backslash G/B, k)$ graded highest weight

Thm (Modular Koszul duality [AR]) If k is a field whose char is good for G , then

$$D^{\text{mix}}(U \backslash G/B, k) \xrightarrow{\sim} D^{\text{mix}}(B^v \backslash G^v/U^v, k)$$

$$\Delta_w^{\text{mix}} \longmapsto \Delta_w^{v, \text{mix}}$$

$$E_w^{\text{mix}} \longmapsto T_w^{v, \text{mix}}$$

$$\vdots$$

$$[1] \longleftrightarrow [1]$$

$$\langle 1 \rangle \longleftrightarrow \{1\} = [1] \langle -1 \rangle$$

$\forall w \in W$

and intertwining

§ New definition of free-monodromic ex

Achar-Riche's approach is only for finite flag var. For modular KD for affine or Kac-Moody flag var, want to first show modular analogue of Bezrukavnikov-Tyurin nodular KD.

Problem: $D^{\text{mix}}(U^v \backslash G^v/U^v)$ involves infinite rank nonsemisimple local system, not obvious how to deal with in $D^{\text{mix}} := K^b \text{Parity}$.

Philosophy: replace geometric complication with homological algebra / category theoretic one

Let $G > B > T, c$ Kac-Moody, k field of char $\neq 2$.

Thm (Achar-M-Riche-Williamson)

\exists category $D^{\text{mix}}(U \backslash G/U, k)$ of free-monodromic ex, nodular additive subcat

$(\text{Tilt}(U \backslash G/U, k), \star)$ of free-monodromic tilting ex.

Thm [AMRW]

\exists additive monoidal equivalence
 $(\text{Parity}(B \backslash G/B, k), \hat{*}) \xrightarrow{\sim} (\text{Tilt}(U \backslash G \backslash U, k), \hat{*})$

Application (for $G = \text{loop group}$) char formula for tilting modules of reductive gp involving p -Kazhdan-Lusztig polynomials.

Defn A free-mon cx is a pair (F, \mathcal{S}) , $F = (F^i)_{i \in \mathbb{Z}}$ is sequence of $F^i \in \text{Parity}(B \backslash G/B, k)$

$\mathcal{S} \in A \otimes_{\mathbb{R}} \text{End}(F) \otimes_{\mathbb{K}} \mathbb{R}^{\vee}$ satisfying ...
 R -bimodule, $R = \text{Sym}_{\mathbb{Z}}(\mathbb{K} \otimes_{\mathbb{Z}} X^*(T)) \simeq H_B^*(pt)$

$A = \text{Koszul cx}$ (resolution of $\mathbb{K} = \text{Erlv mod of } R$) by free R -modules),
 so $A \otimes_{\mathbb{R}} - \simeq \mathbb{K} \otimes_{\mathbb{R}} -$ homological substitute for passing from B -equiv to U -equiv.

R^{\vee} forces Langlands dual $R^{\vee} \simeq H_{B^{\vee}}^*(pt)$ action on the right.

Ex

$$\tilde{T}_s^{\text{mix}} := \begin{matrix} & \nearrow \mathcal{E}_{id} \otimes & \\ -\alpha_s \otimes id & \uparrow \mathcal{E}_s \otimes & \\ & \mathcal{E}_s \otimes & \\ & \downarrow \mathcal{E}_{id} & \\ & \mathcal{E}_{id} & \end{matrix} \circ \text{Id} \otimes \alpha_s^{\vee}$$

Remark (1) Showing that $\hat{*}$ is bifunctorial takes ~50 pages, technical rk 2 calculation
 (2) monoidal functor defined by generators and rels (Elias-Williamson calculus)

Remark (In preparation with Hogancamp) Much better proof of bifunctoriality of $\hat{*}$ and more conceptual definition of $D^{\text{mix}}(U \backslash G \backslash U, k)$: it is the derived cat of A -bimodules in $D^{\text{mix}}(B \backslash G/B, k)$, where $A = \text{Koszul cx}$, viewed as an algebra object in $D^{\text{mix}}(B \backslash G/B, k)$.

Conj (derived A -bimodules in $D^{\text{mix}}(B \backslash G/B, k)$) \simeq $D^{\text{mix}}(B^{\vee} \backslash G^{\vee} / B^{\vee}, k)$
 Hecke cat. Langlands dual Hecke cat.

