CALCULUS III PRACTICE MIDTERM 1 SOLUTIONS

SOLUTIONS BY ZHENGYU XIANG

No calculators will be allowed on the exam.

1. (*5 points*) If $\vec{v} \times \vec{w} = \langle 1, 1, -1 \rangle$ and $\vec{v} \cdot \vec{w} = 2$, find the angle between $\vec{v}$ and $\vec{w}$.

Solution:
since $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$, we have $|\vec{v}||\vec{w}| \sin(\theta) = |\langle 1, 1, -1 \rangle| = \sqrt{3}$.
since $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$, we have $|\vec{v}||\vec{w}| \cos(\theta) = 2$.
It follows that $\tan(\theta) = \frac{sin(\theta)}{cos(\theta)} = \frac{\sqrt{3}}{2}$, so $\boxed{\theta = \arctan \sqrt{3}/2.}$

2. (*10 points*) For which values of parameter a, the vectors $\vec{u} = \langle 1, a, 2 \rangle$ and $\vec{v} = \langle a, 4, 4 \rangle$ are

(a) parallel?
(b) orthogonal?
(c) make $45°$ angle?

Solution:

(a) $\vec{u}$ and $\vec{v}$ are parallel if and only if there exists λ such that $\vec{v} = \lambda \times \vec{u}$. Solving the resulting system of linear equations, we get $a = \lambda, 4 = a \times \lambda, 4 = 2 \times \lambda$. This implies that $a = \lambda = 2$. $\boxed{\text{Answer: } a = 2.}$
(b) $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = a + 4a + 8 = 5a + 8 = 0$. $\boxed{\text{Answer: } a = -\frac{8}{5}.}$
(c) If $\vec{u}$ and $\vec{v}$ make $45°$ angle, then by the formula $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$, we have $5a + 8 = \sqrt{1 + a^2 + 4}\sqrt{a^2 + 16 + 16} \cos(45°)$, which implies $a^4 - 13a^2 - 160a + 32 = 0$. The roots of this polynomial cannot be found by hand. **Ignore this part!**

3. (*15 points*) Find an equation of the following lines (if they exist!):
a. (*5 pts*) The line through $A = (2, 4, 2), B = (3, 7, -2), C = (1, 3, 3)$.
b. (*5 pts*) The line that passes through the points and $(3, 7, 0)$ and $(-5, 5, 1)$.
c. (*5 pts*) The line through the point $(0, 1, 1)$ and perpendicular to the vector $\langle 4, 6, -2 \rangle$.

(a) $\overrightarrow{AB} = \langle 1, 3, -4 \rangle, \overrightarrow{AC} = \langle -1, -1, 1 \rangle$, so $\overrightarrow{AB}$ is not parallel to $\overrightarrow{AC}$. There is no line through the three points.
(b) $(x, y, z) = (3, 7, 0) + t((-5, 5, 1) - (3, 7, 0)) = (3 - 8t, 7 - 2t, t)$
(c) There are infinitely many such lines, actually, for any a, b such that $ab \neq 0$, consider the vector $\langle a, b, 2a + 3b \rangle$, which is perpendicular to the vector $\langle 4, 6, -2 \rangle$. Then $(at, 1 + bt, 1 + (2a + 3b)t)$ is such a line.

4. (*10 points*) Reparametrize the curve $\vec{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle$ with respect to arc length measured from the point $(1, 0, 1)$.

First notice that $\vec{r}(0) = (1, 0, 1)$. Then, we compute

$$|\vec{r}\,'(u)| = \sqrt{3}e^u.$$

At time t, we know that the arc length is given by $s = \int_0^t |\vec{r}'(u)|du = \int_0^t \sqrt{3}e^u du = \sqrt{3}(e^t - 1)$. Solving for t we get

$$t = \ln\left(\frac{s}{\sqrt{3}} + 1\right).$$

Substitute this to the original formula, we get the curve parameterized by arc length as

$$\langle\left(\frac{s}{\sqrt{3}} + 1\right), \left(\frac{s}{\sqrt{3}} + 1\right)\sin\ln\left(\frac{s}{\sqrt{3}} + 1\right), \left(\frac{s}{\sqrt{3}} + 1\right)\cos\ln\left(\frac{s}{\sqrt{3}} + 1\right)\rangle.$$

5. (*10 points*) For the curve

$$\vec{r}(t) = \langle\frac{1}{3}t^3, \frac{1}{2}t^2, t\rangle,$$

find the unit normal and binormal vectors at every point. What is the equation of the osculating plane at $(0,0,0)$ and $(1/3, 1/2, 1)$?

