1. (*10 points*) Let P be the point in the plane given in rectangular coordinates by $(-6, -6\sqrt{3})$. Find the polar coordinates (r, θ) of P (express θ in radians).

Solution: $r = \sqrt{(-6)^2 + (-6\sqrt{3})^2)} = \sqrt{144} = 12$ and $\tan\theta = \sqrt{3}$. Solving for $0 \leq \theta < 2\pi$ gives two solutions $\theta = \pi/3$ or $4\pi/3$. Since the point lies in the quadrant in the xy-plane where both x and y are negative, $\theta = 4\pi/3$. The final answer is $(12, 4\pi/3)$.

2. (*15 points*) Determine whether the following vectors are parallel, perpendicular or neither. Explain why.

a. (*4 pts*) $< 2, -3, 1 >$ and $< 2, 1, -1 >$
Solution: Since $< 2, -3, 1 > \cdot < 2, 1, -1 >= 4 - 3 - 1 = 0$, the vectors are perpendicular.

b. (*4 pts*) $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $-7\mathbf{i} - \frac{7}{2}\mathbf{j} + 14\mathbf{k}$
Solution: Since $(-7\mathbf{i} - \frac{7}{2}\mathbf{j} + 14\mathbf{k}) = (-\frac{7}{2})(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})$, the vectors are parallel.

c. (*4 pts*) $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are arbitrary vectors.
Solution: Since $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) = 0 + 0 = 0$, the vectors are perpendicular.

d. (*3 pts*) $< 1, 2, -1 >$ and $\mathbf{n}$, where $\mathbf{n}$ is any vector perpendicular to the plane defined by $2x - y - z = 1$
Solution: We can choose $\mathbf{n} =< 2, -1, -1 >$. Since $< 1, 2, -1 > \cdot \mathbf{n} = 2 - 2 + 1 = 1 \neq 0$, the vectors are not perpendicular. Furthermore, $< 1, 2, -1 >$ is not a scalar multiplie of $\mathbf{n}$. Therefore, the vectors are neither parallel nor perpendicular.

3. (*15 points*)

a. (*8 pts*) Determine whether the three points $(1, -5, 2)$, $(-1, -3, 3)$ and $(-3, -1, 5)$ lie on the same line.

Solution: Let $\mathbf{a} =< -2, 2, 1 >$ and $\mathbf{b} =< -4, 4, 3 >$ be the vectors originating at the first point and ending at the second point and third point, respectively. The three points lie on the same line if and only if the area of the parallelogram they span is 0, which is true if and only if $\mathbf{a} \times \mathbf{b} = 0$. We compute

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -4 & 4 & 3 \end{vmatrix} = (6-4)\mathbf{i} - (-6-(-4))\mathbf{j} + (-8-(-8))\mathbf{k} = 2\mathbf{i} + 2\mathbf{j}$$

Since $\mathbf{a} \times \mathbf{b}$ is non-zero, we conclude that they do not lie on the same line.

b. (*7 pts*) Determine whether the four points $(1, 1, 0)$, $(1, 1, -2)$, $(0, 2, -1)$ and $(5, -3, 0)$ lie on the same plane.
Solution: Let $\mathbf{a} =< 0, 0, -2 >$, $\mathbf{b} =< -1, 1, -1 >$, and $\mathbf{c} =< 4, -4, 0 >$ be the vectors from the first point to the second, third and fourth, respectively. The four points lie on the same plane if and only if the volume of the parallelepiped that they span is 0. The volume of the parallelepiped is $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})|$ which we will know compute. First

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -2 \\ -1 & 1 & -1 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j}$$

Therefore $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |8 - 8| = 0$ so the four points do lie on the same line.

4. (*15 points*) Let $\mathbf{a} =< -1, 0, 1 >$ and $\mathbf{b} =< 2, 2, 0 >$ be vectors.

a. (*8 pts*) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.
Solution: Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}\ \cos\theta$ where $0 \leq \theta < \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-2}{\sqrt{2}\sqrt{8}} = \frac{-1}{2}$$

so that $\theta = 2\pi/3$.

b. (*7 pts*) Find two unit vectors orthogonal to $\mathbf{a}$ and $\mathbf{b}$.
Solution: We compute that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 2 & 2 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

The length $|\mathbf{a} \times \mathbf{b}| = \sqrt{4+4+4} = 2\sqrt{3}$. Both $\pm(\mathbf{a} \times \mathbf{b})/|\mathbf{a} \times \mathbf{b}|$ are unit vectors perpendicular to $\mathbf{a}$ and $\mathbf{b}$. Therefore the answer is $< -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} >$ and $< \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} >$.

