INTRODUCTION TO HIGHER-DIMENSIONAL ALGEBRAIC GEOMETRY

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Contents

1. Seminar goals

The goal of this seminar is to learn about techniques of higher-dimensional geometry with a view towards the minimal model program's main theorems. The sample of theorems covered in the seminar includes Lefschetz hyperplane theorems, Kodaira vanishing and its generalizations, and the cone theorem.

Every lecture will be accompanied by exercises designed to alleviate lecturers' task and to fill in certain details better left to a privacy of one's room.

2. References, prerequisites and black boxes

The main references for the seminar are:

- [8] *Positivity in algebraic geometry I. Classical setting: line bundles and linear series* by R. Lazarsfeld.
- [7] *Ample Subvarieties of Algebraic Varieties* by R. Hartshorne.

Other references:

- *Algebraic geometry* by R. Hartshorne.

Introduction

The birational geometry of higher-dimensional algebraic varieties had enjoyed exciting developments in the recent years culminating in the proof of the finite generation of the canonical ring

$$R(K_X) := \oplus_{m\geq 0} H^0(X, mK_X)$$

for an arbitrary smooth projective variety and existence of minimal models for varieties of general type ([5], [1]). This has largely completed the so-called Minimal Model Program. One of the main themes of these works is the importance of the role played by the vector space of curves and, dually, divisors up to numerical equivalence and various numerically defined cones inside them. A minimal model of a smooth projective variety X of general type is a mildly singular (in particular, $\mathbb{Q}$-Goresnstein) projective variety Y whose canonical divisor K_Y is nef, i.e., has a non-negative intersection with every irreducible curve on Y.

In one of the approaches to MMP, the major role is played by the Cone Theorem which roughly speaking says that a variety X with K_X not nef can be reduced to a simpler one. More precisely, in the technically simpler case of smooth varieties, we have

T:cone-theorem **Theorem 1** (Cone Theorem)**.** *Let X be a projective smooth variety. Then there are countably many rational curves $C_i \subset X$ such that*

$$0 < -K_X \cdot C_i \leq \dim(X) + 1,$$

and the closed cone of curves $\overline{NE}(X)$ satisfies

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

Moreover, for any K_X-negative extremal face $F \subset \overline{NE}(X)$, there is a unique morphism $cont_F : X \to Z$ to a projective variety such that $(cont_F)_*\mathcal{O}_X = \mathcal{O}_Z$ (i.e., $cont_F$ is its own Stein factorization) and an irreducible curve $C \subset X$ is contracted to a point only if $[C] \in F$.

In addition, any line bundle on X numerically trivial on F is a pullback of some line bundle from Z.

Remark: The second part of the Cone Theorem establishing an existence of a contraction of K_X-negative extremal ray is also called the Contraction Theorem.

The goal of this seminar is modest. We will not attempt to delve into the most recent research papers on MMP. Instead, we will fix ideas and acquaint ourselves with some of the techniques and results of higher-dimensional algebraic geometry. In particular, we will content ourselves with getting to the point where we can prove the cone theorem with as few black boxes as possible.

Our focus will be primarily on geometric aspects of birational geometry and as we try to avoid cohomological methods permeating the literature it is necessary to admit that some results will be completely beyond the scope of this seminar. One of this is Kawamata's theorem on freeness of divisors of special form, which is one of the key ingredients in the proof of the Cone Theorem and which explains why minimal models are so sought-after. Again, we state this theorem only for smooth projective varieties.

T:bpf-thm

Theorem 2 (Base-Point-Free Theorem)**.** *Let X be a smooth projective variety and D be a nef divisor. Suppose that the divisor $aD-K_X$ is big and nef for some rational a. Then for all integer $m \geq 0$, the complete linear system $|mD|$ is base-point free.*

Together with the following result of Zariski one obtains finite generation of canonical rings for varieties X with K_X nef.

fg-zariski

Theorem 3. (Finite generation of semiample line bundles [8, Theorem 2.1.30]) *Suppose L is a semiample line bundle on a normal projective variety X over $\mathbb{C}$. Then the section ring of L*

$$R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$$

is finitely generated as a $\mathbb{C}$-algebra.

One of the goals of the seminar is to develop an appreciation for the idea that many properties of divisors on a complete scheme X are numerical in nature and thus reduce to questions in intersection theory. To fix ideas we introduce for every complete scheme X vector spaces $N_1(X)_{\mathbb{R}} = N_1(X) \otimes \mathbb{R}$ of numerical classes of $\mathbb{R}$-curves and $N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$ of numerical classes of divisors. By construction, we have a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$$

given by an intersection product.

Inside $N_1(X)_{\mathbb{R}}$ one defines a *closed cone of curves* $\overline{\mathrm{NE}}(X)$ to be the closure of the union of rays spanned by classes of irreducible curves in X. One of the foundation results is the following

Theorem 4 (Kleiman's criterion for ampleness, [8] Theorem 1.4.29)**.** *Let X be a smooth complete scheme and D a Cartier divisor. Then D is ample if and only if it is positive on*

$$\overline{NE}(X) \setminus \{0\}.$$

A scheme is projective if it carries at least one ample Cartier divisor. The Kleiman's criterion hence reduces a geometric question of projectivity to a question in intersection theory. It will be discussed and proved in Lecture 4 of this seminar. As an application of above, we prove

Fact: *Any smooth proper surface is projective.*

Proof. Let X be the surface in question. By the Chow's lemma (Lecture 4), there exists a surjective birational morphism $f : X' \to X$ with X' projective. Let E_i be exceptional curves of f. It follows from the projection formula that $f_* : \overline{\mathrm{NE}}(X') \to \overline{\mathrm{NE}}(X)$ is well-defined. Clearly, f_* is a surjective linear map. Let H be a general element of a very ample linear series on X'. We claim that the Weil divisor $D = f_* H$ is ample. Indeed, for any $0 \neq x \in \overline{\mathrm{NE}}(X)$, where $x = \lim_{i\to\infty} C_i$ with C_i effective $\mathbb{R}$-curves, we can consider the sequence $\{D_i\} = \{f_*^{-1}(C_i)\}$ of birational transforms of C_i. By compactness, we can choose a subsequence $\{D_j\}$ converging to $y \in \overline{\mathrm{NE}}(X')$. Then

$$D \cdot x = f^* D \cdot y = \lim_{j\to\infty} (H + \sum a_k E_k) \cdot D_j > 0$$

since D_j share no components with E_k and since H is positive on $\overline{\mathrm{NE}}(X')$.

Q.E.D.

Outline of Lectures

Lecture 1. Divisors and line bundles. First examples
Jan. 30, Speaker: Daniel Disegni

lec1

The material of this lecture should come from [8] – Sections 1.1A-1.1.B, [7] – Chapter 1, Sections 0-2 and Harshorne – Chapter II.6. All references to definitions and results in **PAG** [8] are referred to by their numbers.

Goal. Introduces Cartier, Weil divisors and line bundles on arbitrary schemes. Explore relations among them on reduced and projective schemes.

Black box. Cech cohomology.

Lecture Outline.

Cartier Divisors. Begin with **Def 1.1.1**: Cartier divisors group $CDiv(X) = H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$. Therefore, a Cartier divisor is represented by data $\{(U_i, f_i)\}$ with $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$ for some open covering $\{U_i\}$ of X. Functions f_i are local equations for D on U_i. A principal divisor is a Cartier divisor defined by a global section of $\mathcal{M}_X^*$ (a rational function on X).

Remark. An arbitrary Cartier divisor should be thought of as a difference of a subscheme cut out by the numerator of f_i and a subscheme cut out by the denominator of f_i. This explains why we care about local equation only up to a unit in $\mathcal{O}_X^*$.

The *support* of the divisor D is the set of points $x \in X$ such that the local equation f of D at x is not a unit in $\mathcal{O}_{X,x}$. Note that $\mathcal{O}_{X,x} \subset \mathcal{M}_{X,x}$ and the condition has to be interpreted as saying that x in the support of D if either f does not lie in $\mathcal{O}_{X,x}$ entirely or, if it does, it lies in $\mathfrak{m}_{X,x}$.

Weil divisors. A prime Weil divisor as an irreducible (reduced) subvariety of pure dimension $n-1$ (see Hartshorne, Chapter II.6). A Weil divisor (**Def 1.1.3**) is a linear combination of prime divisors. These form an abelian group $\mathrm{WDiv}(X)$. If X is normal (more generally, regular in codimension 1), we define the cycle map

$$\mathrm{div} : \mathrm{CDiv}(X) \to \mathrm{WDiv}(X)$$

by sending a Cartier divisor $D = \{(U_i, f_i)\}$ to a linear combination $\sum v_Y(D)[Y]$ where Y runs over all prime Weil divisors whose generic point lies in the support of D and where $v_Y(D)$ is a valuation of the local equation of D in the discrete valuation ring $\mathcal{O}_{X,Y}$ (see Hartshorne Chapter II.6 and in particular Proposition 6.11 for more details). For a general scheme, one defines the cycle map by

$$\mathrm{div}(D) = \sum \mathrm{ord}_V(D)[V]$$

where V runs over all prime Weil divisors. For a prime divisor V with a generic point x we take a local equation of D at x to be f and define

$$\mathrm{ord}_V(D) := \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/f_x).$$

The image of the principal Cartier divisor in $\mathrm{WDiv}(X)$ is called *principal* Weil divisor. The quotient of $\mathrm{WDiv}(X)$ by a subgroup of principal divisor is called *divisor class group* of X and is denoted $Cl(X)$.

Proposition 1 (Hartshorne, Proposition 6.11)**.** *If X is a locally factorial, integral, separated Noetherian scheme, then the cycle map is an isomorphism.*

For general schemes, the cycle morphism needs not to be injective or surjective.