Solution: **(see the lecture notes for a better approach)**
We first compute that $\vec{r}'(t) = \langle t^2, t, 1\rangle$ and $\vec{r}''(t) = \langle 2t, 1, 0\rangle$, then $\vec{r}'(t) \times \vec{r}'(t) = \langle -1, 2t, -t^2\rangle$ and $|\vec{r}'(t) \times \vec{r}"(t)| = \sqrt{1 + 4t^2 + t^4}$. So by definition, the binormal vector is given by

$$\vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t) \times \vec{r}''(t)|} = \frac{\langle -1, 2t, -t^2\rangle}{\sqrt{1 + 4t^2 + t^4}}.$$

Now we compute by definition that $\vec{N}(t) = \frac{\vec{B}(t)\times\vec{r}(t)}{|\vec{B}(t)\times\vec{r}(t)|} = \frac{\langle 2t^2 + \frac{1}{2}t^4, -\frac{1}{3}t^5 - t^2, \frac{1}{2}t^3 - \frac{2}{3}t^4\rangle}{\sqrt{\frac{1}{9}t^{10} + \frac{25}{36}t^8 + \frac{9}{4}t^6 + 5t^4}}$.

Finally, notice that $\vec{r}(0) = (0,0,0), \vec{B}(0) = \langle -1, 0, 0\rangle$, so the osculating plane at $(0,0,0)$ is given by $\boxed{x = 0}$;
Notice that $\vec{r}(1) = (\frac{1}{3}, \frac{1}{2}, 1), \vec{B}(1) = \frac{\langle -1,2,-1\rangle}{6}$, so the osculating plane at $(\frac{1}{3}, \frac{1}{2}, 1)$ is given by $\boxed{(x - \frac{1}{3}) - 2(y - \frac{1}{2}) + (z - 1) = 0.}$

6. (*10 points*) An object moves along a parabola in the plane with the position vector given by $\vec{r}(t) = \langle t, t^2\rangle$. At what point in time is the normal component of the acceleration attains the maximal value? What is the curvature at that point?

We first compute that $\vec{v}(t) = \vec{r}'(t) = \langle 1, 2t\rangle$ and $\vec{a}(t) = \langle 0, 2\rangle$, so by the formula of the curvature, we have

$$\kappa(t) = \frac{\vec{a} \times \vec{v}}{|\vec{v}|^3} = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}.$$

Further, we have $\vec{N}(t) = \frac{\langle t, 1\rangle}{\sqrt{1+t^2}}$, so the normal component of the acceleration at t is given by $\vec{a}(t) \cdot \vec{N}(t) = \frac{2}{\sqrt{1+t^2}}$, which attains maximal value when $t = 0$ and when $\kappa(0) = 2$.

7. (*10 points*) Convert the polar equation $r = \sin\phi$ from polar coodinates to cartesian coordinates and identify the curve it describes.

Since $r = \sin\phi$, that $r^2 = r\sin\phi$. Recall that $r^2 = x^2 + y^2, y = r\sin\phi$, we have the cartesian form of the curve is $x^2 + y^2 = y$, which can be written as $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. So it is a circle.

8. (*15 points*)

a. (*10 pts*) Find the plane perpendicular to the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ and passing through the point $P = (1, 0, -2)$.

b. (*5 pts*) What is the distance from P to the plane $2x - 3y + 4z = 5$?

Solution:

(a) The plane we need is spanned by the normal vectors of the two given planes. They are $\langle 1, 1, -1\rangle$ and $\langle 2, -3, 4\rangle$. So the normal direction of the plane we need is $\langle 1, 1, -1\rangle \times \langle 2, -3, 4\rangle = \langle 1, -6, -5\rangle$, so the plane is given by $(x - 1) - 6y - 5(z + 2) = 0$; or after simplification

$$x - 6y - 5z - 11 = 0.$$

(b) The distance is given by

$$\frac{|2 \times 1 - 3 \times 0 + 4 \times (-2) - 5|}{\sqrt{2^2 + (-3)^2 + 4^2}} = \frac{11}{\sqrt{29}}.$$