

# THE FINAL LOG CANONICAL MODEL OF THE MODULI SPACE OF STABLE CURVES OF GENUS FOUR

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ABSTRACT. We describe the GIT quotient of the linear system of  $(3, 3)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  as the final non-trivial log canonical model of  $\overline{\mathcal{M}}_4$ , isomorphic to  $\overline{\mathcal{M}}_4(\alpha)$  for  $8/17 < \alpha \leq 29/60$ . We describe singular curves parameterized by  $\overline{\mathcal{M}}_4(29/60)$ , and show that the rational map  $\overline{\mathcal{M}}_4 \dashrightarrow \overline{\mathcal{M}}_4(29/60)$  contracts the Petri divisor, in addition to the boundary divisors  $\Delta_1$  and  $\Delta_2$ . This answers a question of Farkas.

## 1. INTRODUCTION

The goal of this note is twofold. One is to show that the Petri divisor on  $\overline{\mathcal{M}}_4$  is contracted by a rational contraction, thus answering a question of Farkas. Second is to describe the final non-trivial log canonical model appearing in the Hassett-Keel log minimal model program for  $\overline{\mathcal{M}}_4$ , thus confirming various predictions obtained in [AFS10] for when singular curves replace curves with special linear systems.

We now describe each of these goals in more detail: It is well-known that the hyperelliptic divisor in  $\overline{\mathcal{M}}_3$  is contracted by the rational map to the final non-trivial log canonical model of  $\overline{\mathcal{M}}_3$  given by the GIT quotient of plane quartics; see [HL10b]. Farkas has observed [Far10a, p.281] that on  $\overline{\mathcal{M}}_4$  there is no rational contraction, well-defined away from the hyperelliptic locus, that contracts the Petri divisor  $P \subset \overline{\mathcal{M}}_4$ . Subsequently, Farkas asked [Far10b] whether there are rational contractions, necessarily with a larger indeterminacy locus, that do contract  $P$ . Here, we answer this question in affirmative: The rational map to the GIT quotient of  $(3, 3)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  contracts the Petri divisor to a point. This map is undefined both along the hyperelliptic locus *and* the locus of irreducible nodal curves with a hyperelliptic normalization.

Our second goal is to describe the final non-trivial step in the Hassett-Keel log MMP program for  $\overline{\mathcal{M}}_4$  and to verify that it satisfies the modularity principle of [AFS10]. The aim of the Hassett-Keel program for  $\overline{\mathcal{M}}_g$  is to find an open substack  $\overline{\mathcal{M}}_g(\alpha)$  in the stack of all complete genus  $g$  curves such that  $\overline{\mathcal{M}}_g(\alpha)$  has a good moduli space isomorphic to the log canonical model

$$\overline{\mathcal{M}}_g(\alpha) := \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}_g, [m(K_{\overline{\mathcal{M}}_g} + \alpha\delta)]).$$

As of this writing, the Hassett-Keel program for  $\overline{\mathcal{M}}_4$  has been carried out for  $\alpha \geq 2/3$  using GIT of the Hilbert and Chow schemes of bicanonically embedded curves and the

threshold values at which  $\overline{\mathcal{M}}_4(\alpha)$  changes are  $\alpha = 9/11, 7/10, 2/3$  [HH08, HH09, HL10a]. In particular, Hyeon and Lee [HL10a] construct a small contraction  $\overline{\mathcal{M}}_4(7/10 - \epsilon) \rightarrow \overline{\mathcal{M}}_4(2/3)$  of the locus of Weierstrass genus 2 tails (i.e. curves  $C_1 \cup_p C_2$  where  $C_1$  and  $C_2$  are genus two curves meeting in a node  $p$  such that  $p$  is a Weierstrass point of  $C_1$  or  $C_2$ ) using Kawamata basepoint freeness theorem. An alternative approach to the Hassett-Keel program for  $\overline{\mathcal{M}}_g$  and to a functorial construction of the log canonical model  $\overline{\mathcal{M}}_g(2/3)$  is pursued by Alper, Smyth, and van der Wyck in [ASvdW10]. They define a moduli stack  $\overline{\mathcal{M}}_g(A_4)$  of genus  $g$  curves with at worst  $A_4$  singularities and no Weierstrass genus 2 tails and show that it is weakly proper without using GIT. Once the existence and the projectivity of a good moduli space of  $\overline{\mathcal{M}}_g(A_4)$  is established,  $\overline{\mathcal{M}}_g(A_4)$  will give a modular interpretation of the log canonical model  $\overline{\mathcal{M}}_g(2/3)$  for every  $g \geq 4$ .

We note that there are other threshold values at which  $\overline{\mathcal{M}}_4(\alpha)$  changes for  $0 < \alpha < 2/3$ ; these can be easily obtained from [AFS10]. One of them is  $\alpha = 5/9$  and the corresponding log canonical model has been completely described by Casalaina-Martin, Jensen, and Laza [CMJL11a] as a GIT quotient of the Chow variety of canonically embedded genus 4 curves. In a forthcoming work [CMJL11b], the same authors describe all log canonical models  $\overline{\mathcal{M}}_4(\alpha)$  that arise for  $29/60 < \alpha < 5/9$  by using VGIT on the parameter space of  $(2, 3)$  complete intersections in  $\mathbb{P}^3$  and show that the only threshold value in the interval  $(29/60, 5/9)$  is  $\alpha = 1/2$ , which agrees with predictions of [AFS10].

Here, we describe the final step in the Hassett-Keel program for  $\overline{\mathcal{M}}_4$  which is given by a natural GIT quotient of the linear system of  $(3, 3)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ : A starting point is the classical observation that a canonical embedding of a non-hyperelliptic smooth curve of genus 4 lies on a unique quadric in  $\mathbb{P}^3$ . If the quadric is smooth, the curve is called *Petri-general*, and is realized as an element of the linear system

$$V := |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3)| \simeq \mathbb{P}^{15}$$

of  $(3, 3)$ -curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Moreover, the uniqueness of a pair of  $g_3^1$ 's implies that two smooth curves of class  $(3, 3)$  are abstractly isomorphic if and only if they belong to the same  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ -orbit. Conversely, a smooth genus 4 curve is called *Petri-special* if its canonical image lies on a singular quadric. Petri-special curves form a divisor whose closure in  $\overline{\mathcal{M}}_4$  is called the *Petri divisor*; we denote it by  $P$ .

This said, we consider a linearly reductive group  $G = \text{SL}(2) \times \text{SL}(2) \rtimes \mathbb{Z}_2$  that, while being a finite cover of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ , has the advantage of linearizing  $\mathcal{O}_V(1)$ . Then the GIT quotient  $V^{\text{ss}}//G$  will be a birational model of  $\overline{\mathcal{M}}_4$  as soon as the general curve in  $V$  is GIT stable, which is easy to verify. Our main result is:

**Main Theorem.** *The GIT quotient  $M := |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3)|^{\text{ss}}//(\text{SL}(2) \times \text{SL}(2) \rtimes \mathbb{Z}_2)$  is isomorphic to  $\overline{\mathcal{M}}_4(\alpha)$  for  $8/17 < \alpha \leq 29/60$ . The resulting birational contraction  $f: \overline{\mathcal{M}}_4 \dashrightarrow \overline{\mathcal{M}}_4(29/60)$  contracts the Petri divisor  $P$  to the point parameterizing triple conics, and contracts the boundary divisor  $\Delta_2$  to the point parameterizing curves with two  $A_5$  singularities. The hyperelliptic locus  $\overline{H}_4$  is flipped by  $f$  to the locus*

$$A := \overline{\{\text{curves with an } A_8 \text{ singularity}\}},$$

i.e. the total transform of the generic point of  $\overline{H}_4$  is  $A$ .