Example 1. Take $X = \mathrm{Spec}(k[x,y]/(xy))_{(x,y)}$. Then the Cartier divisor $D = \frac{x+y}{x-y}$ is not trivial, but $\mathrm{div}(D) = 0$.

Example 2.

Line bundles. A line bundle is a rank 1 locally free sheaf on X. To a Cartier divisor $D = \{(U_i, f_i)\}$ we associate a line bundle $\mathcal{L}(D) = \mathcal{O}_X(D)$ defined as a subsheaf of $\mathcal{M}_X$ generated by f_i^{-1} as $\mathcal{O}_X$-module. Clearly, $\mathcal{L}(D)$ is locally free of rank 1. There is one-to-one correspondence between Cartier divisors and locally free rank 1 $\mathcal{O}_X$-submodules of $\mathcal{M}_X$ (see Proposition 6.13 in Hartshorne). Hence the question whether every line bundle comes from a Cartier divisor reduces to question whether every line bundle can be embedded into $\mathcal{M}_X$.

Examples.

Example 3 (A ruling of a singular quadric cone is not Cartier (but $\mathbb{Q}$-Cartier))**.** Consider an affine cone over a smooth plane conic – $X = \mathrm{Spec}\,\mathbb{C}[x,y,z]/(xy - z^2)$. The subscheme L defined by the ideal (x,z) is reduced and irreducible. Hence, it defines a Weil divisor.

Claim. *L is not a Cartier divisor at the closed point P defined by the maximal ideal $\mathfrak{m} = (x,y,z)$. In other words, the ideal (x,z) is not principal in $\mathcal{O}_{X,P}$.*

Indeed, suppose (x,z) is principal. Then its image in $\mathfrak{m}/\mathfrak{m}^2 \cong (\bar{x}, \bar{y}, \bar{z})$ should be one-dimensional $\mathbb{C}$-vector space. This is visibly false – both $\bar{x}$ and $\bar{z}$ are in the image.

Finally note that the Cartier divisor $x = 0$ vanishes to order 2 along L (i.e., its equation has degree 2 in the discrete valuation ring $\mathcal{O}_{X,L}$, which is generated by z).

Example 4 (Kleiman's example of a line bundle not coming from any Cartier divisor)**.** Let X be the Hironaka's smooth complete and non-projective threefold and l and m two disjoint curves satisfying $l + m \equiv_{alg} 0$ on X. Introduce embedded points on l and m by considering a coherent $\mathcal{O}_X$-sheaf $\mathcal{I} = \mathcal{O}_X/(\mathcal{I}_P \cap \mathcal{I}_Q) \cong \mathbb{C} \oplus \mathbb{C}$ where P and Q are points on l and m. Consider the threefold $Y = \mathit{Spec}(\mathcal{O}_X \oplus \mathcal{I})$ where the $\mathcal{O}_X$-algebra structure on $\mathcal{O}_X \oplus \mathcal{I}$ is given by $(a,i)\cdot(b,j) = (ab, aj + bi)$. Note that Y, the so-called *extension of X by $\mathcal{I}$*, is a proper non-reduced threefold with embedded points supported at P and Q. Note that X is a closed subscheme of Y cut out by a square-zero ideal $0 \oplus \mathcal{I} \subset \mathcal{O}_X \oplus \mathcal{I}$ abstractly isomorphic to $\mathcal{I}$ as an $\mathcal{O}_X$-module.

Now prove that $\mathrm{Pic}(X) = \mathrm{Pic}(Y)$ by observing that if $\mathcal{I}$ is a nilradical of $\mathcal{O}_Y$, then one has a short exact sequence of sheaves of abelian groups

$$0 \to \mathcal{I} \to \mathcal{O}_Y^* \to \mathcal{O}_X^* \to 0,$$

where the second arrow is given by $i \mapsto \exp(i)$. Deduce that every line bundle on X extends to Y. Take a line bundle $\mathcal{L}$ on X with $\deg(\mathcal{L}_{|l}) > 0$ whose existence follows by construction. Then the extension of $\mathcal{L}$ to Y cannot correspond to any Cartier divisor D, since any effective D_i in the support of D cannot pass through embedded points and so we must have $D_i \cdot l$ and $D_i \cdot m \geq 0$. Together with $l+m \equiv 0$ this gives $D_i \cdot l = 0$ for all $D_i \in \text{Supp}(D)$. A contradiction.

Exercise. Work through the Hironaka's construction to ascertain that we can find a 3-fold X with two effective curve classes l and m whose sum is algebraicly equivalent to 0. Construct a divisor D on X with $D \cdot l > 0$.

Lecture 2. Cohomological characterization of ampleness
Feb. 6, Speaker: Yanhong Yang

Goal. State cohomological characterization of ampleness and give examples where it works. Introduce intersection numbers of Cartier divisors with subschemes and hint at why these depend only on numerical equivalence classes of divisors.

Black box. Sheaf cohomology. We will also need a a particular case of the general Leray spectral sequence:

E:Leray

$$E_{p,q}^2 = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}) \tag{1}$$

for any morphism $f : X \to Y$ of schemes and any quasicoherent $\mathcal{F}$ on X. In the case when f is a finite (or more generally affine) morphism, the Leray spectral sequence takes its simplest form:

$$H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F})$$

for any quasicoherent $\mathcal{F}$ on X.

We might make use of the Hirzebruch-Riemann-Roch formula:

$$\chi(X, \mathcal{F}) = \mathrm{ch}(\mathcal{F}) \cdot \mathrm{Td}(X).$$

Lecture Outline.

*. Cohomological criteria of ampleness Let X be a scheme. Recall that a line bundle L is very ample if it is a pullback of $\mathcal{O}_{\mathbb{P}}(1)$ under an embedding of X into some projective space $\mathbb{P} = \mathbb{P}^N$. A line bundle L is ample if $L^{\otimes m}$ is very ample for some m.

The ampleness is a subtle property. Most statements about ampleness that can be formulated in a general form (i.e., for any complete scheme) are amendable to proofs employing various cohomological characterizations.

serre-vanishing

Theorem 5 ([8] Theorem 1.2.6)**.** *Let L be a line bundle on a complete scheme X. TFAE:*

1. *L is ample.*
2. *For any coherent $\mathcal{F}$ there exist m_0 such that for all $m \geq m_0$ we have $H^i(X, \mathcal{F} \otimes L^{\otimes m}) = 0$ for all $i > 0$.*
3. *For any coherent $\mathcal{F}$ there exist m_0 such that for all $m \geq m_0$ the sheaf $\mathcal{F} \otimes L^{\otimes m}$ is generated by global sections (i.e., there is a surjection $\mathcal{O}_X^N \twoheadrightarrow \mathcal{F} \otimes L^{\otimes m}$ for some N).*

Corollary. *Suppose $f : X \to Y$ is a finite morphism. If $\mathcal{L}$ is ample on Y, then $f^* \mathcal{L}$ is ample on X.*

Proof. This an example of application of Leray spectral sequence. Observe that since f is finite, higher direct images of coherent sheaves under f all vanish. Therefore,

$$H^i(X, \mathcal{F} \otimes f^* L^{\otimes m}) = H^i(Y, f_*(\mathcal{F} \otimes f^* L^{\otimes m})) = H^i(Y, f_* \mathcal{F} \otimes L^{\otimes m}).$$

Therefore we can apply Serre's vanishing on Y to obtain necessary vanishing on X (remember that $f_* \mathcal{F}$ is coherent due to f being finite!). ☐

This result will play a key role in the following subsection. Note that it can be reinterpreted as saying that if a linear series $V \subseteq H^0(X, L)$ defines a finite morphism to a projective space then L is ample.

Corollary. *If D is an ample Cartier divisor and E is any divisor, then $mD + E$ is ample for all $m \gg 0$.*

P:red-irred

Proposition 2 ([8] Prop. 1.2.16)**.** *Let X be complete scheme and $\mathcal{L}$ a line bundle on X. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{red}$ is ample on X_{red}. Also, $\mathcal{L}$ is ample if and only if $\mathcal{L}$ is ample on each irreducible component of X.*

Ampleness in families. Here we prove

ample-in-family

Theorem 6 ([8](Theorem 1.2.17) Ampleness in families)**.** *Let $f : X \to T$ be a proper (possibly non-flat) morphism of arbitrary schemes and L is a line bundle on X such that L_0 is ample on X_0 for some point $0 \in T$. Then there exists an open $U \subset T$ containing 0 such that L_t is ample on X_t for all $t \in U$.*

Proof. Replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes m}$ we can assume that $\mathcal{L}_0$ is very ample and moreover that $H^1(X_0, \mathcal{L}_0) = 0$. The idea of the proof is to construct a family of finite morphisms from fibers X_t to projective spaces. Observe that if $g : X \to Y$ is a T-morphism which is finite on X_0, then it is finite over an open neighborhood of $0 \in T$. Therefore, it sufficient to produce a T-morphism from X to a family of projective spaces over T in such a way that it is an embedding defined by $\mathcal{L}_0$ when restricted to X_0.

