Severi varieties and the moduli space of curves

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Abstract

We study Severi varieties parameterizing genus g curves in a fixed divisor class on a rational surface. Corresponding to every such variety, there is a one-parameter family of genus g stable curves whose numerical invariants we compute. Building on the work of Caporaso and Harris, as well as Vakil, we derive a recursive formula for the degrees of the Hodge bundle on the families in question. In the case when a surface is isomorphic to $\mathbb{P}^2$, we produce moving curves in the moduli space $\overline{M}_g$ of Deligne-Mumford stable curves. We use these to derive lower bounds on the slopes of effective divisors on $\overline{M}_g$. Another application of our results is to various enumerative problems for planar curves.

Contents

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1 Introduction

In this thesis we study parameter spaces of curves on rational surfaces. The rational surfaces we consider fall into two categories: those isomorphic to the complex plane $\mathbb{P}^2$ and rational ruled surfaces. Our main focus will be on the relationship between these spaces and the moduli space M_g of abstract smooth curves of fixed genus g. The central role here will be played by the celebrated Deligne and Mumford's compactification of M_g by the moduli space of stable curves $\overline{M}_g$.

Classically, abstract smooth curves of any genus were studied via their embeddings into the projective plane as divisors of sufficiently high degree with at worst nodes as singularities. We exploit this viewpoint in both direction. On one hand, we obtain a better understanding of the variety of embedded curves by studying the induced map into the moduli space of abstract curves. Our results in this direction are mostly enumerative in nature. The principal result of this thesis is Theorem 1 of Section 5.1. On the other hand, we obtain new information about $\overline{M}_g$ as a consequence of our investigation. Namely, new lower bounds on the slopes of effective divisors on $\overline{M}_g$ are obtained in Section 13. We refer the reader to Section 2 for the definition of the slope of an effective divisor on $\overline{M}_g$.

The motivating problem of this thesis is enumerative in nature: can we calculate numerical invariants of families of plane curves using degeneration techniques?

1.1 Severi varieties of curves on surfaces

Let S be a smooth projective surface. Every Weil divisor is Cartier on S, so we will not make a distinction among Weil and Cartier divisors. Fix an effective divisor class D and consider the associated linear system

$$|D| = |\mathcal{O}_S(D)|.$$

The functor $\mathrm{LinSys}_D(S)$ of flat families of effective Cartier divisors linearly equivalent to D is representable by $\mathbb{P}H^0(S, \mathcal{O}_S(D))$ (see Proposition 2 in Lecture 13 of [26]). From now on we will not distinguish between the linear system $|D|$ and the projective space $\mathbb{P}H^0(S, \mathcal{O}_S(D))$ whose closed points parameterize elements of the linear system.

For a smooth surface S, the parameter space $\mathrm{CDiv}(S)$ of effective Cartier divisors is isomorphic to the Hilbert scheme of divisors by [23, Theorem 1.13, p. 18]. Suppose that S is rational. By Serre's duality, $h^1(S, \mathcal{O}_S) = h^1(S, K_S) = 0$ and $h^2(S, \mathcal{O}_S) = h^0(S, K_S) = 0$. Therefore, for any effective divisor C in $|D|$, the tangent space to the Hilbert scheme $\mathrm{Hilb}(S)$ at the point $[C]$ has dimension

$$h^0(C, \mathcal{O}_C(C)) = h^0(C, \mathcal{O}_S(C)) - 1.$$

As this equals to the dimension of $\mathbb{P}H^0(S, \mathcal{O}_S(D))$, it follows that the component $\mathrm{Hilb}_D(S)$ of the Hilbert scheme, containing a divisor from $|D|$ is isomorphic to $\mathbb{P}H^0(S, \mathcal{O}_S(D))$.

Inside $|D|$ we consider a locally closed subvariety $U^{D,\delta}$ parameterizing curves with exactly δ nodes and no other singularities and a subvariety $U^{D,\delta}_{con}$ of $U^{D,\delta}$ parameterizing *irreducible* curves.[1]

Definition 1.1. The *Severi variety* $V^{D,\delta}$ is the closure of $U^{D,\delta}$ inside $|D|$. We also denote by $V^{D,\delta}_{con}$ the closure of $U^{D,\delta}_{con}$.

The geometric genus of any curve in $U^{D,\delta}$ is

$$g = p_a(D) - \delta = \frac{1}{2} D \cdot (D + K_S) + 1 - \delta. \tag{1}$$

The variety of $V^{D,\delta}$ is unpleasant for several reasons. First, it is defined as a closure of a locally closed subvariety – an unsatisfactory procedure from the point

[1] The subscript *con* indicates a *connected* normalization.

of view of moduli theory. Second, it is extremely singular in general and its local geometry is difficult to study because the variety lacks a modular interpretation. However, it is possible to analyze the geometry of $V^{D,\delta}$ in codimension one. This is our general philosophy.