Main Theorem is proved in Section 3.1. We give a roadmap to its proof: That  $f$  is a contraction is proved in Proposition 3.1. That  $f$  contracts  $P$  and  $\Delta_2$  to a point is proved in Theorem 3.13. That  $\overline{H}_4$  is flipped to  $A$  is established in Theorem 3.16. Finally, the identification of  $M$  with log canonical models of  $\overline{M}_4$  is made in Corollary 3.6.

We also obtain a strengthening of the genus 4 case of [Far10a, Theorem 5.1], whose terminology we keep:

**Theorem 1.1.** *The moving slope of  $\overline{M}_4$  is  $60/7$ .*

Theorem 1.1 is proved in Corollary 3.7.

**Remark 1.2.** A birational contraction of the Petri divisor inside  $\overline{M}_{4,1}$  is constructed in [Jen10] using GIT on the universal curve over  $V$ . However, Jensen does not give a modular interpretation to this contraction.

1.1. **Preliminaries.** We recall some definitions and results that will be used throughout this work. We work over  $\mathbb{C}$ .

1.1.1. *Varieties of stable limits.* Let  $C$  be an l.c.i. integral curve of arithmetic genus  $g$  and  $p \in C$  be a singular point. Recall that the *variety of stable limits* of  $C$  is the closed subvariety  $\mathcal{T}_C \subset \overline{M}_g$  consisting of stable limits of all possible smoothings of  $C$ . Namely, if  $\text{Def}(C) \xleftarrow{p} Z \xrightarrow{q} \overline{M}_g$  is the graph of the rational moduli map  $\text{Def}(C) \dashrightarrow \overline{M}_g$ , then we set  $\mathcal{T}_C := q(p^{-1}(0))$ .

Suppose that  $p \in C$  is the only singularity. Let  $b$  be the number of branches of  $p \in C$  and  $\delta(p) = \dim_{\mathbb{C}} \mathcal{O}_{\tilde{C}}/\mathcal{O}_C$  be the  $\delta$ -invariant. Then curves in  $\mathcal{T}_C$  are of the form  $\tilde{C} \cup T$ , where  $(\tilde{C}, q_1, \dots, q_b)$  is the pointed normalization of  $C$  and  $(T, p_1, \dots, p_b)$  is a  $b$ -pointed curve of arithmetic genus  $\gamma = \delta(p) - b + 1$ . The pointed stable curve  $(T, p_1, \dots, p_b)$  is called the *tail of a stable limit*. Tails of stable limits are independent of  $\tilde{C}$  and depend only on  $\hat{\mathcal{O}}_{C,p}$ . It follows that we can define the variety of tails of stable limits of  $\hat{\mathcal{O}}_{C,p}$  as a closed subvariety  $\mathcal{T}_{\hat{\mathcal{O}}_{C,p}} \subset \overline{M}_{\gamma,b}$  (see [Has00, Proposition 3.2]).

We recall the following results concerning the varieties of tails of stable limits of  $A$  and  $D$  singularities (see [Has00, Sections 6.2, 6.3] and [Fed10]).

**Proposition 1.3** (Varieties of stable limits of AD singularities).

- (A<sub>odd</sub>) *The variety of tails of stable limits of the  $A_{2k+1}$  singularity  $y^2 = x^{2k+2}$  is the locus of  $(C, p_1, p_2) \in \overline{M}_{k,2}$  such that a semistable model  $C'$  of  $C$  admits an admissible hyperelliptic cover  $\varphi: C' \rightarrow R$ , where  $R$  is a rational nodal curve, and  $\varphi(p_1) = \varphi(p_2)$ .*
- (A<sub>even</sub>) *The variety of tails of stable limits of the  $A_{2k}$  singularity  $y^2 = x^{2k+1}$  is the locus of  $(C, p) \in \overline{M}_{k,1}$  such that a semistable model  $C'$  of  $C$  admits an admissible hyperelliptic cover  $\varphi: C' \rightarrow R$ , where  $R$  is a rational nodal curve, and  $p$  is a ramification point of  $\varphi$ .*

- (D<sub>odd</sub>) *The variety of tails of stable limits of the  $D_{2k+1}$  singularity  $x(y^2 - x^{2k-1}) = 0$  is the locus  $(C, p_1, p_2) \in \overline{M}_{k,2}$  such that  $p_1 \neq p_2$  and the stabilization of  $(C, p_1)$  is as in (A<sub>even</sub>).*
- (D<sub>even</sub>) *The variety of tails of stable limits of the  $D_{2k}$  singularity  $x(y^2 - x^{2k-2}) = 0$  is the locus  $(C, p_1, p_2, p_3) \in \overline{M}_{k-1,3}$  such that  $p_3 \notin \{p_1, p_2\}$  and the stabilization of  $(C, p_1, p_2)$  is as in (A<sub>odd</sub>).*

*Proof.* This is the content of [Fed10, Main Theorem 1(2) and Main Theorem 2(2)].  $\square$

1.1.2. *Deformations of curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ .* Suppose that  $p_1, \dots, p_n$  are singular points of a reduced curve  $C$ . Then

$$(1.1) \quad \text{Def}(C) \rightarrow \prod_{i=1}^n \text{Def}(\hat{\mathcal{O}}_{C,p_i})$$

is smooth because  $H^2(C, \text{Hom}(\Omega_C, \mathcal{O}_C)) = (0)$ . If  $C$  is a  $(3, 3)$  curve on  $\mathbb{P}^1 \times \mathbb{P}^1$ , then in fact all deformations of  $\hat{\mathcal{O}}_{C,p_i}$  are realized by embedded deformations of  $C$ :

**Proposition 1.4.** *Let  $C$  be a reduced curve in class  $(3, 3)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $p_1, \dots, p_n$  are singular points of  $C$ . Then the natural map  $\text{Hilb}(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow \prod_{i=1}^n \text{Def}(\hat{\mathcal{O}}_{C,p_i})$  is smooth.*

*Proof.* This is standard. Set  $X = \mathbb{P}^1 \times \mathbb{P}^1$ . Since  $\text{Hilb}(X)$  and  $\prod_{i=1}^n \text{Def}(\hat{\mathcal{O}}_{C,p_i})$  can be taken to be of finite type over  $\mathbb{C}$ , it suffices to establish formal smoothness. Since  $C \hookrightarrow X$  is locally unobstructed, it suffices to check that a collection of local first-order deformations can always be glued to a global embedded deformation. A sufficient condition for this is the vanishing of  $H^1(C, \mathcal{N}_{C/X})$  and  $H^1(C, T_X \otimes \mathcal{O}_C)$ . We now compute:

$$\begin{aligned} H^1(C, \mathcal{N}_{C/X}) &= H^0(C, \omega_C \otimes \mathcal{N}_{C/X}^\vee)^\vee = H^0(C, \mathcal{O}(-2, -2)|_C)^\vee = (0), \\ H^1(C, T_X \otimes \mathcal{O}_C) &= H^1(C, \mathcal{O}_C(2, 0) \oplus \mathcal{O}_C(0, 2)) = H^0(C, \mathcal{O}_C(-1, 1) \oplus \mathcal{O}_C(1, -1))^\vee = (0). \end{aligned}$$

$\square$

## 2. GIT OF $(3, 3)$ CURVES ON $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we classify semistable points of  $V := |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3)|$  under the action of  $G := (\text{SL}(2) \times \text{SL}(2)) \rtimes \mathbb{Z}_2$  by applying the Hilbert-Mumford numerical criterion [MFK94, Chapter 2.1]. In Section 2.1 we describe equations of (semi)stable and nonsemistable points. The geometric consequences of these results are then collected in Section 2.2.