A natural guess is that the alleged morphism should be defined on fibers by sections of $\mathcal{L}$ extending sections of $\mathcal{L}_0$. By Proposition 7.12 of [6] we need to find a vector bundle $\mathcal{E}$ on T and a surjective map of sheaves $f^*\mathcal{E} \to \mathcal{L}$. Since everything in sight is coherent and f is proper, it is sufficient to have a map of sheaves which is surjective when restricted to X_0. A candidate for a vector bundle $\mathcal{E}$ is a coherent sheaf $f_*\mathcal{L}$ (since $f_*\mathcal{L}$ is coherent and the problem is local, after shrinking T we can always find a surjective map $\mathcal{O}_X^{\otimes N} \twoheadrightarrow f_*\mathcal{L}$). There is a natural map $f^*f_*\mathcal{L} \to \mathcal{L}$ which factors

$$(f^*f_*\mathcal{L}) \otimes \mathcal{O}_{X_0} \to H^0(X_0, \mathcal{L}_0) \otimes_{\mathbb{C}} \mathcal{O}_{X_0} \to \mathcal{L}_0$$

on the fiber. Since $\mathcal{L}_0$ is generated by global sections, the second arrow is surjective. The surjectivity of the first arrow is equivalent to the surjectivity of a natural map

$$(f_*\mathcal{L}) \otimes \mathcal{O}_{T,0}/\mathfrak{m} \to H^0(X_0, \mathcal{L}_0).$$

In other words, we want every section in $H^0(X_0, L_0)$ to extend to a neighborhood of X_0.

We have an exact sequence

$$f_*\mathcal{L} \to f_*\mathcal{L}_0 \to R^1f_*(\mathcal{L} \otimes \mathcal{I}_{X_0}),$$

where $\mathcal{I}_{X_0} = \mathfrak{m} \cdot \mathcal{O}_X$ is an ideal of the central fiber. It remains to show that $R^1f_*(\mathcal{L} \otimes \mathcal{I}_{X_0}) = 0$. See Lazarsfeld's book for details.

∎

Locally free resolution. Here we establish the existence of a resolution of any coherent sheaf by a direct sum of line bundles.

Intersection theory of divisors. Suppose $Z \subseteq X$ is a closed subscheme of dimension d and $L_1, \ldots, L_d$ are line bundles on X. We define intersection number $(L_1 \cdot L_2 \cdots L_d \cdot Z)$ as the coefficient of $m_1 \cdots m_d$ in the polynomial

$$\chi(Z, \mathcal{O}_Z(m_1L_1 + \cdots + m_dL_d)).$$

The point being that by the Hirzerbruch-Riemann-Roch, the Euler characteristic $\chi(Z, \mathcal{O}_Z(m_1L_1 + \cdots + m_dL_d))$ is a polynomial in m_i of total degree d.

Lecture 3. Numerical characterization of ampleness. Nakai's criterion

Feb. 13, Speaker: Thibaut Pugin

Goal. Discuss numerical criteria of ampleness. Introduce **nef** divisors.

Lecture Outline.

2.1. **Numerical properties of ample divisors.** Recall that two divisors D_1 and D_2 are *numerically equivalent* if

$$D_1 \cdot C = D_2 \cdot C$$

for any irreducible curve $C \subseteq X$.

Exercise. Convince yourself that the intersection numbers $(D_1 \cdots D_k \cdot V)$ defined in the previous lecture depend only on the numerical equivalence classes of the divisors D_i.

By one of the definitions of intersection numbers, if $V \subset X$ is a closed subscheme of pure dimension d and $l_1, \dots, L_d$ are ample divisors, then

$$(L_1 \cdots \mathcal{L}_d \cdot V) > 0.$$

In particular when L is very ample ample, then the intersection number $L^d \cdot V$ is simply the degree of V under the embedding of X by L into a projective space and hence positive. It turns out that the property of having positive top intersection on any subvariety characterizes ample divisors and gives a simple numerical criterion for ampleness.

Exercise. Suppose X is a complete integral (possibly singular) curve. Show that any Cartier divisor D on X satisfying $D \cdot C > 0$ is ample.

Hint: Reduce to the normalization of X where the statement follows from Riemann-Roch.

Proof. From the Riemann-Roch on a smooth proper curve, we know that a linear series $|mD|$ is non-empty, separates points and tangent vectors. Suppose C is not smooth. Let $\eta : \tilde{C} \to C$ be the normalization.

$$\chi(\tilde{C}, \eta^*D) = \chi(C, \eta_*\mathcal{O}_{\tilde{C}}(D)) = \chi(C, \mathcal{L}) - \chi(C, (\eta_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) \otimes \mathcal{L}),$$

and so $\deg_{\tilde{C}} \eta^*\mathcal{L} = \deg_C \mathcal{L}$.

Hence, we know that $\eta^*\mathcal{L}$ is ample on $\tilde{C}$. We would like to use Serre's vanishing 5, to show that for any coherent $\mathcal{F}$ on C higher cohomologies of $\mathcal{F} \otimes \mathcal{L}^m$ vanish eventually.

A natural map of coherent sheaves $\phi : \mathcal{F} \to \eta_*\eta^*\mathcal{F}$ is an isomorphism at the generic point of C and gives rise to

$$\phi_m : \mathcal{F} \otimes \mathcal{L}^m \to \eta_*(\eta^*(\mathcal{F} \otimes \mathcal{L}^m)).$$

We know that $\eta^*\mathcal{L}$ is ample so higher cohomologies of $\eta^*(F \otimes \mathcal{L}^m)$ vanish. Applying now the Leray spectral sequence and observing that $\ker(\phi_m)$ and $\operatorname{coker}(\phi_m)$ have zero-dimensional support, we obtain vanishing of higher cohomologies for $\mathcal{F} \otimes \mathcal{L}^m$. □

Nakai's criterion.

Theorem 7 (Nakai's criterion)**.** *A Cartier divisor D is ample on a complete scheme X if and only if for all integral closed subschemes $V \subset X$, we have $D^k.V > 0$.*

As a corollary, we can discuss ample numerical equivalence classes of divisors.

Proof. First, we reduce to the case of integral X using Proposition 2. The proof then proceeds by induction on the dimension. The case of $\dim(X) = 1$ is established in the exercise above.

Suppose the Nakai's criterion holds for all lower dimensions. The hardest part is to show that some multiple of D is actually an effective divisor. Assuming that D is effective, we can show that the linear system $|mD|$ is base-point-free for $m \gg 0$. Indeed, $|mD|$ has no base points outside of the support of D and by induction assumption the linear system $|\mathcal{O}_D(mD)|$ is base-point-free and so $|mD|$ will have no base points once we establish surjectivity of

$$\rho_m : H^0(X, \mathcal{O}_X(mD)) \to H^0(D, \mathcal{O}_D(mD)).$$

The map fits into an exact sequence

$$H^0(X, \mathcal{O}_X(mD)) \to H^0(D, \mathcal{O}_D(mD)) \to H^1(X, (m-1)D) \to H^1(X, mD) \to H^1(D, (m-1)D).$$

Again, by the induction assumption $h^1(D, (m-1)D) = 0$ and so either ρ_m is surjective or $h^1(X, mD) \leq h^1(X, (m-1)D) - 1$. This shows that ρ_m is eventually surjective.

Now the base point free linear system $|mD|$ defines a proper morphism ϕ_{mD} to a projective space. Since D is positive on every curve, ϕ_{mD} does not contract a curve and hence is finite. We are done.

Finally, we explain why we can assume that some multiple of D is effective. First of all it follows from the asymptotic form of Hirzebruch-Riemann-Roch that $\chi(X.\mathcal{O}_X(mD)) \to \infty$ as $n \to \infty$. Hence we would like to bound the growth of higher cohomology groups. On a projective variety X any Cartier divisor can be written as a difference of two effective (in fact very ample) divisors, i.e., $D = A - B$ or $\mathcal{O}_X(D) \otimes \mathcal{O}_X(-A) = \mathcal{O}_X(-B)$. While it might not be true for an arbitrary complete integral scheme, there exists an ideal sheaf $I := \mathcal{O}_X(-D) \cap \mathcal{O}_X$ such that $J := I \otimes \mathcal{O}_X(D)$ is still an ideal sheaf. The ideal sheaves I and J cut out proper subschemes in X:

$$Y := \mathrm{Spec}(\mathcal{O}_X/I), \quad Z := \mathrm{Spec}(\mathcal{O}_Y/J).$$

We can assume by induction assumption that all higher cohomologies of $\mathcal{O}_Y(mD)$ and $\mathcal{O}_Z(mD)$ vanish for $m \gg 0$. It follows from a short diagram chase that $h^i(X, mD) = h^i(X, (m+1)D)$ for $i \geq 2$ and $m \gg 0$. So cohomology groups in degree above 1 stabilize. □

Corollary. *An effective divisor D on a surface is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all integral curves C .*

Exercise. An effective Cartier divisor D on a surface is ample if and only if $D \cdot C > 0$ for all integral curves on X.

Nef divisors.

Definition. A divisor D on a complete scheme X is called *nef* if $D.C \geq 0$ for all integral curves C on X.

Examples: Mumford's example of a line bundle positive on every effective curve but not ample.

Lecture 4. Nef divisors and Kleiman's theorem
Feb. 20, Speaker: Alice Rizzardo

Goal. Nefness and Kleiman's theorem.

2.2. **Lecture Outline.**

Chow's lemma. This lemma allows to reduce certain questions about proper morphisms to questions about projective.

Proposition 3 (Chow's lemma). *Suppose $f : X \to T$ is a proper morphism over a noetherian T, then there exists a T-morphism $g : X' \to X$ so that X' is projective over T and g is an isomorphism over a dense open $U \subset X$.*

Nef divisors. Basic properties of nef divisors (e.g., L is nef on Y if and only if f^*L is nef on X for any surjective morphism $f : X \to Y$.)

kleiman-nef-thm

Kleiman's theorem. On a complete scheme, a nef divisor D satisfies $D^k.V \geq 0$ for all integral closed subschemes $V \subset X$. As a corollary of Kleiman's and Nakai's theorems, a nef divisor is a limit of ample divisors.