5. (*15 points*) Let P be the plane perpendicular to $< 1, 2, 3 >$ passing through the point $< 1, 0, 1 >$.

a. (*5 pts*) Find the scalar equation for the plane P.
Solution: The scalar equation is $(x - 1) + 2y + 3(z - 1) = 0$ or $x + 2y + 3z = 4$.

b. (*5 pts*) Is the plane P parallel to the plane defined by $2x + 3y - 4z = 2$? Is it perpendicular?
Solution: Plane P has as a normal vector $\mathbf{n}_1 =< 1, 2, 3 >$ while the plane $2x + 3y - 4z = 2$ has as a normal vector $\mathbf{n}_2 =< 2, 3, -4 >$. Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are not scalar multiplies, the planes are not parallel. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 + 6 - 12 = -4 \neq 0$, the planes are not perpendicular.

c. (*5 pts*) Does the line given by the parameterization $x(t) = 3t + 1, y(t) = 3$ and $z(t) = -t + 3$ intersect the plane P?
Solution: The line is in the direction of the vector $\mathbf{v} =< 3, 0, -1 >$. Since $\mathbf{v} \cdot \mathbf{n}_1 = 3 - 3 = 0$, $\mathbf{v}$ is perpendicular to $\mathbf{n}_1$. Therefore either the line lies on the plane or is parallel to the plane but does not intersect. By setting $t = 0$, we see that the point $(3, 3, 3)$ is on the line. But this point does not line on the plane as it does not satisfy the equation $x + 2y + 3z = 4$. Therefore, the answer is no!

6. (*15 points*)

a. (*10 pts*) Find parametric equations for the line of interesection of the planes $x + y = 1$ and $y + z = 1$.

Solution: The point $(0, 1, 0)$ is on both planes. Set $\mathbf{r}_0 =< 0, 1, 0 >$. The first plane has $\mathbf{n}_1 =< 1, 1, 0 >$ as a normal vector and the second plane has $\mathbf{n}_2 =< 0, 1, 1 >$ as a normal vector. Therefore, the line of intersection is the direction of the vector $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. We compute

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

The line is paramaterized by the vector equation $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ which gives the parametric equations

$$x(t) = t$$
$$y(t) = 1 - t$$
$$z(t) = t$$

b. (*5 pts*) Find symmetric equations in x, y and z for the same line.

Solution: By solving for t, we get the symmetric equations:

$$x = 1 - y = z$$

7. (*15 points*) The curve in space is given by the vector function $\vec{r}(t) = \langle 2\sin(t), \sin(t), \sqrt{5}\cos(t)\rangle$.

a. (*5 pts*) Find the unit tangent and the unit normal vectors.

Solution: $\vec{v}(t) = \vec{r}\,'(t) = \langle 2\cos t, \cos t, -\sqrt{5}\sin t\rangle$. The magnitude of $\vec{v}(t)$ is

$$\sqrt{4\cos^2 t + \cos^2 t + 5\sin^2 t} = \sqrt{5}.$$

Therefore, the unit tangent vector is $\vec{T}(t) = \vec{v}(t)/\sqrt{5} = \langle \frac{2}{\sqrt{5}}\cos t, \frac{1}{\sqrt{5}}\cos t, -\sin t\rangle$.

The unit normal vector is $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$. We compute

$$\vec{T}'(t) = \langle -\frac{2}{\sqrt{5}}\sin t, -\frac{1}{\sqrt{5}}\sin t, -\cos t\rangle.$$

Luckily, the magnitude of $\vec{T}'(t)$ is already 1. We get

$$\vec{N}(t) = \langle -\frac{2}{\sqrt{5}}\sin t, -\frac{1}{\sqrt{5}}\sin t, -\cos t\rangle.$$

b. (*5 pts*) Compute the curvature of the curve.

Solution:

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{\sqrt{5}}.$$

c. (*5 pts*) Find the equation of the osculating plane to the curve at the point $(2, 1, 0)$.

Solution: The normal vector of the osculating plane is the binormal vector $\vec{B}$. From above, the unit tangent and the unit normal vectors at the point $(2, 1, 0)$ (corresponding to $t = \pi/2$) are

$$\vec{T} = \langle 0, 0, -1\rangle, \quad (1)$$

$$\vec{N} = \langle -2/\sqrt{5}, -1/\sqrt{5}, 0\rangle. \quad (2)$$

It follows that

$$\vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ -2/\sqrt{5} & -1/\sqrt{5} & 0 \end{vmatrix} = \langle -1/\sqrt{5}, 2/\sqrt{5}, 0\rangle.$$

The equation of the osculating plane is

$$-\frac{1}{\sqrt{5}}(x-2) + \frac{2}{\sqrt{5}}(y-1) = 0.$$