**2.1. Numerical criterion.** Choose projective coordinates  $X, Y, Z, W$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Consider then one-parameter subgroup  $\rho_{u,v}: \text{Spec } \mathbb{C}[t, t^{-1}] \rightarrow G$  that acts via

$$t \cdot (X, Y, Z, W) = (t^u X, t^{-u} Y, t^v Z, t^{-v} W),$$

where we assume that  $u \geq v \geq 0$ .

With respect to  $\rho_{u,v}$ , the monomial  $X^i Y^{3-i} Z^j W^{3-j}$  has weight  $(2i - 3)u + (2j - 3)v$ . The monomials of positive weight are those with:

- (1)  $i = 3, j \geq 1$ .
- (2)  $i = 3, j = 0$  if  $u > v$ .

- (3)  $i = 2, j \geq 2$ .
- (4)  $i = 2, j = 1$  if  $u > v$ .
- (5)  $i = 2, j = 0$  if  $u > 3v$ .
- (6)  $i = 1, j = 3$  if  $u < 3v$ .

It follows that the general *nonsemistable* point is nonsemistable either with respect to the one-parameter subgroup  $\rho_{2,1}$  or with respect to  $\rho_{4,1}$ . We record that in the affine coordinates  $x = X/Y$  and  $z = Z/W$  the general nonsemistable point with respect to  $\rho_{2,1}$  has equation

$$(2.1) \quad x(cz^3 + a_0xz + a_1xz^2 + a_2xz^3 + b_1x^2 + b_2x^2z + b_3x^2z^2 + b_4x^2z^3) = 0.$$

Similarly, the general nonsemistable point with respect to  $\rho_{4,1}$  is

$$(2.2) \quad x^2(1 + a_0z + a_1z^2 + a_2z^3 + b_1x + b_2xz + b_3xz^2 + b_4xz^3) = 0.$$

We proceed to describe strictly semistable points. It is clear from the above list of monomials that the only one-parameter subgroups  $\rho_{u,v}$  with respect to which there are monomials of degree 0 are  $\rho_{1,1}$  and  $\rho_{3,1}$ . The degree 0 monomials with respect to  $\rho_{3,1}$  are

- (1)  $X^2YW^3$ , which become positive if  $u > 3v$ .
- (2)  $XY^2Z^3$ , which becomes negative if  $u > 3v$ .

Since there are only two monomials of weight 0 when  $u = 3v$ , any curve which is strictly semistable with respect to  $\rho_{3,1}$  and has a closed orbit is unique up to automorphisms of  $\mathbb{P}^1 \times \mathbb{P}^1$  and is defined by the equation

$$(2.3) \quad X^2YW^3 + XY^2Z^3 = XY(XW^3 + YZ^3) = 0.$$

The degree 0 monomials with respect to  $\rho_{1,1}$  are

- (1)  $X^3W^3$ , which becomes positive for  $u > v$  and negative for  $u < v$ .
- (2)  $X^2YZW^2$ , which becomes positive for  $u > v$  and negative for  $u < v$ .
- (3)  $XY^2Z^2W$ , which becomes negative for  $u > v$  and positive for  $u < v$ .
- (4)  $Y^3Z^3$ , which becomes negative for  $u > v$  and positive for  $u < v$ .

Thus, strictly semistable points with respect to  $\rho_{1,1}$  have form

$$ax^3 + bx^2z + cxz^2 + dz^3 + ex^2z^2 + fxz^3 + gx^2z^3 + hx^3z^3 = 0$$

Following such curves to the flat limit under  $\rho_{1,1}$ , we see that every strictly semistable point isotrivially specializes to a curve  $ax^3 + bx^2z + cxz^2 + dz^3 = 0$ , or, in projective coordinates, to a curve

$$(2.4) \quad L_1L_2L_3 = aX^3W^3 + bX^2W^2ZY + cXWZ^2Y^2 + dZ^3Y^3 = 0,$$

where  $(a, b)$  are not simultaneously zero and  $(c, d)$  are not simultaneously zero. (Here,  $L_i$  are homogeneous forms of bidegree  $(1, 1)$ .)

**Proposition 2.1.** *The orbit closure of every semistable curve with a multiplicity 3 singularity contains its “tangent cone,” which is a curve described by Equation (2.4). In particular, the orbit closure of every semistable curve with a  $D_8$  singularity contains a double conic, i.e. a curve defined by  $(ax + bz)^2(cx + dz) = 0$ .*

*Proof.* Let  $C$  be a semistable curve with a multiplicity 3 singularity. Choose coordinates  $X, Y, Z, W$  so that the equation of  $C$  in the affine coordinates  $x := X/Y$  and  $z := Z/W$  is  $f_3 + f_{\geq 4} = 0$ , where  $f_3$  is a homogeneous polynomial of degree 3 in  $x$  and  $z$ , and  $f_{\geq 4} \in (x, z)^4$ . Since  $C$  is semistable, we have that  $x^2, z^2 \nmid f_3$ ; otherwise  $C$  would be defined by Equation (2.1). Then the flat limit of  $C$  under  $\rho_{1,1}$  is  $f_3(x, z) = 0$  or, in projective coordinates,

$$aX^3W^3 + bX^2W^2ZY + cXWZ^2Y^2 + dZ^3Y^3 = 0,$$

where  $(a, b)$  are not simultaneously zero and  $(c, d)$  are not simultaneously zero.

If the triple point is a  $D_8$  singularity, the tangent cone is a union of a double line and a transverse line, i.e.  $f_3(x, z) = (ax + bz)^2(cx + dz)$ .  $\square$

**2.2. Stability analysis.** We summarize the calculations of the previous section and reinterpret them in a geometric language. First, we describe nonsemistable curves.

**Proposition 2.2** (Nonsemistable curves). *A curve  $C$  is nonsemistable if and only if one of the following holds:*

- (1)  $C$  contains a double ruling.
- (2)  $C$  contains a ruling and the residual curve  $C'$  intersects this ruling in a unique point that is also a singular point of  $C'$ .

*Proof.* Nonsemistable curves are precisely those curves that are defined by Equations (2.1) and (2.2) for some choice of coordinates. Equation (2.1) defines a reducible curve  $C = C_1 \cup C_2$ , where  $C_1$  is a ruling of  $\mathbb{P}^1 \times \mathbb{P}^1$  that intersects the residual (2, 3) curve  $C_2$  with multiplicity 3 at the singular point  $x = z = 0$  of  $C_2$ . Finally, Equation (2.2) defines a curve with a double ruling.  $\square$

**Corollary 2.3.** *The only non-reduced semistable curves are:*

- (1) *Triple conics; these are strictly semistable and have closed orbits.*
- (2) *A union of a smooth double conic and a conic which is nonsingular along the double conic; these are strictly semistable and have closed orbits.*

*Proof.* The proof is immediate: By Proposition 2.2 and degree considerations, any non-reduced structure has to be supported along a smooth conic. If the generic multiplicity is 3, then the curve is a triple conic. If it is 2, then the residual curve cannot be a union of two rulings meeting along the double conic, also by Proposition 2.2.  $\square$

**Remark 2.4** (Double conics). For brevity we call the curve described in Part (2) of Corollary 2.3 a *double conic*. Given a double conic  $C = 2C_1 + C_2$ , we consider the horizontal rulings  $L_1$  and  $L_2$  passing through two triple points  $C_1 \cap C_2$ . Then the four points of intersection of the general vertical ruling with  $C_1, C_2, L_1, L_2$  define a cross-ratio which is an  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ -invariant of  $C$ . We call it the *cross-ratio of the double conic  $C$* .