2.3. **Seshadri's criterion.** Sometimes it is not so clear how to show that a given divisor is in the interior of the ample cone. The following theorem can become handy in those cases.

Theorem 8 (Seshadri's criterion). *Suppose D is a Cartier divisor on a complete scheme X. Then D is an ample divisor if and only if there exists $\varepsilon > 0$ such that for every curve $C \subset X$ we have*

$$D \cdot C \geq \varepsilon \,\mathrm{mult}_x(C)$$

where x varies over all points of C.

Example 5. The divisor $aH - bE$ on $\mathrm{Bl}_p\mathbb{P}^2$ is ample if and only if $a > b > 0$.

Example 6. Suppose $f : \mathcal{C} \to B$ is a flat family of stable curves of genus $g \geq 2$, then the relative dualizing sheaf $\omega_{\mathcal{C}/B} = \mathcal{O}_{\mathcal{C}}(K_{\mathcal{C}} - f^*K_B)$ is ample.

2.4. **Nefness in families.**

Theorem 9 (**PAG** 1.4.14). *Let $f : X \to T$ be a surjective proper (possibly non-flat) morphism of varieties and L is a line bundle on X such that L_0 is nef on X_0 for some point $0 \in T$. Then there exists a countable union of proper subvarieties V_i of T, not containing 0, such that for $U = T \setminus (\cup V_i)$ containing 0 such that L_t is nef on X_t for all $t \in U$.*

Lecture 5. Ample and nef cones. Closed cone of curves
Feb. 27, Speaker: Chenyan Wu

2.5. **Lecture Outline.**

*. Cones

Let X be an arbitrary complete scheme. Recall that

$$N^1(X)_{\mathbb{R}} := (\mathrm{CDiv}(X)/\,\mathrm{Num}(X)) \otimes_{\mathbb{Z}} \mathbb{R}$$

This is the $\mathbb{R}$-vector space spanned by the numerical equivalence classes of Cartier divisors. The dimension of $N^1(X)$ is called the *Picard number of* X and is usually denoted $\rho(X)$.

We introduce ample and nef cones inside $N^1(X)$ by setting $\mathrm{Amp}(X)$ to be a convex cone[1] of ample $\mathbb{R}$-divisors (or $\mathbb{Q}$-divisors) and by $\mathrm{Nef}(X)$ the closed cone of nef $\mathbb{R}$-divisors.

Note that the fact that $\mathrm{Amp}(X)$ is a cone follows from the fact that the sum of two integral ample divisors is ample.

We also define

$$N_1(X)_{\mathbb{R}} := (A_1(X)/\,\mathrm{Num}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

By definition of numerical equivalence we have a non-degenerate pairing

$$N^1(X) \times N_1(X) \to \mathbb{R}$$

and so $N_1(X)$ is a dual of $N^1(X)$. Inside of $N_1(X)$ there is a cone spanned by non-negative combinations of integral curves (effective 1-cycles) denoted by $\mathrm{NE}(X)$.We define its closure

$$\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbb{R}}$$

and call it the *closed cone of curves* of X.

Ampleness via cones. We can now reinterpret Kleiman's theorem 2.2 as saying that the ample cone $\mathrm{Amp}(X)$ is an interior of $\mathrm{Nef}(X)$ and $\mathrm{Nef}(X)$ is the closure of $\mathrm{Amp}(X)$.

Convex geometry.

Definition. A subcone F of a convex cone K inside a vector space is called a *extremal* if for any two x and y in K such that $x + y \in F$ we have $x, y \in F$.

A one-dimensional extremal subcone is called an extremal ray.

See Kollár [10] Definitions 4.9, 4.11 on page 127. Draw a picture of an ample/nef cone sitting inside of $\overline{\mathrm{NE}}(X)$ sitting inside $N_1(X)$ for $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Bl}_p\mathbb{P}^2$.

Hodge index theorem. Recall the following

hodge-thm **Theorem 10** (Hodge Index Theorem)**.** *Suppose X is a smooth projective surface. The intersection form on $N^1(X) = N_1(X)$ has exactly one positive eigenvalue.*

The proof of the Hodge Index Theorem can be found in Hartshorne [6].

Exercise. Using Riemann-Roch for surfaces show that if D is a divisor on a smooth surface S such that $D^2 > 0$ then either $h^0(S, mD) > 0$ or $h^0(S, -mD) > 0$ for $m \gg 0$. In other words, a line spanned by a divisor with a positive self-intersection intersects $\mathrm{NE}(X)$.

[1] cone here is allowed to not contain the origin

2.6. **Cone of curves on a surface.** Après Kollár. Look at Chapter II.4 *Rational curves* and pick examples appealing to you.

By the Hodge Index Theorem the signature of the intersection form on $N_1(X)$ is $(1, n-1)$. Let $E_1, E_2, \dots, E_n$ be the orthogonal basis in which the intersection form is

$$x_1^2 - x_2^2 - \cdots - x_n^2.$$

We can assure that E_1 is an effective $\mathbb{R}$-divisor (Gram-Schmidt and the exercise above).

Any ample divisor $H = x_1E_1 + x_2E_2 + \cdots + x_nE_n$ intersects E_1 positively and hence must have $x_1 > 0$.

The circular cone

$$Q = \{z = (x_1, \dots, x_n) \in N_1(X) \quad | \quad z^2 > 0\}$$

naturally breaks into two parts: Q^+ where $x_1 > 0$ and Q^- where $x_1 < 0$. The Lemma 4.12 on p. 128 in [10] shows that effectivity of divisors in Q is detected by the sign of x_1:

$$Q^+ \subseteq \mathrm{NE}(X).$$

Examples: Cone of curves on an abelian surface (e.g., $E \times E$). Surface with infinitely many (-1)-curves. Case of K3's.

Lecture 6. Castelnuovo-Mumford Regularity
Mar. 6, Speaker: Joe Ross

Goal. Discuss Castelnuovo-Mumford regularity.

2.7. **Lecture Outline.**

2.8. **Definition of Castelnuovo-Mumford regularity.** A coherent sheaf $\mathcal{F}$ is Castelnuovo-Mumford m-regular if $H^i(\mathbb{P}, \mathcal{F}(m-i)) = 0$ for all i.

2.9. **Mumford's theorem.** A coherent sheaf $\mathcal{F}$ is m-regular $\Rightarrow$ $\mathcal{F}(m)$ is globally generated and $\mathcal{F}$ is $(m+k)$-regular.

2.10. **Resolutions.** Discuss linear resolutions of m-regular sheaf.

2.11. **Boundedness for coherence sheaves.** Prove boundedness of coherent sheaves with a fixed Hilbert polynomial. Possibly discuss Gotzmann's regularity theorem: if the Hilbert polynomial P of a subscheme X is represented as

$$P(k) = \binom{k+a_1}{a_1} + \binom{k+a_2-1}{a_2} + \cdots + \binom{k+a_s-(s-1))}{a_s}$$

then I_X is s-regular.

Discuss how this is used in the construction of the Hilbert scheme.

*. Examples Give an example of a polynomial P such that there is no bound for the regularity of coherent sheaves with Hilbert polynomial P.

Lecture 7. Linear series, finite generation of section rings and birational geometry
Mar. 13, Speaker: Jian Wang

Goal. Linear series and birational geometry.

Lecture Outline.

(1) Itaka dimension of a line bundle.
(2) Algebraic fiber space (discuss Stein's factorization and connectedness theorem).
(3) Section ring associated to a line bundle.
(4) Discuss finite generation of canonical rings as applicable to MMP.
(5) Semiample line bundles.
(6) Semiample fibrations.

2.12. **Itaka dimension of a line bundle.** Given a divisor D on a projective variety X and an associated line bundle $L = \mathcal{O}_X(D)$ we consider a rational map

$$\phi_L : X \dashrightarrow \mathbb{P}H^0(X, L)$$

defined by the complete linear series $|L|$. The closure of the image of X under ϕ_L will be denoted by $\phi_L(X)$ or by $\phi_D(X)$. The *semigroup* $N(X, L)$ *of* L consists of non-negative integers m such that $H^0(X, L^{\otimes m}) \neq 0$. We define *Itaka dimension* of D and $\mathcal{L}(D)$ to be

$$\kappa(X, D) := \max_{m \in N(X,L)} \{\dim \phi_{mD}(X)\}$$

and $\kappa(X, D) = -\infty$ if no multiple of D has a global section.

The *Kodaira dimension* of X is defined to be the number $\kappa(X) := \kappa(X, K_X)$ for a nonsingular variety X. When X is singular, we define $\kappa(X)$ to be the Kodaira dimension of the desingularization of X.

Example 7. A projective cone in $\mathbb{P}^3$ over a smooth plane curve of degree $d \geq 4$ is a normal variety (a plane curve is projectively normal) with the relative dualizing sheaf $\omega_X = \mathcal{O}(d - 4)$ of Itaka dimension $\kappa(X, \omega_X) \geq 0$. However, the desingularization of X is a ruled variety and hence $\kappa(X) = -\infty$.

2.13. **Complete linear series associated to multiples of a divisor.** For the most part we will work with *normal* projective varieties defined over $\mathbb{C}$. Recall that an *algebraic fiber space* is a proper surjective morphism of varieties

$$f\colon X \to Y$$

satisfying $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Recall that every proper morphism $f : X \to Y$ has a *Stein factorization*

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

where $f'\colon X \to Y'$ is an algebraic fiber space and g is a finite morphism. By abuse of language, Y' is occasionally called the Stein factorization of f and we say that the morphism f is its own Stein factorization if g is trivial.

