Calculus III, Practice Midterm

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1. (*15 points*) Consider the quadric surface in 3-space defined by the equation

$$-x^2 - y^2 + \frac{z^2}{4} = 1$$

a. (*5 pts*) What are the traces for $z = 4$? For $z = 0$? For $z = 2$? For the three cases, you should both write down the equation for the trace and describe the set of solutions.

Solution: Substituting $z = 4$ in the equation of the surface, we obtain: $\boxed{x^2 + y^2 = 3}$; which is an equation **circle** in the plane $z = 4$ of radius $\sqrt{3}$.
Substituting $z = 0$, we obtain $\boxed{x^2 + y^2 = -1}$; which describes an **empty set**.
Substituting $z = 2$, we obtain $\boxed{x^2 + y^2 = 0}$; which is the equation of a **single point** $(0, 0, 2)$ in the plane $z = 2$.

b. (*5 pts*) Describe the intersection of the surface and the tangent plane to the surface at the point $(0, 0, 2)$. What is the projection of the said intersection to the (x, y)-plane?

Solution: Rewrite the equation of the surface as

$$F(x, y, z) = -x^2 - y^2 + \frac{z^2}{4} - 1 = 0.$$

Then $\nabla F(x, y, z) = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z\rangle = \langle -2x, -2y, \frac{z}{2}\rangle$. The normal direction of the tangent plane at the point $(0, 0, 2)$ is $\nabla F(x, y, z)_{|(0,0,2)} = \langle 0, 0, 1\rangle$, and the equation of the plane is

$$0 \cdot (x - 0) + 0 \cdot (y - 0) + 2 \cdot (z - 2) = 0,$$

or if simplified, $\boxed{z = 2}$.
The intersection of this tangent plane with the surface is the horizontal trace for $z = 2$, which is the point $(0, 0, 2)$ by above.

c. (*5 pts*) Find all points on this quadric surface where the tangent plane is parallel to $x + y + z = 0$.

Solution: The normal vector of $x + y + z = 0$ is $\langle 1, 1, 1\rangle$, so we need to find points (x, y, z) satisfying the equation $F(x, y, z) = -x^2 - y^2 + \frac{z^2}{4} - 1 = 0$ and such that $\nabla F(x, y, z)$ is a multiple of $\langle 1, 1, 1\rangle$. This implies $-2x = -2y = \frac{z}{2}$. Substitute these into the equation of the surface, we get two points $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -2\sqrt{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\sqrt{2})$.

2. (*15 points*)

a. (*5 pts*) Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{xy^3 + y^6}{x^2 + y^6}$$

Solution: If $(x,y) \to (0,0)$ along the line $x = y$, the limit is 0. However, if $(x,y) \to (0,0)$ along the curve $x = y^3$, the limit is 1. It follows that the limit **does not exist**.

b. (*5 pts*) Compute the following limit:

$$\lim_{(x,y)\to(1,1)} \frac{\sin(x^2 + y^2 - 2)}{(x^2 + y^2 - 2)}.$$

Write $F(t) = \dfrac{\sin t}{t}$, $G(x,y) = x^2 + y^2 - 2$. We have $G(x,y)$ is continuous at $(1,1)$ with limit 0, and $\lim_{t\to 0} F(t)$ exists and equals 1. So we have

$$\lim_{(x,y)\to(1,1)} \frac{\sin(x^2 + y^2 - 2)}{(x^2 + y^2 - 2)} = \lim_{(x,y)\to(1,1)} F(G(x,y)) = 1.$$

c. (*5 pts*) Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2y + xy^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at all points.