Any other curve that does not fit the description of Proposition 2.2 is semistable. Recall that there is a unique closed orbit of semistable curves which is strictly semistable with respect to  $\rho_{3,1}$ . This curve is defined by Equation (2.3). We call such a curve the

*maximally degenerate  $A_5$ -curve.* It consists of two lines in the same ruling  $(1, 0)$  and a smooth  $(1, 3)$  curve meeting each line at an  $A_5$  singularity (i.e. tangent with multiplicity 3); see Figure 1.

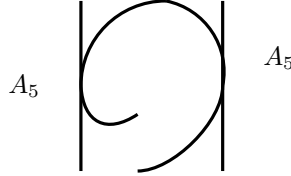


FIGURE 1. A maximally degenerate  $A_5$ -curve

Recall from above that every semistable curve which is strictly semistable with respect to  $\rho_{1,1}$  and has a closed orbit is given by Equation (2.4). Every such curve is a union of 3 conics in the class  $(1, 1)$ , all meeting at points  $0 \times 0$  and  $\infty \times \infty$ . In addition, by scaling  $X$  and  $Z$ , we can assume that either  $a = d = 1$ , or  $a = d = 0$  and  $b = c = 1$ . This leaves us with a 2-dimensional family of strictly semistable points

$$x^3 + bx^2z + cxz^2 + z^3 = 0,$$

or  $x^2z + xz^2 = xz(x + z) = 0$ . We call such curves *D-curves*, because the generic D-curve has two ordinary triple point ( $D_4 : x^3 = y^3$ ) singularities.

We can now restate Proposition 2.1 in a geometric language:

**Proposition 2.5.** *The orbit closure of every semistable curve with a multiplicity 3 singularity contains a D-curve described by Equation (2.4), i.e. either a union of three conics at two  $D_4$  singularities, or a double conic, or a triple conic.*

**Remark 2.6.** We also note that every non-reduced semistable curve is a D-curve: *triple conics* arise from Equation 2.4 by taking  $L_1 = L_2 = L_3$  and *double conics* arise by taking  $L_1 = L_2$ .

**2.3. Geometry of semistable curves.** We refine the GIT analysis to obtain a list of geometrically meaningful strata inside the semistable locus. We begin with strictly semistable points in the highest stratum:

**2.3.1. D-curves:** A D-curve is a curve defined by Equation (2.4). These are precisely strictly semistable curves (with closed orbits) which consist of three conics in class  $(1, 1)$  passing through two points of  $\mathbb{P}^1 \times \mathbb{P}^1$  not on the same ruling, i.e. three conics meeting in two  $D_4$  (ordinary triple points) singularities (see Figure 2). By Proposition 1.3 ( $D_{\text{even}}$ ) and Proposition 1.4, the variety of stable limits of the general D-curve is the locus of *elliptic triboroughs* in  $\overline{M}_4$ , i.e. nodal unions of two elliptic components along three points.

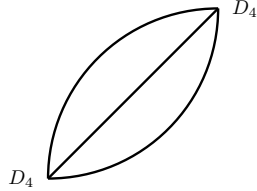


FIGURE 2. A D-curve

2.3.2. *Curves with separating  $A_5$  singularities:* Such curves are necessarily of the form  $C = C_1 \cup C_2$  where  $C_1$  is a ruling in class  $(1,0)$  and  $C_2$  is a curve in class  $(2,3)$  that intersects  $C_1$  with multiplicity 3 at a smooth point of  $C_2$ . It is easy to see that all such curves contain the maximally degenerate  $A_5$ -curve in their orbit closure. From Proposition 1.3 ( $A_{\text{odd}}$ ) and Proposition 1.4, we conclude that the variety of stable limits of the maximally degenerate  $A_5$ -curve is all of  $\Delta_2 \subset \overline{M}_4$ .

2.3.3. *Double conics:* These are defined by the equation  $L_1^2 L_2 = 0$ , where  $L_1$  is an irreducible form of bidegree  $(1,1)$  and  $L_2$  is a form of bidegree  $(1,1)$  that meets  $L_1$  in two distinct points. Double conics form a closed locus inside the locus of D-curves.

2.3.4. *Curves with  $D_8$  singularities:* Consider a curve  $C = C_1 \cup C_2$ , where  $C_2$  is a nodal curve in class  $(2,2)$  and  $C_1$  is a smooth curve in class  $(1,1)$  that intersects one of the branches of the node of  $C_2$  with multiplicity 3. These curves do not have closed orbits: they isotrivially specialize to double conics by Proposition 2.1. The reason we single out this class of curves is that it follows immediately from Proposition 1.3 ( $D_{\text{even}}$ ) that the variety of stable limits of a  $D_8$ -curve is the closure of the locus of irreducible nodal curves with a hyperelliptic normalization. Denote this locus by  $\Delta_0^{\text{hyp}}$ . Then  $\Delta_0^{\text{hyp}}$  is divisorial inside the Petri divisor  $P$ : it is the locus of canonical genus 4 curves lying on a singular quadric and passing through its vertex. Note that  $\Delta_0 \cap P$  has two irreducible components, with  $\Delta_0^{\text{hyp}}$  being one of them.

Since  $D_8$ -curves specialize isotrivially to double conics,  $\Delta_0^{\text{hyp}}$  also lies in the variety of stable limits of double conics. Observing that double conics have moduli (see Remark 2.4), we conclude that the rational map from  $\overline{M}_4$  to the GIT quotient  $V^{\text{ss}}//G$  is undefined along  $\Delta_0^{\text{hyp}} \subset P$ .

2.3.5. *Triple conics:* These form a single (closed) orbit of curves defined by the equation  $L^3 = 0$ , where  $L$  is an irreducible form of bidegree  $(1,1)$ . The corresponding point in  $V^{\text{ss}}//G$  lies in the closure of double conics. Among semistable curves whose orbit closure contains the orbit of the triple conic are curves with a  $J_{10}$  singularity  $y^3 = x^6$  defined by the equation  $L_1 L_2 L_3 = 0$ , where  $L_i$  are forms of bidegree  $(1,1)$  such that the corresponding conics all meet in a single point:

$$(x + z + c_0xz)(x + z + c_1xz)(x + z + c_2xz) = 0.$$

Evidently, all such curves isotrivially specialize to the triple conic.

**2.3.6. Curves with  $E_6, E_7, E_8$  singularities:** None of these have closed orbits: These arise as deformations of curves with  $J_{10}$  singularities (see [Arn76, Section I.1] or [Jaw87]) and in fact isotrivially specialize to the triple conic by Proposition 2.1. (Note, however, that the variety of stable limits of  $E_6$  is the locus of  $[C_1 \cup C_2] \in \Delta_1$  such that  $C_1 \cap C_2$  is a hyperflex of the genus 3 curve  $C_2$ ; see [Has00, Theorem 6.2].)

We proceed to describe geometrically meaningful strata inside the stable locus:

**2.3.7. Curves with  $A_8$  and  $A_9$  singularities:** Consider an  $A_8$ -curve  $C$  (see [FJ11] for a general background on canonical  $A$ -curves) defined parametrically by

$$t \mapsto [1 - 3t + 3t^3, (1 - 3t + 3t^3)(t^2 + t^3), t^2(1 - 2t), t^4(1 - t - 2t^2)].$$

This curve is a complete intersection of the smooth quadric  $z_0z_3 = z_1z_2$  and a cubic in  $\mathbb{P}^3$ . The only singularity of  $C$  is of type  $A_8$  and  $C$  has a rational normalization. Locally around  $[C]$ , the locus of curves with  $A_8$  singularities is the fiber of a smooth map  $\text{Hilb}(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow \text{Def}(A_8)$  (see Proposition 1.4). Denote by  $A$  the closure of the locus of  $A_8$ -curves in  $V^{\text{ss}}//G$ . Counting dimensions, we conclude that  $\dim(A) = 1$ . Since  $J_{10}$  singularity deforms to  $A_8$  by [Jaw87], we see that  $A$  passes through the triple conic.