Recall the following facts:

(1) (**Connectedness Theorem**) Fibers of an algebraic fiber space are connected. See [6, II.11.3].
(2) If $f\colon X \to Y$ is birational and Y is normal, then f is an algebraic fiber space. See [6, II.11.4].
(3) (**Exercise**) If $f : X \to Y$ is an algebraic fiber space and X is normal, then Y is normal. See [8, 2.1.15].

Definition. A line bundle L is called *semiample* if some multiple mL is free, i.e., the complete linear series $|mL|$ is base point free.

Given a semiample line bundle one wonders whether morphisms there is an infinite collection of morphisms ϕ_{mL}. The following theorem states that this is not a case. Note that of course the target projective space is getting larger. By abuse of notation we also consider the image $\phi_{mL}(X)$ to be the target of ϕ_{mL}. The following theorem says that $\{\phi_{mL}(X)\}$ regarded as sequence of abstract varieties stabilizes.

T:semiample-fibration **Theorem 11.** *Let X be a normal projective variety and L a semiample line bundle on X, then there exists an algebraic fiber space*

$$f : X \to Y$$

such that $f = \phi_{mL}$ for all sufficiently large m such that mL is free. Moreover, mL is a pullback of an ample line bundle on Y.

Definition. A *section ring* of a line bundle L on a scheme X is a graded $H^0(X, \mathcal{O}_X)$-algebra (when X is proper over $\mathbb{C}$, we have $H^0(X, \mathcal{O}_X) = \mathbb{C}$)

$$R(X, L) := \bigoplus_{m \geq 0} H^0(X, mL).$$

We say that a line bundle L is finitely generated if $R(X, L)$ is a finitely generated $\mathbb{C}$-algebra.

Proposition 4. *If X is a normal projective variety and L is a semiample line bundle, then L is finitely generated and moreover*

$$f : X \to \operatorname{Proj} R(X, L)$$

is an algebraic fiber space of Theorem 11.

Exercise. For a graded ring $R = \bigoplus_{k \geq 0} R_k$, the following are equivalent:

1. R is a finitely generated R_0-algebra.
2. The Veronese subring $R^{(s)} := \bigoplus_{k \geq 0} R_{sk}$ is finitely generated R_0-algebra.

Hints: Show that for any finitely generated graded R, there exists r, such that $R^{(r)}$ is generated by R_r. Also, observe that R is integral over $R^{(s)}$.

An algebraic fiber space $f : X \to Y$ is characterized uniquely by the set of curves it contracts.

P:contraction **Proposition 5.** *] Suppose X is a normal projective variety and $f : X \to Y$ and $f' : X \to Y'$ are two algebraic fiber spaces such that every curve in X contracted by f is contracted by f'. Then f' factors through f.*

Proof. Note that since fibers of an algebraic fiber space are connected the assumption of the proposition implies that $f'(f^{-1}(y))$ is a point for any point $y \in Y$. Consider an incidence correspondence

$$I = \{(x, f(x), f'(x))\} \subset X \times Y \times Y'.$$

Denote its reduced image in $Y \times Y'$ by Γ. It is easy to check that every set-theoretic fiber of $\Gamma \to Y$ is a point. Since Γ is reduced and Y is normal we must have $\Gamma \cong Y$. This defines a requisite morphism $Y \to Y'$. □

The morphism $f \colon X \to Y$ between two projective normal varieties satisfying $f_* \mathcal{O}_X = \mathcal{O}_Y$ is also called a *contraction*.

Exercise. Show that numerical classes of curves in $\overline{\mathrm{NE}}(X)$ contracted by f form an extremal face, i.e., there is an extremal face F of $\overline{\mathrm{NE}}(X)$ such that a curve $C \subset X$ maps to a point in Y if and only if $[C] \in F$.

Proposition 5 can be reinterpreted as saying that a contraction is uniquely defined by its extremal face. A fundamental problem in birationally geometry is when an extremal face corresponds to a contraction. The Cone Theorem 1 provides one such criterion; on the other hand, see below for an example of a variety having an extremal face not associated to any contraction.

It is worthy to mention that Proposition 5 is often used to identify various spaces. For example, suppose you would like to show that a naturally defined morphism (with connected fibers) $f : X \to Y$ between two normal projective varieties obtained possibly in some indirect way (say when Y is a coarse moduli space for *somethings* and X parametrizes a family of *somethings*) is defined by a natural line bundle on X. Then a way to proceed would be to find a base point free (or simply semiample) line bundle L on X such that it is numerically trivial precisely on curves contracted by f. Then the above results imply that in fact

$$Y = \operatorname{Proj} R(X, L),$$

and f is defined by a sufficiently divisible multiple of L. A more direct way to proceed would be to find an ample line bundle on Y whose pullback to X will be a multiple of L. However, often one has more control of the Picard group of X then of that of Y.

We now mention the following result generalizing Theorem 11 for an arbitrary line bundle

T:Itaka-fibration

Theorem 12 (Itaka Fibration, Theorem 2.1.33 in [8])**.** *Let X be a normal variety and L an arbitrary line bundle such that the Itaka dimension $\kappa(X, L) > 0$. Then for all sufficient large and divisible k, the rational maps $\phi_k\colon X \dashrightarrow \phi_k(X)$ are birationally equivalent to a fixed algebraic fiber space*

$$\phi_\infty\colon X \to Y.$$

Draw a diagram! No proof of this result will be given in lectures. Please refer to [8] for details.

2.14. **Examples.**

An extremal face not corresponding to any contraction. (See [11, Example 1.27]) Let X be a $\mathbb{P}^2$ blown-up at 12 points $p_1, \dots, p_{12}$ lying on a smooth cubic C. The strict transform of C is a curve of self-intersection -3 and hence spans an extremal face of $\overline{\mathrm{NE}}(X)$ (see Lecture 5). However, there is no contraction $f\colon X \to Y$ such that $f(C)$ is a point. Indeed, suppose such a contraction exists. Then for any non-trivial line bundle M on Y we have

$$f^*M = dH + \sum_i a_i E_i$$

where H is a pullback of a hyperplane classs to X and E_i are exceptional divisors of the blow-up and not all numbers $d, a_1, \dots, a_{12}$ are zero. We have $(f^*M)_{|C} = \mathcal{O}_C$ and so $\sum a_i p_i \sim dH$ on C or equivalently

$$\sum_{i=1}^{12} a_i p_i = 0$$

in the group law of C. This is clearly impossible for a general choice of points p_i.

This argument shows that a contraction cannot exist in the category of projective varieties. A slight variation accomplishes the same result in a category of schemes:

take a non-trivial (but possibly principal) Cartier divisor D on Y with $\mathrm{Supp}(D)$ not passing through $f(C)$. Then f^*D is a Cartier divisor with support disjoint from C. Clearly, such a divisor must be trivial on X. A contradiction.

Finally, a contraction of C exists in analytic category by Grauert's Contraction Theorem (see [4, Theorem 20, p.72]).

Zariski's example of a big and nef divisor whose section ring is not finitely generated. (See [8, 2.3.A])Take X as in a previous example and $D = 4H - \sum_{i=1}^{12} E_i$. Observe that $mD \cdot C = 0$ and $\mathcal{O}_C(mD) = \mathcal{O}_C\left(m(4H - p_1 - \cdots - p_{12})\right)$ is not a trivial line bundle once the line bundle $\mathcal{O}_C(4H - p_1 - \cdots - p_{12})$ is chosen to be non-torsion in $\mathrm{Pic}^0(C)$. It follows that C must lie entirely in the base locus of $|mD|$. In other words, every element of $|mD|$ has to vanish to order at least 1 along C. On the other hand observe that

$$mD - C = (4m-3)H - (m-1)\sum_{i=1}^{12} E_i$$

is base point free.

Exercise. Prove it.

Proof. We attempt to supply the details for the inductive argument on page 158 of [8]. For $m = 1$, we have $D - C = H$ is base point free. From a short exact sequence,

$$0 \to \mathcal{O}_X(mD - 2C) \to \mathcal{O}_X(mD - C) \to \mathcal{O}_C(mD - C) \to 0$$

we gather that

$$H^0(X, mD - C) \to H^0(C, \mathcal{O}_C(mD - C)) \to H^1(X, mD - 2C).$$

Now $|O_C(mD - C)|$ is a base point free linear series on C of degree 3. We now observe that $H^1(X, mD - 2C) = H^1(X, K_X + mD - C)$. Since $mD - C$ is nef and big ($mD - C = mH + (m-1)C$), we have $H^1(X, K_X + mD - C) = 0$ by Kawamata-Viehweg Vanishing Theorem 23. It follows that $|mD - C|$ has no base points along C and since $mD - 2C = (m-1)D - C + H$, it has no base points outside of C by induction hypothesis. □

Now, if $R(X, D)$ were finitely generated by elements in degree less or equal to m_0, then any element of degree m would have to be a polynomial in generators of $R(X, D)$ with every monomial of degree at least m/m_0 and hence every element of $|mD|$ would have to vanish to degree at least m/m_0. A contradiction!

We observe for the future that D is nef and the transcendence degree of $R(X, D)$ is 2 (i.e., $h^0(X, mD) \sim m^2/2$) and so the divisor D provides an example of a big and nef divisor which is not semiample and whose section ring is not finitely generated.

Lecture 8. Big divisors
Mar. 27, Speaker: Helge Pedersen

Goal. Discuss big divisors.

Lecture Outline.

(1) Definition of a big divisor.
(2) Numerical characterization of big divisors.
(3) Pseudoeffective and big cones.
(4) Fujita's vanishing theorem.
(5) Wilson's theorem.
(6) Finite generation of section rings for big and nef divisors.