Solution: $f(x,y)$ is continuous at all the points $(x,y) \neq (0,0)$, since it is a rational function with non-zero denominator. So we only need to check that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + xy^2}{x^2 + y^2} = 0.$$

Switch to polar coordinates: $x = r\cos\phi, y = r\sin\phi$. Then

$$x^2y + xy^2 = xy(x+y) = r^3 \cos\phi \sin\phi(\cos\phi + \sin\phi)$$

and

$$x^2 + y^2 = r^2.$$

It follows that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + xy^2}{x^2 + y^2} = \lim_{r\to 0} r \sin\phi\cos\phi(\cos\phi + \sin\phi).$$

Since, $r \to 0$ as $r \to 0$ and $-2 \leq \sin\phi\cos\phi(\cos\phi + \sin\phi) \leq 2$,

$$\lim_{r\to 0} r \sin\phi\cos\phi(\cos\phi + \sin\phi) = 0,$$

by the Squeeze Theorem.

3. (*15 points*)

a. (*8 pts*) Let $f(x, y)$ be a function in two variables with continuous second partial derivatives. Define $g(t) = f(t^2, t^3)$. Express $g'(t)$ in terms of the partial derivatives of f.

Apply the Chain Rule I:

$$g'(t) = 2t\frac{\partial f}{\partial x}(t^2, t^3) + 3t^2\frac{\partial f}{\partial y}(t^2, t^3)$$

b. (*7 pts*) With $g(t)$ defined as above, express $g''(t)$ in terms of the first and second order partial derivatives of f.

$$g''(t) = 2\frac{\partial f}{\partial x}(t^2, t^3)+2t\left[2t\frac{\partial^2 f}{\partial x^2}(t^2, t^3) + 3t^2\frac{\partial^2 f}{\partial x \partial y}(t^2, t^3)\right]+6t\frac{\partial f}{\partial y}(t^2, t^3)+3t^2\left[2t\frac{\partial^2 f}{\partial y \partial x}(t^2, t^3) + 3t^2\frac{\partial^2 f}{\partial y^2}(t^2, t^3)\right].$$

This simplifies to

$$2\frac{\partial f}{\partial x}(t^2, t^3) + 6t\frac{\partial f}{\partial y}(t^2, t^3) + 4t^2\frac{\partial^2 f}{\partial x^2}(t^2, t^3) + 12t^3\frac{\partial^2 f}{\partial x \partial y}(t^2, t^3) + 9t^4\frac{\partial^2 f}{\partial y^2}(t^2, t^3)$$

4. (*10 points*) Let $f(x, y) = x^2y^2 - x^2 - y^2$.

a. (*5 pts*) Find all points points such that

$$f(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0.$$

(What can you say about the tangent plane at the above points?)

We can compute that

$$\frac{\partial f}{\partial x}(x, y) = 2xy^2 - 2x = 2x(y^2 - 1),$$

and

$$\frac{\partial f}{\partial y}(x, y) = 2x^2y - 2y = 2y(x^2 - 1).$$

Combining the three equations we have

$$\begin{cases} x^2y^2 - x^2 - y^2 = 0 \\ 2x(y^2 - 1) = 0 \\ 2y(x^2 - 1) = 0. \end{cases}$$

To solve, we observe that if $x^2 = 1$, then we get $-1 = 0$ in the first equation. Hence, $x^2 \neq 1$. Similarly, $y^2 \neq 1$. From the second and third equation we find that $(x, y) = (0, 0)$. This is the only solution.

At this point, the tangent plane is **horizontal**.

b. (*5 pts*) Find the partial derivatives $\dfrac{\partial f(r\cos t, r\sin t)}{\partial t}$ and $\dfrac{\partial f(r\cos t, r\sin t)}{\partial r}$

Solution: A cumbersome way to proceed is via the Chain Rule II. Instead, observe that

$$x^2 + y^2 = r^2,$$

$$x^2y^2 = (r^2\cos t\sin t)^2 = \left(\frac{r^2\sin(2t)}{2}\right)^2 = \frac{1}{4}r^4\sin^2(2t).$$

Hence, $\boxed{f(r\cos t, r\sin t) = \frac{1}{4}r^4\sin^2(2t) - r^2.}$ Therefore,

$$\frac{\partial f(r\cos t, r\sin t)}{\partial t} = r^4\cos(2t)\sin(2t), \quad (1)$$

$$\frac{\partial f(r\cos t, r\sin t)}{\partial r} = r^3\sin^2(2t) - 2r. \quad (2)$$