By Proposition 1.3 ( $A_{\text{even}}$ ), the variety of stable limits of an  $A_8$ -curve is the hyperelliptic locus  $\overline{H}_4 \subset \overline{M}_4$ . We will see in Theorem 3.16 that  $f: \overline{M}_4 \dashrightarrow V^{\text{ss}}//G$  flips  $\overline{H}_4$  to  $A$ .

There is another distinguished point in  $A$ , which corresponds to a union of  $(2, 1)$  and  $(1, 2)$  curve at an  $A_9$  singularity. Up to projectivities, there is a unique  $A_9$ -curve. It is defined parametrically by

$$\begin{pmatrix} 1 & s & s^2 & s^3 \\ 1 - 3t + 3t^2 + t^3 & t(1 - 2t + t^2) & t^2 - t^3 & t^3 \end{pmatrix}.$$

The variety of stable limits of the  $A_9$ -curve is  $\overline{H}_4 \subset \overline{M}_4$  by Proposition 1.3 ( $A_{\text{odd}}$ ).

**2.3.8. Curves with  $A_6$  or  $A_7$  singularities:** Curves with non-separating  $A_7$  singularities ( $y^2 = x^8$ ) replace curves in  $\Delta_0 \cap \overline{H}_4$ , i.e. curves whose normalization is hyperelliptic and such that points lying over the node are conjugate.

Curves with separating  $A_7$  singularities (smooth  $(1, 1)$  and  $(2, 2)$  curves meeting with multiplicity 4 at a single point) replace curves in  $\Delta_1$  with a hyperelliptic genus 3 component.

Curves with  $A_6$  singularities replace curves in  $\Delta_1$  with a hyperelliptic genus 3 component attached at a Weierstrass point.

**2.3.9.** Finally, using Proposition 1.3 we see that: Curves with non-separating  $A_5$ -singularities replace hyperelliptic admissible covers with two irreducible components of genus 1 and 2. Curves with  $A_4$  singularities replace curves with Weierstrass genus 2 tails, i.e., curves that are a nodal union of two genus 2 curves in a point which is a Weierstrass

point on one of the components. Curves with  $A_3$  singularities replace curves with elliptic bridges. Curves with  $A_2$  singularities replace curves with elliptic tails.

2.3.10. We summarize this section by collecting the observations regarding which singular curves in  $M$  replace which geometrically meaning loci in  $\overline{M}_4$  under the rational map  $f: \overline{M}_4 \dashrightarrow M$ . The list of singular curves discussed in this section, together with their varieties of stable limits is given in Table 1. We note that this list is not exhaustive and, for example, does not include all possible boundary strata flipped by  $f$ .

Singularity type introduced	Locus removed
$A_2$	elliptic tails attached nodally
$A_3$	elliptic bridges attached nodally
$A_4$	genus 2 tails attached nodally at a Weierstrass point
non-separating $A_5$	genus 2 bridges attached nodally at conjugate points
the maximally degenerate $A_5$ -curve	$\Delta_2$
$A_6$	hyperelliptic genus 3 tails attached nodally at a Weierstrass point
$A_7$	hyperelliptic genus 3 tails attached nodally
$D_4$	elliptic triboroughs
double conics	curves in $\Delta_0$ with a hyperelliptic normalization
$A_8, A_9$	hyperelliptic curves
the triple conic	the Petri divisor $P$

TABLE 1. Replacing varieties of stable limits by singular curves

### 3. BIRATIONAL GEOMETRY OF $\overline{M}_4$

As before,  $V = |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, 3)|$ ,  $G = \mathrm{SL}(2) \times \mathrm{SL}(2) \rtimes \mathbb{Z}_2$ , and  $M = V^{\mathrm{ss}} // G$ . The GIT quotient  $\phi: V^{\mathrm{ss}} \rightarrow M$  has been described in detail in the previous section, where we have seen that the natural map  $f: \overline{M}_4 \dashrightarrow M$  is birational. We now describe how  $M$  fits into the Hassett-Keel program for  $\overline{M}_4$ .

**Proposition 3.1.** *The rational map  $f: \overline{M}_4 \dashrightarrow M$  is a contraction, i.e.  $f^{-1}$  does not contract divisors.*

*Proof.* It suffices to show that  $f^{-1}$  is defined away from codimension 2 locus. Consider the  $G$ -invariant closed subscheme

$$B := \{[C] \in V \mid C \text{ has worse than nodal singularities}\}.$$

Clearly, curves in  $V^{\mathrm{ss}} \setminus B$  are moduli semistable. The existence of the stabilization morphism implies that the morphism  $V^{\mathrm{ss}} \setminus B \rightarrow \overline{M}_4$  is well-defined. Since this morphism is  $G$ -invariant and  $\phi: V^{\mathrm{ss}} \rightarrow M$  is a GIT quotient, we see that  $f^{-1}$  is a regular morphism

on  $M \setminus \phi(B)$ . As the discriminant divisor in  $V$  is irreducible with the geometric generic point a nodal curve, we deduce that every irreducible component of  $B$  has codimension at least 2 inside  $V^{\text{ss}}$ . Thus  $\text{codim}(\phi(B), M) \geq 2$ .  $\square$

**Proposition 3.2.** *On  $V \simeq \mathbb{P}^{15}$ , we have  $\delta = \mathcal{O}_V(34)$  and  $\lambda = \mathcal{O}_V(4)$ .*

*Proof.* Let  $\mathcal{C} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times V$ , together with the projection  $\text{pr}_3: \mathcal{C} \rightarrow V$ , be the universal  $(3, 3)$  curve. Set  $H_i = \text{pr}_i^* \mathcal{O}(1)$ . Since  $\mathcal{C}$  is a smooth  $(3, 3, 1)$ -divisor on  $\mathbb{P}^1 \times \mathbb{P}^1 \times V$ , we obtain by adjunction that  $\omega_{\mathcal{C}/V} = \mathcal{O}_{\mathcal{C}}(1, 1, 1)$ . By pushing forward via  $\text{pr}_3$  the exact sequence

$$0 \rightarrow \mathcal{O}(-2, -2, 0) \rightarrow \mathcal{O}(1, 1, 1) \rightarrow \mathcal{O}_{\mathcal{C}}(1, 1, 1) \rightarrow 0,$$

we deduce that

$$\lambda = c_1((\text{pr}_3)_* \mathcal{O}_{\mathcal{C}}(1, 1, 1)) = c_1((\text{pr}_3)_* \mathcal{O}(1, 1, 1)) = 4H_3.$$

We also compute that

$$\kappa = (\text{pr}_3)_*(\omega_{\mathcal{C}/V}^2) = (\text{pr}_3)_*((3H_1 + 3H_2 + H_3)(H_1 + H_2 + H_3)^2) = 14H_3.$$

Using Mumford's formula  $\lambda = (\kappa + \delta)/12$ , we conclude that  $\delta = 34H_3$ . The claim follows.  $\square$

**Corollary 3.3.** *On  $M = V^{\text{ss}}//G$ , we have  $\delta = \mathcal{O}_M(34)$  and  $\lambda = \mathcal{O}_M(4)$ , where  $\mathcal{O}_M(1)$  is the GIT polarization coming from  $\mathcal{O}_V(1)$ .*

*Proof.* Since  $\lambda$  and  $\delta$  are  $G$ -invariant divisor classes, the results of Proposition 3.2 descend to  $M$ .  $\square$

**3.1. Proof of Main Theorem.** We have seen that  $f: \overline{M}_4 \dashrightarrow M$  is a birational contraction in Proposition 3.1. Theorem 3.13 below shows that  $f$  contracts the Petri divisor  $P$ , and the boundary divisors  $\Delta_1, \Delta_2$ . Thanks to the GIT analysis in Section 2,  $f$  is well-defined at all points of  $\overline{M}_4 \setminus (\Delta \cup P)$  and is well-defined at the generic point of  $\Delta_0$ . It follows that  $f$  is defined at the generic point of every irreducible divisor in  $\overline{M}_4$  with the exception of  $P, \Delta_1$ , and  $\Delta_2$ .