2.15. **Definition and first properties.** A divisor D (respectively, a line bundle $\mathcal{L} = \mathcal{O}_X(D)$) is *big* if Itaka's dimension of D is maximal, i.e., $\kappa(X, D) = \dim X$.

By Itaka Fibration Theorem 12, for a normal X bigness of D is equivalent to the fact that some multiple of D defines a *birational* rational map $\phi_{mD} \colon X \dashrightarrow \mathbb{P}H^0(X, mD)$.[2]

Remark. A variety X whose canonical class K_X is big is said to be of *general type*.

It is easy to see that the growth of global sections of a big divisor is maximal possible. In other words,

$$h^0(X, mD) \geq cm^n$$

for some positive constant c and all m large enough in $N(X, D)$.

2.16. **Numerical Properties.** We begin with an observation. For any effective divisor F we can find m such that $mD - F$ is effective. This is easily seen from a short exact sequence

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to O_F(mD) \to 0$$

and the fact that $h^0(X, mD) \geq cm^n$ and $h^0(F, mD) = O(m^{n-1})$. From the observation we deduce the following very important characterization of big divisors. The divisor D is big if and only if one of the following equivalent condition holds:

(1) $mD \sim A + E$ where A is some ample integral divisor and E is some effective integral divisor.
(2) $D \equiv A + E$ where A is some ample $\mathbb{Q}$-divisor and E is some effective $\mathbb{Q}$-divisor.

We conclude that bigness is a property of numerical equivalence class of divisors.

For a nef divisor D the following are equivalent:

(1) D is big.
(2) $D^n > 0$.

Clearly, positivity of top self-intersection is not sufficient to guarantee bigness in general. Take a divisor $H + aE$ on the blow-up of $\mathbb{P}^2$. It is big, but its square is negative. On the other hand, the divisor $-H$ has positive self-intersection, but is not even effective.

2.17. **Big and pseudoeffective cone.** Since being big is numerical property, we can consider a subset of $N^1(X)$ consisting of big divisor classes. By the characterization above, this subset is an open convex cone (without apex) inside $N^1(X)$ called *big cone* and is denoted $\mathrm{Big}(X)$. We also define a closed convex *pseudo-effective* cone $\overline{\mathrm{Eff}}(X)$ to be the closure of the subset of all numerical equivalence

[2]In some old papers, this property was dubbed birationally ample.

classes represented by effective divisors.[3] It also follows by the characterization of big divisors as sums of ample and effective that $\mathrm{Big}(X)$ is an interior of $\overline{\mathrm{Eff}}(X)$.

The importance of big cone stems from the fact that big divisors on X correspond to birational models of X. A theorem of Boucksom-Demailly-Paun-Peternell provides description of the pseudoeffective cone as dual of the cone of movable curves.

Definition. A curve C on a projective variety X of dimension n is *strongly movable* if there is a birational morphism $\mu\colon X' \to X$ such that $C = \mu_*(H_1 \cdot H_2 \cdots H_{n-1})$ where H_i are generic (in their respective linear series) very ample divisors.

Definition. A reduced irreducible 1-cycle on a projective variety X is *movable* if there is a deformation of C as a curve which covers X.

BDPP prove that the closures of the cones of very movable and movable cones coincide. Moreover, by [2, Theorem 2.2] a divisor class is pseudoeffective if and only if it lies in the dual of the cone of strongly movable curves. Using the bend-and-break as given by

Theorem 13 (see Theorem 5.8 [10])**.** *Suppose that C is a curve lying completely in a smooth locus of a projective variety X. If $K_X \cdot C < 0$, then through every point of C there is a rational curve.*

we see that if a canonical divisor K_X of a smooth projective variety is not pseudoeffective, then a variety is necessarily uniruled.

When a canonical class is big it follows by a theorem of Birkar-Cascini-Hacon-McKernan [1] and by Kawamata's Base-Point-Free Theorem 2 that the canonical ring $R(X, K_X)$ is finitely generated and that there is a birational model X' of X such that K'_X is semiample. An outstanding conjecture in higher-dimensional algebraic geometry asks to describe what happens in a gray area where the canonical class is simply pseudoeffective but not big (i.e., lies on the boundary of the closure of the effective cone).

Conjecture (Abundance Conjecture)**.** If K_X is pseudoeffective, then $\kappa(X) \geq 0$ (i.e., some multiple of K_X is effective). If moreover K_X is nef, then K_X is semi-ample.

2.18. **Boundedness of stable base locus.**

Theorem 14 (Wilson's theorem)**.** *Let X be an irreducible projective variety of dimension n, then for a big and nef divisor D, there exists an effective divisor N such that*

$$|mD - N|$$

is free for all $m \geq m_0$ for some fixed m_0.

Proof. The proof uses the following theorem which we quote without proof (see [8, Theorem 1.4.35]).

Theorem 15 (Fujita's vanishing)**.** *If X is complex projective variety and H an ample divisor then for any coherent $\mathcal{F}$ we have $H^i(X, \mathcal{F}(mH + D)) = 0$ for all $i > 0$, all nef divisors D and all m large enough.*

[3] Caution: being effective is not numerical property as a difference of two fibers of a $\mathbb{P}^1$-bundle over an elliptic curve demonstrates.

□

The immediate corollary of Wilson's theorem is that the multiplicity

$$\operatorname{mult}_x |mD| := \min_{D' \sim mD} \{\operatorname{mult}_x D'\}$$

of the linear series $|mD|$ at any point $x \in X$ is bounded from above by $\operatorname{mult}_x N$.

2.19. **Finite generation of section rings.** As a result of Wilson's theorem, we have the following criterion for finite generation of big and nef divisors ([8, Theorem 2.3.15]).

Theorem 16. *A section ring of big and nef divisor D on a normal projective X is finitely generated if and only if D is semiample (i.e., some multiple of D is free).*

Example 8. This theorem gives a quick way to produce divisors which are not finitely generated with the first instance being example of Zariski from the previous lecture. Take a surface X and a curve C on X of negative self-intersection and such that $\operatorname{Pic}(X) \to \operatorname{Pic}(C)$ is injective. Then for a very ample divisor H on X, we construct an effective divisor

$$D = -(C.C)H + (H.C)C.$$

Clearly, for each positive integer m, we have $mD.C = 0$ and since since $\mathcal{O}_X(mD)$ is not trivial, we must have that C lies in the base locus of $|mD|$. In other words, D is not semiample. Hence, by above theorem $R(X, D)$ is not finitely generated.

2.20. **Additional topics.** Possible topics include the volume of a big divisor ([8, 2.2.C]) and Zariski Decomposition ([8, 2.3.E]).

Stable base locus. Recall that the base locus of a divisor D is a common locus of vanishing of all divisors in a complete linear series $|D|$. This locus can be given a scheme structure by defining the ideal

$$I = \operatorname{Im}(H^0(X, D) \otimes \mathcal{O}_X(-D) \to \mathcal{O}_X)$$

and the base locus scheme

$$\operatorname{Bs}(D) = \mathbb{V}(I).$$

Definition. The *stable base locus* of a divisor D is the set-theoretic intersection

$$\mathbf{B}(D) := \bigcap_{m \geq 1} \operatorname{Bs}(mD).$$

Zariski Decomposition.

T:zariski-decomp

Theorem 17 (Zariski Decomposition)**.** *Any pseudoeffective integral divisor D on a smooth projective surface X can be written as uniquely as*

$$D = P + N$$

for $\mathbb{Q}$-divisors P and N such that

(1) *P is nef.*
(2) *$N = \sum a_i E_i$ effective ($a_i \geq 0$ and E_i irreducible curves) and the intersection matrix $(E_i \cdot E_j)$ is negative definite (if $E \neq 0$).*
(3) *P is orthogonal to E_i.*

A nice theorem Bauer-Kuronya-Szemberg says that the stable base locus and the negative part of Zariski decomposition are essentially numerical invariants.

Theorem 18. *Let X be complex smooth projective surface. Then there is a decomposition of* $\mathrm{Big}(X)$ *into rational locally polyhedral subcones (chambers) such that:*

(1) *In each chamber, the support of the negative part of the Zariski decomposition is constant.*
(2) *In the interior of each chamber, the stable base loci are constant.*

Example 9. Let $X = \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at two points. Construct Zariski chamber decomposition of $\overline{\mathrm{Eff}}(X)$.

Lecture 9. Topology of affine varieties and Lefschetz hyperplane theorems

Apr. 3, Speaker: Zhengyu Xiang

Goal. The lecture should be devoted to a proof of Artin-Grothendieck's theorem, which states that cohomology of constructible sheaves vanishes in degree higher than $\dim(X)$ for any affine X, and to its corollaries such as Lefschetz hyperplane theorem.

Lecture Outline.

Constructible sheaves. Recall that a sheaf $\mathcal{F}$ of abelian groups on an algebraic variety X is *constructible* if there is a decomposition

$$X = \bigsqcup_{i=1}^{N} X_i$$

of X into disjoint finite union of Zariski locally closed subvarieties such that $\mathcal{F}_{|X_i}$ is locally constant in the classical topology[4] on X_i.

Theorem 19 (Artin-Grothendieck theorem)**.** *Let V be an affine variety of dimension n and $\mathcal{F}$ a constructible sheaf on V, then $H^i(V, \mathcal{F}) = 0$ for all $i > n$.*

Using Noether's normalization lemma and the proposition below the Artin-Grothendieck theorem is reduced to the case when V is an affine space $\mathbb{C}^n$.

Proposition 6. *Suppose $f\colon X \to Y$ is a morphism of finite type between two algebraic varieties, then for any constructible sheaf $\mathcal{F}$ on X all higher direct images $R^i f_* \mathcal{F}$ are constructible on Y.*

2.20.1. *Lefschetz hyperplane theorems.*

Theorem 20 (Weak Lefschetz hyperplane theorem)**.** *Let X be a nonsingular irreducible complex projective variety X and D an effective ample divisor. Then the restriction maps*

$$H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

are isomorphism for $i \leq n-2$ and injective when $i = n-1$.