Soergel bimodules

$G \supset B \supset T, c$ connected reductive, k field of char $\neq 2, 3$

$W \circledast \mathfrak{h} := \mathbb{K} \otimes_{\mathbb{Z}} X_*(T) \rightarrow W \circledast R := \text{Sym}_{\mathbb{K}}(\mathfrak{h}^{\vee})$, \mathbb{Z} -graded with $\text{deg } \mathfrak{h}^{\vee} = 2$

$R\text{-grad-}R := \mathbb{Z}$ -graded R -bimodules (and $\text{deg } 0$ R -bimodules)

\uparrow
 (1) = shift grading down

For $w \in W$, choose reduced exp $\underline{w} = (s_1, \dots, s_{l(w)})$,

$$B_{\underline{w}} := R \otimes_{R^{s_1}} \dots \otimes_{R^{s_{l(w)}}} R \quad \in R\text{-gmod-}R$$

$U \oplus$
 $B_w =$ unique "largest" direct summand

Facts

- indep of \underline{w}
- [Soergel] - $(\text{SBim}(\mathfrak{h}, W), \otimes_{\mathbb{R}})$ categorifies Hecke alg
- \oplus of $()$ of B_w 's
- Monoidal equiv.
- $\text{HT}^* : (\text{Partty}(B \backslash G/B, k), \uparrow_{\mathbb{B}}) \xrightarrow{\sim} (\text{SBim}(\mathfrak{h}, W), \otimes_{\mathbb{R}})$
- $\mathcal{E}_w \longleftarrow B_w$
- $\{1\} \longleftarrow (1)$

Ex ($G = SL_2$) $W = S_2 \curvearrowright R = k[\alpha_s]$, $\deg \alpha_s = 2$
 $s(\alpha_s) = -\alpha_s$, $R^s = k[\alpha_s^2]$
 $B_{id} = R$ regular bimodule, $B_s = R \otimes_{R^s} R(1)$
 $B_s \otimes_R B_s \xrightarrow{\sim} B_s(1) \oplus B_s(-1)$ categorifies quadratic relation in Hecke alg

$\text{Partty}_B(\mathbb{P}^1, k) \xrightarrow{\sim} \text{SBim}(\mathfrak{h}, S_c)$
 $\mathcal{E}_{id} = k_{\mathbb{P}^1} \longleftarrow R$
 $\mathcal{E}_s = k_{\mathbb{P}^1} \{1\} \longleftarrow B_s$

$\text{End}^*(R) \simeq R$, $\text{Hom}^*(B_s, R) = R \cdot \text{mult}$, $\text{Hom}^*(R, B_s) = R \cdot \text{comult}$
 $\text{End}^*(B_s) = R \cdot \text{id} \oplus R \cdot (\text{comult} \circ \text{mult}) \xrightarrow{\sim} R \oplus R(-2) \simeq B$ -equiv cohomology of \mathbb{P}^1

Rmk SBim is not well-behaved (does not categorify the Hecke alg) in general, e.g. for affine flag var and positive char coeff. Must use Partty instead.

Ext-enhanced Soergel bimodules [Hogancamp-M.]

$\text{SBim} \subset R\text{-gmod-}R \subset D^b(R\text{-gmod-}R)$, $\Gamma_{1,1} :=$ cohomological shift additive abelian

Def Category SBim^{Ext} of Ext-enhanced Soergel bimodules: obj are \oplus of $B^i(n)$,
 $B \in \text{SBim}$, $n \in \mathbb{Z}$. Hom spaces are $\mathbb{Z} \times \mathbb{Z}$ -graded, determined by
 $\text{Hom}_{\text{SBim}^{\text{Ext}}}^{i,j}(B^i(n), B^j(m)) := \text{Ext}_{R\text{-gmod-}R}^{i+m-n}(B, B'(j))$
 (contains SBim as (not full) subcategory, $i=0$.)