**Lemma 3.4** (Petri divisor). *The divisor class of the Petri divisor is*

$$P = 17\lambda - 2\delta_0 - 7\delta_1 - 9\delta_2.$$

*Proof.* This is an instance of Theorem 2 of [EH83] for  $g = 4$ .  $\square$

**Proposition 3.5.** *We have*

$$(3.1) \quad f^* \lambda = f_* f^* \lambda = \lambda + \delta_1 + 3\delta_2 + 7P,$$

$$(3.2) \quad f^* \delta = f_* f^* \delta = \delta_0 + 12\delta_1 + 30\delta_2 + 60P = \delta + 11\delta_1 + 29\delta_2 + 60P.$$

*Proof.* We begin by writing

$$\begin{aligned} f^* \lambda &= \lambda + a_0 \delta_1 + b_0 \delta_2 + c_0 P, \\ f^* \delta &= \delta_0 + a_1 \delta_1 + b_1 \delta_2 + c_1 P. \end{aligned}$$

We will now find indeterminate coefficients by using judiciously chosen test families. A care needs to be exercised to use curves  $T \subset \overline{\mathcal{M}}_4$  such that  $f$  is defined along  $T$  and such that  $f(T)$  is a point. We prove that our test families satisfy these requirements in Theorem 3.13 below.

### Test families.

3.1.1. *Elliptic tails:* Our first test family  $T_1$  is obtained by attaching the family of varying elliptic tails to the general pointed curve of genus 3. The rational map  $f: \overline{\mathcal{M}}_4 \dashrightarrow M$  is defined in the neighborhood of  $T_1$  and contracts  $T_1$  to a point by Theorem 3.13 (2). We also have:

$$(3.3) \quad \lambda \cdot T_1 = 1, \quad \delta_0 \cdot T_1 = 12, \quad \delta_1 \cdot T_1 = -1, \quad \delta_2 \cdot T_1 = P \cdot T_1 = 0.$$

It follows that  $a_0 = 1$  and  $a_1 = 12$ .

3.1.2. *Genus 2 tails:* Consider now the family  $T_2$  of irreducible genus 2 tails attached at non-Weierstrass points. The intersection numbers of this family are standard and are written down in [FS10] Table 4.2.2:

$$(3.4) \quad \lambda \cdot T_2 = 3, \quad \delta_0 \cdot T_2 = 30, \quad \delta_1 \cdot T_2 = 0, \quad \delta_2 \cdot T_2 = -1, \quad P \cdot T_2 = 0.$$

By Theorem 3.13 (3) we have that  $f$  is defined in the neighborhood of  $T_2$  and  $f(T_2)$  is a point. It follows that  $b_0 = 3$  and  $b_1 = 10$ .

3.1.3. *Petri curves:* We now take the family  $T_3$  of Petri-special curves on  $\mathbb{P}(1, 1, 2)$  defined by the weighted homogeneous equation

$$y^3 = x^6 + axyz^3 + bz^6,$$

where  $[a : b] \in \mathbb{P}(1, 2)$ . Evidently, these curves are at worst nodal and avoid the vertex  $[0 : 0 : 1]$  of the singular quadric. Thus  $f$  is defined in the neighborhood of  $T_3$  by Theorem 3.13 (1) and  $f(T_3)$  is a point. The intersection numbers of this family are computed in [AFS10, Proposition 6.6] and are as follows:

$$(3.5) \quad \lambda \cdot T_3 = 7, \quad \delta_0 \cdot T_3 = 60, \quad \delta_1 \cdot T_3 = \delta_2 \cdot T_3 = 0.$$

From Lemma 3.4, compute  $P \cdot T_3 = 119 - 120 = -1$ . Thus  $c_0 = 7$  and  $c_1 = 60$ .  $\square$

**Corollary 3.6.** *We have  $\overline{\mathcal{M}}_4(\alpha) \simeq M$  for all  $\frac{8}{17} < \alpha \leq 29/60$ . Moreover,  $\overline{\mathcal{M}}_4(8/17)$  is a point and  $\overline{\mathcal{M}}_4(\alpha) = \emptyset$  if  $\alpha < \frac{8}{17}$ .*

*Proof.* Let  $f: \overline{\mathcal{M}}_4 \dashrightarrow M$  be the rational contraction. Then

$$f_*(K_{\overline{\mathcal{M}}_4} + \alpha\delta) = f_*(13\lambda - (2 - \alpha)\delta) = 13\lambda - (2 - \alpha)\delta.$$

We now compute using Proposition 3.5

$$\begin{aligned} (K_{\overline{\mathcal{M}}_4} + \alpha\delta) - f^*f_*(K_{\overline{\mathcal{M}}_4} + \alpha\delta) &= (13\lambda - (2 - \alpha)\delta) - f^*f_*(13\lambda - (2 - \alpha)\delta) \\ &= -13(\delta_1 + 3\delta_2 + 7P) + (2 - \alpha)(11\delta_1 + 29\delta_2 + 60P) \\ &= (29 - 60\alpha)P + (19 - 29\alpha)\delta_2 + (9 - 11\alpha)\delta_1. \end{aligned}$$

This is an effective exceptional divisor as long as  $\alpha \leq 29/60$ . It follows that for  $\alpha \leq 29/60$ :

$$\begin{aligned} \overline{M}_4(\alpha) &= \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{\mathcal{M}}_g, m(K_{\overline{\mathcal{M}}_g} + \alpha\delta)) \\ &= \text{Proj} \bigoplus_{m \geq 0} H^0(M, mf_*(K_{\overline{\mathcal{M}}_g} + \alpha\delta)) \\ &= \text{Proj} \bigoplus_{m \geq 0} H^0(M, m(13\lambda - (2 - \alpha)\delta)) = \text{Proj} \bigoplus_{m \geq 0} H^0(M, \mathcal{O}_M(34\alpha - 16)), \end{aligned}$$

where we have used Corollary 3.3 in the last step. The statement now follows from the fact that  $\mathcal{O}_M(34\alpha - 16)$  is ample on  $M$  for  $\alpha > 8/17$  and is a zero line bundle for  $\alpha = 8/17$ .  $\square$

**Corollary 3.7** (cf. Theorem 1.1). *There is a moving divisor of class  $60\lambda - 7\delta_0 - 24\delta_1 - 30\delta_2$  on  $\overline{M}_4$ . Furthermore, any moving divisor  $D = a\lambda - b_0\delta_0 - b_1\delta_1 - b_2\delta_2$  satisfies  $a/b_0 \geq 60/7$ . In particular, the moving slope of  $\overline{M}_4$  is  $60/7$ .*

*Proof.* By Proposition 3.5,

$$f^*(60\lambda - 7\delta) = 60\lambda - 7\delta_0 - 24\delta_1 - 30\delta_2.$$

By Corollary 3.3 the divisor  $60\lambda - 7\delta = \mathcal{O}_M(2)$  is ample. Since  $f$  is a rational contraction, the divisor  $f^*(60\lambda - 7\delta)$  is moving on  $\overline{M}_4$ .