(1) Prove Lefschetz hyperplane theorem.
(2) Observe that ample divisors are always connected once $dim(X) \geq 2$.

[4]A classical topology is locally induced by Euclidean metric on complex affine spaces.

(3) State Lefschetz theorem on homotopy groups:

$$\pi_i(D) \to \pi_i(X)$$

is bijective if $i \leq n-2$ and surjective when $i = n-1$.

(4) Lefschetz theorem for Hodge groups.

(5) Lefschetz theorem for Picard groups.

Examples: A counterexample to the Lefschetz hyperplane theorem when X is singular.

Lecture 10. Kodaira vanishing theorem

Apr. 10, Speaker: Sho Tanimoto (tanimoto@cims.nye.edu)

Goal. Prove Kodaira's vanishing theorem and its generalizations.

The following result originally due to Hironaka will be used as a **black box**.

Theorem 21 (Resolution of singularities)**.** *If X is an irreducible complex algebraic variety and $D \subset X$ is an effective Cartier divisor, then there exists a projective birational*

$$\eta : X' \to X,$$

such that X' is nonsingular and

$$\eta^* D + Exc(\eta)$$

is a divisor (!) with simple normal crossings support. Moreover, η can be taken to be a sequence of blow-ups with smooth centers and to be an isomorphism over $X - Sing(X) - Sing(D)$.

T:K-vanishing **Theorem 22** (Kodaira vanishing)**.** *Let X be a smooth irreducible projective variety of dimension n and A an ample divisor on X, then for all $i > 0$,*

$$H^i(X, K_X + A) = 0.$$

Main ingredients of the proof are:

(1) Lefschetz hyperplane theorem (discussed in one of the previous lectures).

(2) Hodge decomposition of the complex cohomology.

(3) The case when there is an effective non-singular representative in $|A|$ follows trivially from a long exact sequence in cohomology and injectivity of $H^i(X, \mathcal{O}_X) \to H^i(D, \mathcal{O}_D)$ for $i \leq n-2$.

(4) The general case makes use of a cyclic covering trick.

(5) *The cyclic covering trick*: If $\mathcal{L} = \mathcal{M}^{\otimes n}$ on X and D an effective divisor with $\mathcal{O}_X(D) = \mathcal{L}$, then there exists a finite flat morphism $f : Y \to X$ such that $f^*D = nD'$ where D' is an effective divisor on Y.

(6) *Injectivity lemma*: if $f : X \to Y$ is a finite surjective morphism to a normal variety Y, then for any vector bundle E on Y, the homomorphism

$$H^j(Y, E) \to H^j(X, f^*E)$$

is injective.

2.21. **Assorted applications of Kodaira vanishing.**

2.21.1. *Bound for the index of a Fano variety.* Recall that a smooth variety X is *Fano* if $-K_X$ is ample. The *index* of a Fano variety X is the largest integer number f such that $-K_X = fA$ for some integral Cartier divisor A. Note that A is necessarily ample. It follows from the Bend-and-Break that X is covered by rational curves C such that $-K_X \cdot C \leq \dim(X) + 1$. In particular, we immediately obtain the bound $f \leq \dim(X) + 1$.

Kodaira vanishing gives an alternative proof of this. Consider a polynomial

$$P(m) = \chi(X, K_X + mA)$$

of degree $\dim(X)$. By Kodaira vanishing, for all $m \geq 1$,

$$P(m) = h^0(X, K_X + mA) = h^0(X, (m - f)A).$$

In particular, $P(m) = 0$ for $1 \leq m \leq f - 1$. Since the polynomial of degree $\dim(X)$ has at most $\dim(X)$ roots, we conclude that $f \leq \dim(X) + 1$.

2.21.2. *Constancy of Euler characteristic in numerical equivalence class.* (See [11, Proposition 2.57]) Let $D \equiv D'$ be two numerically equivalent Cartier divisors on smooth X. Without invoking Hirzebruch-Riemann-Roch, we prove that

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D')).$$

Fix an ample H. Consider the polynomial in u and v defined by the formula

$$P(u, v) = \chi(X, D' + u(D - D') + vH).$$

For $v \geq m_0$ (m_0 is independent of u!), we have that

$$D' + vH - K_X \equiv D' + u(D - D') + vH - K_X$$

is an ample class and so $h^i(X, D' + u(D - D') + vH) = 0$ for $i \geq 1$. We conclude that $P(u, v) = h^0(X, D' + u(D - D') + vH)$ for all $v \geq m_0$ and all u. Simple induction on the dimension of X shows that in fact $h^0(X, D' + u(D - D') + vH)$ is a bounded function of u for a fixed number v. We conclude that $P(u, v) = P(v)$ for $v \geq m_0$ and so for all v.

Taking $v = 0$, we see that $P(u, 0)$ is a constant function and so

$$\chi(X, \mathcal{O}_X(D')) = P(0, 0) = P(1, 0) = \chi(X, \mathcal{O}_X(D)).$$

Lecture 11. Vanishing for big and nef line bundles
Apr. 17, Speaker: MingMin Shen

Goal. Generalize Kodaira vanishing to big and nef line bundles.

T:KV-vanishing

Theorem 23 (Vanishing for big and nef divisors)**.** *Let X be a smooth irreducible projective variety of dimension n and D a nef and big divisor on X, then for all $i > 0$,*

$$H^i(X, K_X + D) = 0.$$

(1) State and prove Norimatsu's lemma.
(2) Prove the main theorem of the lecture.
(3) Ramunajan's vanishing is a corollary of the above.

Examples: An example where vanishing for nef and big divisors plays role in the classification of K3 surfaces (from Reid's notes).

Lecture 12. The cone theorem
Apr. 24, Speaker: Matt DeLand

Goal. State the cone theorem and discuss some aspects of its proof. The technical part should consist of the proof of the Base-Point-Free Theorem. The discussion can stay in the realm of smooth varieties, but the importance of working with fractional divisors has to be emphasized.

The main reference for this lecture is [11, Chapter 3].

Lecture outline.

Kawamata log terminal singularities. Recall that a log pair (X, Δ) is a datum of a normal projective variety X and a $\mathbb{Q}$-Cartier divisor $\Delta = \sum a_i D_i$ where D_i are *distinct* integral Weil divisors. We say that (X, Δ) has *Kawamata log terminal* singularities if $a_i < 1$ and there is an embedded resolution $f\colon Y \to X$ of $(X, \sum D_i)$ such that exceptional divisors E_i of the birational morphism f satisfy

$$K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum a(X, E_i)E_i$$

and all coefficients satisfy $a(X, E_i) > -1$.

Example 10. Give a non-trivial example of a klt pair naturally arising while running MMP.

Three Theorems. In this section we assume that (X, Δ) is a klt pair and D is a *nef integral Cartier* divisor. Recall that in particular $\Delta = \sum a_i D_i$ where $a_i \in \mathbb{Q} \cap (-\infty, 1)$ and D_i are distinct.

T:non-vanishing

Theorem 24 (Non-vanishing Theorem)**.** *Suppose there a positive rational number a such that*

$$aD - (K_X + \Delta)$$

is big and nef. Then

$$H^0(X, mD + \lceil -\Delta \rceil) \neq 0$$

for all $m \gg 0$.

Remark. Note that when Δ is effective, then $\lceil -\Delta \rceil = 0$ since $a_i < 1$ by definition of klt pair. In this case, the theorem really says that all large multiples of D have sections.

We also recall Kawamata Base-Point-Free theorem 2 in fuller generality.

T:bpf-thm-klt

Theorem 25 (Base-Point-Free Theorem)**.** *Let (X, Δ) be a klt pair as above and let Δ be effective now. Suppose there a positive rational number a such that*

$$aD - (K_X + \Delta)$$

is big and nef. Then $|mD|$ is base-point-free for all $m \gg 0$.

With the same assumptions on (X, Δ) is in the previous theorem, we have

T:rationality

Theorem 26 (Rationality Theorem)**.** *Suppose $K_X + \Delta$ is not nef. Then for any integral big and nef divisor H the number*

$$r = \max\{t \in \mathbb{R} \ : \ H + t(K_X + \Delta) \in \mathrm{Nef}(X)\}$$

is rational.

Proof of the Cone Theorem: an exercise. We are now ready to approach Theorem 1. We get a contraction part for free from the above three theorems. Indeed, for $K_X + \Delta$-negative extremal face F we can always find an ample divisor H such that the supporting hyperplane of F is defined by $H + t(K_X + \Delta)$. By Theorem 26, the number t is rational. Possibly after scaling, we obtain a nef Cartier divisor $D = H + t(K_X + \Delta)$. Since $D - K_X - \Delta = H$ is ample, and so big and nef, $|mD|$ is base-point-free for all $m \gg 0$ by Theorem 25. The algebraic fiber space $f\colon X \to Y$ existing by Theorem 11 is a requisite contraction of the extremal face F.

The decomposition

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

for *some* curves C_i is a formal consequence of the Rationality Theorem 26 in the convex geometry.

Proving that the class C_i is generated by a rational curve is trickier. By the contraction theorem there exists a birational morphism

$$f := \mathrm{contr}_{C_i}\colon X \to Y$$

such that any contracted curve lies in the class $[C_i]$. If Y is smooth then then the positive dimensional fibers of f are covered by rational curves[5] and we are done.