Ex $\text{Hom}_{\text{SBim}^{\text{Ext}}}^{0,0}(B, B') = \text{Hom}_{\text{SBim}}(B, B')$ if $B, B' \in \text{SBim}$
 $\text{Hom}_{\text{SBim}^{\text{Ext}}}^{i,j}(R, B) = \text{HH}^i(B(j)) = i^{\text{th}}$ Hochschild, $B \in \text{SBim}$

$End_{SBim^{Ext}}^{+,+}(R) = \bigwedge_{\mathbb{K}} \otimes R$ ← symm alg in \mathfrak{h}^+ , $deg \mathfrak{h}^+ = (0, 2)$

↑ exterior alg in \mathfrak{h}^+ , $deg \mathfrak{h}^+ = (1, -2)$

$End^{1,-+}(B_s) \cong \mathbb{K}$, (one-diml self Ext of B_s)

The same homological procedure

$(Parity(B \setminus G/B, \mathbb{K}), \ast_B) \rightsquigarrow (Tilt(U \setminus G \setminus U, \mathbb{K}), \hat{\ast})$

can be repeated starting with $SBim^{Ext}$ to produce

$(SBim^{Ext}(\mathfrak{h}, W), \infty_R) \rightsquigarrow (Tilt^{Ext}(U \setminus G \setminus U, \mathbb{K}), \hat{\ast})$

Remark Expect such defn also for "Parity^{Ext}(B \setminus G/B, K)," but Parity(B \setminus G/B, K) does not sit in natural abelian category ...

Conj (Hogancamp-M)

∃ monoidal additive equiv.

$(Parity^{Ext}(B \setminus G/B, \mathbb{K}), \ast_B) \rightsquigarrow (Tilt^{Ext}(U \setminus G \setminus U, \mathbb{K}), \hat{\ast})$

for any symmetrizable Kac-Moody G/\mathbb{C} , \mathbb{K} field of char $\mathbb{K} \neq 2$. Contains [AMRW] monoidal KD as subcategories.

In [HM], we define Parity^{Ext} for GL_2 using $SBim^{Ext}$, then define such monoidal functor by generator and relations. For this, we give diagrammatic presentation of $SBim^{Ext}$ for GL_2 extending the Elias-Khovanov diagrammatics.

new generators:

\boxed{r} $r \in \check{\Lambda}$, $\begin{array}{c} \text{---} \\ | \\ \circ \\ | \\ \text{---} \end{array} : B_s \longrightarrow B_s \Gamma_{1,1}(-4)$

new rels:

$\begin{array}{c} \text{---} \\ | \\ \circ \\ | \\ \text{---} \end{array} = 0$, $\begin{array}{c} \text{---} \\ | \\ \circ \\ | \\ \text{---} \end{array} = \boxed{\check{\alpha}_s}$, $\check{\alpha}_s \in \check{\Lambda}$, etc.

Checking relations for GL_2 already involves complicated computations that seem to work out miraculously ...

§ Motivation: symmetry in HOMFLY-PT link homology

via Hecke cat [Khovanov] [Rozansky-Khovanov]

HOMFLY-PT homology $HHH^{...}(L) = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ -graded v.s.

decat $\begin{array}{c} \text{---} \\ | \\ \circ \\ | \\ \text{---} \end{array}$ (Poincaré poly $P_L(a, q, t)$, $P_L|_{t=-1} = P_L$, up to normalization.)

link $L \xrightarrow{\text{via Hecke alg (GL}_n\text{) [Jones]}}$ HOMFLY-PT polynomial $P_L(a, q)$

(7)

Symmetry $P_L(a, q) = P_L(a, q^{-1}) \longleftrightarrow$ involution τ of Hecke alg
 no such symmetry for HOMFLY-PT homology \longleftrightarrow related, no Koszul self-duality of (Ext-enhanced) Hecke category

Indeed,
 $L =$ closure of braid $\beta \rightsquigarrow$ Rouquier cx $F_\beta \in K^b \text{SBim}^{(GL_n)} \hookrightarrow K^b \text{SBim}^{\text{Ext}}$
 $\text{HHH}^{\dots}(L) = \bigoplus_{i,j,k} \text{Hom}_{K^b \text{SBim}^{\text{Ext}}}(R, F_\beta[i](j) \uparrow k)$.

So Koszul self-duality of $\mathcal{D}^{\text{mix, Ext}}(B \backslash GL_n / B) := K^b(\text{SBim}^{\text{Ext}}(GL_n))$
 fixing R, F_β would immediately imply symmetry.

Instead, expect Ext-enhanced Koszul duality involving (Langlands-dual) free-monodromic cx.

Remark Ginzburg-Hogancamp defined γ -ified Soergel bimodules and conjectured its Koszul self-duality, which would imply symmetry of γ -ified homology defining HHH.

Work in progress with Hogancamp: relation between γ -ified and free-monodromic, Koszul duality for γ -ified conjectured for arbitrary G .