Suppose now  $D$  is a moving divisor. Since families  $T_1, T_2, T_3$  constructed in the proof of Proposition 3.5 are covering families for divisors  $\Delta_1, \Delta_2$ , and  $P$ , respectively, we have

$$\begin{aligned} D \cdot T_3 \geq 0 &\implies \frac{a}{b_0} \geq \frac{60}{7}, \\ D \cdot T_2 \geq 0 &\implies 3a - 30b_0 + b_2 \geq 0, \\ D \cdot T_1 \geq 0 &\implies a - 12b_0 + b_1 \geq 0. \end{aligned}$$

The statement follows.  $\square$

**3.2. A theorem on indeterminacy locus.** Here, we describe loci where the rational map  $f: \overline{M}_4 \dashrightarrow M = V^{\text{ss}}//G$  is regular, completing the proof of Proposition 3.5. Recall that  $\phi: V^{\text{ss}} \rightarrow M$  is the GIT quotient.

**Definition 3.8.** Let  $\overline{M}_4 \xleftarrow{p} Z \xrightarrow{q} M$  be the graph of  $f$ . Recall that the variety of stable limits of  $[X] \in V^{\text{ss}}$  is  $\mathcal{T}_X = p(q^{-1}(\phi([X])))$ . We now define the *variety of GIT-semistable limits* of  $[C] \in \overline{M}_4$  to be  $\mathcal{D}_C := q(p^{-1}([C]))$ .

**Remark 3.9.** We allow  $[X]$  to be a non-closed point of the GIT stack  $[V^{\text{ss}}/G]$ .

**Lemma 3.10.** *Suppose that for  $[C] \in \overline{M}_4$ , we have  $[X] \in \mathcal{D}_C$ . Suppose further that for every closed point  $[X']$  in a small punctured neighborhood of  $[X]$  in  $M$ , we have that  $[C] \notin \mathcal{T}_{X'}$ . Then  $f$  is defined at  $[C]$  and  $f([C]) = [X]$ .*

*Proof.* Because both  $\overline{M}_4$  and  $M$  are normal varieties (the latter by [MFK94, Theorem 1.1]),  $\mathcal{D}_C$  is connected. Using the obvious implication

$$[X'] \in \mathcal{D}_C \implies [C] \in \mathcal{T}_{X'},$$

we conclude that  $\mathcal{D}_C$  does not meet a small punctured neighborhood of  $[X]$ . It follows that  $\mathcal{D}_C = [X]$ . Thus,  $f$  is defined at  $[C]$  and  $f([C]) = [X]$ .  $\square$

**Definition 3.11.** We define by  $P^\circ \subset \overline{M}_4$  the locally closed subset of stable curves whose canonical embedding lies on a singular quadric, but which do not pass through the vertex of the quadric.

We define by  $\Delta_2^\circ \subset \overline{M}_4$  the locally closed subset of stable curves  $[E_1 \cup E_2] \in \Delta_2$  such that  $E_1$  and  $E_2$  are irreducible and  $E_1 \cap E_2$  is not a Weierstrass point<sup>1</sup> either on  $E_1$  or  $E_2$ .

**Remark 3.12.** If  $[C] \in P^\circ \cap \Delta_0$  is a curve with a single node, then  $\tilde{C}$  is not hyperelliptic. Conversely, if  $[C] \in \Delta_0 \setminus \overline{H}_4$  is a curve with a single node and  $\tilde{C}$  is hyperelliptic, then  $[C] \in P \setminus P^\circ = \Delta_0^{\text{hyp}}$  because the node maps under the canonical embedding to the vertex of the singular quadric containing  $C$ .

**Theorem 3.13.** *The rational map  $f: \overline{M}_4 \dashrightarrow M$  is regular at the following points:*

- (1) *All curves in  $P^\circ$ . Moreover,  $f$  maps  $P^\circ$  to the triple conic.*
- (2) *All curves in  $\Delta_1$  whose genus 3 component is the general curve in  $\overline{\mathcal{M}}_{3,1}$ . Moreover, if the pointed genus 3 component is fixed, all curves with varying elliptic tails are mapped to the same cuspidal curve in  $M$ .*
- (3) *All curves in  $\Delta_2^\circ$ . Moreover,  $f$  maps  $\Delta_2^\circ$  to the maximally degenerate  $A_5$ -curve.*

*Proof.* To show that  $f$  is defined at a point of  $\overline{M}_4$ , we employ three different techniques: In the case of  $P^\circ$ , we define the map explicitly; in the case of  $\Delta_1$ , we use the moduli space of pseudostable curves (see [Sch91, HH09]); lastly, in the case of  $\Delta_2^\circ$ , we use varieties of stable limits.

*Curves in  $P^\circ$ :* Suppose  $C$  is a stable curve lying on a rank three quadric and avoiding the vertex. We prove that  $f$  is defined at  $C$  by showing that for *every* smoothing of  $C$  away from the Petri locus, the GIT-semistable limit is the triple conic. Indeed, let  $\mathcal{C} = \{C_t\}$  be a smoothing of  $C$ , with  $C_0 = C$  and  $C_t$  smooth Petri-general curves for all  $t \neq 0$ . Realize  $\{C_t\}$  as a family of canonically embedded curves by choosing a trivialization of the Hodge bundle. Associated to this family of canonical curves is the family of quadrics  $\{Q_t\}$ , with  $Q_0$  a singular quadric and  $Q_t$  a smooth quadric for  $t \neq 0$ . We choose coordinates so that the equation for  $Q_t$  is

$$Q_t : \{z_0^2 + z_1^2 + z_2^2 + t^a z_3^2 = 0\},$$

where we arrange  $a$  to be even using a finite base change. Consider now the one-parameter family  $\rho: \mathbb{C}[t, t^{-1}] \rightarrow \text{GL}(4)$  given by  $\rho(t) = \text{diag}(1, 1, 1, t^{a/2-1})$ . Then the family of

<sup>1</sup>A Weierstrass point of an irreducible stable curve of genus 2 is a ramification point of the canonical  $2:1$  map onto  $\mathbb{P}^1$ .

canonical curves  $C'_t := \rho(t)(C_t)$  lies on the family of quadrics defined by the equation

$$Q'_t : \{z_0^2 + z_1^2 + z_2^2 + t^2 z_3^2 = 0\}.$$

Abstractly,  $C'_t \simeq C_t$  for  $t \neq 0$  and so the stable limit of  $\{C'_t\}$  at  $t = 0$  is still  $C_0$ . The flat limit  $C'_0 := \lim_{t \rightarrow 0} C'_t$  remains  $C_0$  if  $a = 2$ , and is the triple conic  $z_3 = 0$  on  $Q'_0 = Q_0$  if  $a \geq 4$ . In either case,  $C'_0$  does not pass through the vertex of  $Q'_0$ . Consider now the blow-up  $\mathcal{X}' := \text{Bl}_p \mathcal{X}$  of the total space  $\mathcal{X}$  of  $\{Q'_t\}$  at  $p = [0 : 0 : 0 : 1]$ . The exceptional divisor  $E$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and meets the strict transform of  $Q'_0$  in a smooth conic  $O$ . The strict transform of  $Q'_0$  is isomorphic to  $\mathbb{F}_2$ , with  $O$  being the  $(-2)$  curve. We now blow-down  $\mathbb{F}_2$  down to  $O$ . The resulting threefold is the total space of the family of  $\mathbb{P}^1 \times \mathbb{P}^1$ 's with central fiber  $E$ . The flat limit of  $\{C'_t\}_{t \neq 0}$  in  $E$  is now the triple conic  $O$ .