Suppose now that for some $p \in Y$ the fiber $f^{-1}(p)$ is positive dimensional and *normal*. Then there exists a map

$$g\colon C \to f^{-1}(p)$$

from a smooth curve to Y whose image avoids singular locus of X. Since $g(C)$ is contracted by f, we have

$$g(C) \cdot K_X < 0.$$

By the Bend-and-Break, there is a rational curve Γ such that numerically $g(C)$ is positive combination of Γ and another effective curve. Since $g(C)$ generates an extremal ray, we conclude that Γ belongs to the same ray.

More generally, we collect the following simple facts about the extremal contraction f:

(1) If the exceptional locus $\mathrm{Exc}(f)$ contains a divisorial irreducible component E, then $\mathrm{Exc}(f) = E$. In particular, $\mathrm{Exc}(f)$ is irreducible divisor. We call such f a *divisorial contraction*.
(2) If the general fiber of f has positive dimension, then it is a (log) Fano variety. In this case, f is called a *Fano fibration*.
(3) If f is a divisorial contraction or a Fano fibration, then the image of f is $\mathbb{Q}$-factorial.
(4) If $\mathrm{codim}(\mathrm{Exc}(f), X) \geq 2$, then f is called a *small contraction*.
(5) The image of small extremal contraction is necessarily not $\mathbb{Q}$-Gorenstein.

Black boxes.

Theorem 27 (Bend-and-Break)**.** *Let X be a projective variety and H any ample divisor on X. Suppose $f\colon C \to X$ is a morphism from a smooth curve such that*

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X is smooth along $f(C)$ and $\deg_C(f^*K_X) < 0$. Then through any point $x \in f(C)$ passes a rational curve Γ with

$$H \cdot \Gamma \leq 2\dim(X)\frac{H \cdot C}{-K_X \cdot C} \quad .$$

Moreover, if X is smooth, $2\dim(X)$ above can be replaced by $\dim(X)+1$.

Lecture 13. Extremal Contractions

May. 1, Speaker: Yi Zhu (yzhu@math.sunysb.edu)

Goal. Illustrate the Contraction Theorem in action. Give examples of Mori fiber space, divisorial and small contractions and examples of flips. The main reference for this lecture will be Section 6.5-6.6 of [3].

Lecture Outline.

Elementary properties of extremal contractions. For now, X is a smooth projective variety.

In the last lecture, we have seen that whenever the canonical divisor K_X is not nef, the K_X-negative part of the closed cone of curves $\overline{\mathrm{NE}}(X)$ is spanned by countably many *extremal rays* with each ray being generated by a class of a rational curve.

Pick now a *negative extremal ray* $\mathbb{R}_{\geq 0}[C]$ where C is a rational curve on X. By the proof of Contraction Theorem, there exists a divisor D of the form $K_X + tA$, where A is an ample divisor and t is a rational number, such that the linear series $|mD|$ defines a contraction morphism

$$f := \mathrm{contr}_C\colon\ X \to Y.$$

By construction of f the following is true:

(1) An effective curve $D \subset X$ satisfies $f_*D = 0$ if only if $D \in \mathbb{R}_{\geq 0}[C]$.
(2) A line bundle $\mathcal{L}$ on X satisfies $\mathcal{L} \cdot C = 0$ if only if it is a pullback of a line bundle from Y. In other words, there are exact sequences[6]

$$0 \to \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\cdot C} \mathbb{Z}$$

$$0 \to \mathrm{Num}(Y)_{\mathbb{R}} \xrightarrow{f^*} \mathrm{Num}(X)_{\mathbb{R}} \xrightarrow{\cdot C} \mathbb{R} \to 0.$$

There are three qualitatively different behaviors of the extremal contraction f:

(1) If $\dim(Y) < \dim(X)$, then f is called a *Mori fiber space.*
(2) If $\dim(Y) = \dim(X)$ and there is a divisorial component of the exceptional locus $\mathrm{Exc}(f)$, then f is called a *divisorial contraction.*
(3) If $\dim(Z) = \dim(X)$ and $\dim(\mathrm{Exc}(f)) \leq \dim(X) - 2$, then f is called a *small contraction* or a *flipping contraction.*

We collect now elementary properties of the extremal contraction f:

(1) $\rho(X/Y) := \rho(X) - \rho(Y) = 1$.

(2) If f is a small contraction, then the image Y is not $\mathbb{Q}$-Gorenstein. Indeed, suppose mK_Y is a Cartier divisor for some m. Then necessarily $f^*(mK_Y) = mK_X$ and we arrive at contradiction with $K_X \cdot C < 0$.

[6] Exactness on the left is a general property of an algebraic fiber space. Exactness on the right in the second s.e.s. follows from existence of an ample divisor on X.

Remark. There are of course small contraction with Gorenstein image as illustrated by both small resolutions of the 3-fold rational double point singularity $\{xy - zw = 0\} \subset \mathbb{C}^4$.

(3) If f is a Mori fiber space or a divisorial contraction, then Y is $\mathbb{Q}$-factorial. This is Corollary 3.18 of [11].

(4) If f is a divisorial contraction, then the exceptional locus $\mathrm{Exc}(f)$ is an irreducible divisor. In particular, it is connected.

Proof. Take an irreducible divisor $E \subset \mathrm{Exc}(f)$.

Claim: $C \cdot E < 0$.

One finds an ample divisor H on Y such that $f^*H = f_*^{-1}H + aE$ where a is a positive integer. Since, $C \cdot f^*H = 0$, it remains to show that $(f_*^{-1}H) \cdot C > 0$. As f^*H is a connected Cartier divisor, $f_*^{-1}H$ intersects E non-trivially and so meets a fiber of f that is covered by curves numerically equivalent to C.

We conclude that any curve contracted by f intersects E negatively and hence is contained in E. Thus $\mathrm{Exc}(f)$ has no other components. □

Examples: Extremal divisorial contractions. Last time we saw that a blow-up along a smooth subvariety is a divisorial contraction. For an example of a divisorial contraction that is not a smooth blow-up consider the Abel-Jacobi map

$$\pi\colon\ C_d \to \mathrm{Pic}^d(C)$$

from a symmetric power of a smooth curve C to the Jacobian of C. The fiber of π over $\mathcal{L} \in \mathrm{Pic}^d(C)$ is $\mathbb{P}H^0(C, \mathcal{L})$.

We now show that π is K_{C_d}-negative contraction when $d \geq g$.

Proof. A line $B \cong \mathbb{P}^1$ in the fiber $\pi^{-1}(\mathcal{L})$ corresponds to a pencil of divisors in $\mathbb{P}H^0(C, \mathcal{L})$. The tangent space to a divisor $D \in C_d$ is naturally identified with $H^0(D, \mathcal{O}_D(D))$. We can therefore identify the restriction of T_{C_d} to B as

$$\mu_*(\mathcal{O}_{C\times B}(\Gamma))$$

where $\Gamma \subset C \times B$ is the restriction of the universal divisor $\Delta \subset C \times C_d$ and the universal family $C \times C_d \to C_d$ to the line B. It is easy to see that Γ is in fact a union of the graph of the map $C \to B$ defined by the base-point-free part of the pencil B and the horizontal sections $\Sigma_1, \dots, \Sigma_k$ corresponding to the base points of B.

Grothendieck-Riemann-Roch computation now gives:

$$ch(\pi_! \mathcal{O}(\Gamma)) = \pi_* \left((1 + \Gamma + \Gamma^2/2)(1 + (1-g)\Sigma)\right)$$

where Σ is a fiber of the projection to C. Hence,

$$-K_{C_d} \cdot B = -c_1(T_{C_d|B}) = -(\Gamma^2/2 + (1-g)\Sigma) = g - d - 1.$$

□

When $d \geq g+1$, the Abel-Jacobi map has positive dimensional fibers but it is not a projective bundle since dimensions of fibers jump. When $d = g$, the Abel-Jacobi map is a divisorial contraction but not a smooth blow-up for $g \geq 6$.

Example: Small contraction and a flip. This is Example 1.36 from [3]. Define

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(-1)^{\oplus(r+1)}$$

and

$$Y_{r,s} = \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(-1)^{\oplus(r+1)}).$$

Let P be the section of $\pi\colon Y_{r,s} \to \mathbb{P}^s$ corresponding to a trivial subbundle of $\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(1)$.

Claim: The normal bundle to P is $\mathcal{O}_{\mathbb{P}^s}(-1)^{\oplus(r+1)}$.

Proof. The way to see this is to recall that the vertical tangent bundle of $Y_{r,s} \to \mathbb{P}^s$ along P is

$$\underline{Hom}(\mathcal{O}_{\mathbb{P}^s}, \mathcal{E}/\mathcal{O}_{\mathbb{P}^s}) = \mathcal{O}_{\mathbb{P}^s}(-1)^{\oplus(r+1)}.$$

□

We conclude that

$$K_{Y_{r,s}|P} = -(c_1(T_P) + (r+1)c_1(\mathcal{O}_P(-1))) = (r-s)c_1(\mathcal{O}_P(1)).$$

It follows that when $s \geq r+1$ the class of a line in P generates a $K_{Y_{r,s}}$-negative ray. The corresponding contraction contr_P contracts the whole plane P to the point. Clearly, this is a small contraction.

To produce the flip of contr_P, we consider the blow-up $X_{r,s}$ of $Y_{r,s}$ along P. The exceptional divisor E of $Y_{r,s}$ is isomorphic to $P \times \mathbb{P}^r = \mathbb{P}^s \times \mathbb{P}^r$. It can be checked that the class of the curve mapping to the line in P is $K_{X_{r,s}}$-negative and so can be contracted.

A small contraction with a disconnected exceptional locus.

A divisorial contraction with singular image.

A flip in dimension 3.

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