**Remark 3.14.** It is clear what goes wrong when  $[C_0] \in \Delta_0^{\text{hyp}}$ , i.e. when  $C_0$  passes through the vertex of  $Q_0$ . The exceptional divisor  $E$  then meets the strict transform  $\overline{\{C_t\}}$  in a conic  $O'$ . As a result, after the blow-down, the flat limit of  $\{C'_t\}_{t \neq 0}$  in  $E$  is a union of the double conic  $O$  and  $O'$ . The indeterminacy of  $f$  along  $\Delta_0^{\text{hyp}}$  arises because the cross-ratio (see Remark 2.4) of the double conic curve depends on the smoothing  $\mathcal{C}$ .

*Curves with elliptic tail:* Since there exists a morphism  $\overline{M}_4 \rightarrow \overline{M}_4^{\text{ps}}$  (see [HH09]), it suffices to show that the morphism from  $\overline{M}_4^{\text{ps}}$  to  $M$  is well-defined at the general cuspidal curve. This immediately follows from the fact that a cuspidal curve  $C$  whose pointed normalization is the general curve in  $\overline{\mathcal{M}}_{3,1}$  is embedded by  $\omega_C$  into a smooth quadric in  $\mathbb{P}^3$ .

*Curves in  $\Delta_2^\circ$ :* Consider the maximally degenerate  $A_5$ -curve  $X$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . By Proposition 1.3 and Proposition 1.4, the variety of stable limits of  $X$  contains all curves  $[E_1 \cup E_2] \in \Delta_2$  such that  $E_1 \cap E_2$  is not a Weierstrass point on either  $E_1$  or  $E_2$ .

Consider now a small deformation  $X'$  of  $X$ . If one of the  $A_5$ -singularities of  $X$  is preserved, then the singularity remains separating on  $X'$ . It follows that  $X'$  is a union of a  $(1,0)$ -ruling and a residual  $(2,3)$ -curve tangent to the ruling with multiplicity 3 at a smooth point. Such a curve is necessarily defined by Equation (2.3) and hence  $[X'] = [X] \in M$ . Suppose both  $A_5$  singularities are smoothed in  $X'$ . Then  $X'$  has at worst  $A_1, A_2, A_3, A_4$  singularities. By Proposition 1.3, the tails of stable limits arising from  $A_2$  and  $A_3$  singularities can have irreducible components of arithmetic genus at most 1 and the tails of stable limits arising from  $A_4$  singularities can have irreducible components of arithmetic genus 2 only if the component is attached to the rest of the curve at a Weierstrass point. This shows that  $\mathcal{T}_{X'} \cap \Delta_2^\circ = \emptyset$ . We are done by Lemma 3.10.  $\square$

We finish the discussion of the indeterminacy locus of the rational map  $f$  by proving the following lemma used in the proof of Theorem 3.16.

**Lemma 3.15.** *Let  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a double conic. Then the variety of stable limits of  $C$  is contained in  $\Delta_0$ .*

*Proof.* Consider a smooth quadric  $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q \subset \mathbb{P}^3$  and choose projective coordinates  $[z_0 : z_1 : z_2 : z_3]$  on  $\mathbb{P}^3$  so that the double conic  $C$  is cut out by  $z_0^2 z_1 = 0$  on  $Q$  and so that  $[1 : 0 : 0 : 0] \notin Q$ . Consider now a one-parameter subgroup  $\rho: \text{Spec } \mathbb{C}[t, t^{-1}] \rightarrow \text{PGL}(4)$  acting by

$$t \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : tz_1 : tz_2 : tz_3].$$

Then  $C_0 := \lim_{t \rightarrow 0} \rho(t) \cdot C$  is a genus 4 curve lying on a singular quadric  $Q_0 = \lim_{t \rightarrow 0} \rho(t) \cdot Q$  with a vertex at  $[1 : 0 : 0 : 0]$ . Evidently,  $C_0$  is a union of a double conic (note that the double conic of  $C$  is fixed under  $\rho(t)$ ) and two rulings of  $Q_0$  meeting in a node at  $[1 : 0 : 0 : 0]$ . Since  $C_0$  is a flat degeneration of  $C$  in an isotrivial family, it follows that the variety of stable limits of  $C$  is contained inside the variety of stable limits of  $C_0$ . It remains to observe that since  $C_0$  has a node and the partial normalization of  $C_0$  at this node has arithmetic genus 3, every stable limit of  $C_0$  also has a non-separating node. This finishes the proof.  $\square$

**Theorem 3.16** (Flip of the hyperelliptic locus). *The hyperelliptic locus  $\overline{H}_4$  is flipped by  $f$  to the one dimensional locus  $A := \overline{\{\text{curves with an } A_8 \text{ singularity}\}}$ , i.e. the total transform of the generic point of  $\overline{H}_4$  is  $A$ .*

*Proof.* Recall from Section 2.3.7 that  $A \subset M$  is a curve passing through the triple conic and through the unique  $A_9$ -point, and smooth away from these two points. (It is not hard to see that  $A$  is a rational curve, but we do not use this fact.) As we have already observed, the variety of stable limits of every  $A_8$  curve is  $\overline{H}_4$  by Proposition 1.3 ( $A_{\text{even}}$ ). It remains to show that for any curve  $X \in M \setminus A$ , the variety of stable limits  $\mathcal{T}_X$  does not pass through the generic point of  $\overline{H}_4$ . This is analogous to the proof of Theorem 3.13 (3): Every closed semistable curve not in  $A$  is either a double conic, or a  $D_4$ -curve, or has at worst  $A_7$  singularities. But by Proposition 1.3, the general hyperelliptic curve does not lie in the variety of stable limits of a  $D_4$ -curve or a  $(3, 3)$  curve with at worst  $A_7$  singularities. Finally, the variety of stable limits of a double conic is contained in  $\Delta_0$  by Lemma 3.15 and hence also does not contain the general hyperelliptic curve.  $\square$

**Remark 3.17.** As we have seen in Section 2.3.8, the flipping loci of special closed subvarieties of  $\overline{H}_4$  do lie outside of  $A$ . For example, hyperelliptic curves in  $\Delta_1$  are flipped to curves with  $A_6$  singularities, etc.

#### 4. CONCLUDING REMARKS

We conclude that the log canonical model  $\overline{M}_4(\alpha)$  satisfies the modularity principle for the log MMP for  $\overline{M}_4$  for  $\alpha \in [0, 29/60)$ . The final non-trivial log canonical model  $\overline{M}_4(29/60)$  also exhibits behavior that we expect of log canonical models with  $\alpha > 29/60$ . Namely, for all  $8/17 < \alpha < 5/9$ :

- (1) Hyperelliptic curves are replaced by curves with  $A_8$  (and  $A_9$ ) singularities.
- (2) Curves with elliptic triboroughs are replaced by curves with two  $D_4$  singularities.
- (3) General curves in  $\Delta_2$  are replaced by a maximally degenerate  $A_5$ -curve.

We finish by noting that although this paper confirms the above assertions only for  $8/17 < \alpha \leq 29/60$ , some of its results are readily extended to higher values of  $\alpha$ . For example, the canonically embedded maximally degenerate  $A_5$ -curve (see Equation (2.3)) is defined by the ideal  $(z_0z_3 - z_1z_2, z_1^2z_3 + z_2^2z_0)$  in  $\mathbb{P}^3$  and has a semistable  $m^{\text{th}}$  Hilbert point for all  $m \geq 3$  [AFS11]. This suggests that the maximally degenerate  $A_5$ -curve replaces  $\Delta_2$  in all of GIT quotients  $\text{Hilb}_{4,1}^{m,ss} // \text{SL}(4)$ , and hence in all log canonical models  $\overline{M}_4(\alpha)$  with  $8/17 < \alpha \leq 5/9$ .

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