1.2 Classical problems

The two most basic and, at the same time, important numerical attributes of the Severi variety $V^{D,\delta}$ are the number of its irreducible components and its projective degree inside $|D|$.

1.2.1 Irreducibility

For surfaces considered in this thesis, the variety $V^{D,\delta}_{con}$ will always be an irreducible component of $V^{D,\delta}$. Often there will be other components of $V^{D,\delta}$ parameterizing divisors with several irreducible components. One of the first examples is the variety of 3-nodal quartics in the plane. It has two components: one consisting of irreducible rational quartics and another consisting of unions of a smooth cubic and a line. Therefore, the genuine question is whether the Severi varieties $V^{D,\delta}_{con}$ are irreducible.

Classically, this question attracted considerable interest. In the case of the projective plane $\mathbb{P}^2$, the irreducibility statement goes back to Severi, who was also the first to make an attempt at proving it. Although his proof was incomplete, the approach proposed by Severi was at least partially vindicated by a theorem of Harris [15] establishing irreducibility of Severi varieties of plane curves. At about the same time, Ran established irreducibility of more general subvarieties of the space of plane curves [28]. We also note the recent result of Tyomkin saying that Severi varieties of irreducible curves on Hirzebruch surfaces are irreducible [29]. The irreducibility results will be important for the slope estimates in Section 13.

1.2.2 Enumerative questions

Definition 1.2. We denote degrees of $V^{D,\delta}$ and $V^{D,\delta}_{con}$ inside $|D|$ by $N^{D,\delta}$ and $N^{D,\delta}_{con}$, respectively.

The problem of finding the degrees of $V^{D,\delta}_{con}$ and $V^{D,\delta}$ is enumerative in nature. It reduces to the following question.

Problem. *How many elements of* $V^{D,\delta}_{con}$ *pass through* $\dim V^{D,\delta}_{con}$ *points on* S*?*

When $S \cong \mathbb{P}^2$, this, and in fact a more general, question was answered, using a degeneration approach, by Caporaso and Harris [2] (see also [3]). Subsequently, their method was applied by Vakil to enumerate curves on rational surfaces with sufficiently positive anticanonical bundle [31]. Our study of Severi varieties is motivated by these papers. Using their methods, supplemented by a careful study of versal deformations of higher-order tacnodes, we will be able to address more refined questions about the Severi varieties. However, even to state these questions we need to introduce a plethora of new definitions. We do this in Section 3. Section 2 contains a quick overview of the divisor theory of the moduli space of stable curves necessary for our purposes. Finally, the main theorem of this thesis is stated in Section 5 and proved in Section 11.

1.3 Notations

We work over the field of complex numbers $\mathbb{C}$. Given a scheme X, we denote its normalization by X^ν. The Zariski tangent space to X at a point x is denoted by $\mathbb{T}_x X$.

The *tacnode* of order m is a planar curve singularity analytically isomorphic to the singularity of $y^2 - x^{2m+2} = 0$ at the origin. For the sake of uniformity, we will not distinguish between a node and a tacnode of order 0. Note that two branches of the m^{th}-order tacnode have order of contact m and the intersection multiplicity $m+1$.

A smooth *log surface* is a pair (S, L) consisting of a smooth projective surface S and a smooth divisor $L \subset S$. We call a divisor on a smooth surface δ-*nodal* if its only singularities are precisely δ nodes; we call a divisor *nodal* if its δ-nodal for some δ. By abuse of language, a smooth divisor is nodal.

Given a sequence of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots)$, with finitely many non-zero elements, we define

$$|\alpha| = \sum_i \alpha_i \ ,$$

$$I\alpha = \sum_i i\alpha_i \ ,$$

$$I^\alpha = \prod_i i^{\alpha_i} \ .$$

For two such sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$, set

$$\binom{\alpha}{\beta} = \prod_i \binom{\alpha_i}{\beta_i}.$$

Throughout the paper, δ_{ij} stands for the Kronecker's delta. We reserve the symbol ϵ for the generator of the ring of dual numbers $\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$.

We denote by $\operatorname{card}(\Omega)$ the cardinality of a set Ω.

2 Generalities on $\overline{M}_{g,n}$

One motivation for the study of the Severi variety in this thesis comes from the moduli space of Deligne-Mumford stable curves $\overline{M}_{g,n}$. Since $\overline{M}_{g,n}$ is both an object of study and one of the main tools of this work, we recall the most important definitions related to this moduli space. For the construction of $\overline{M}_{g,n}$ in the case $n = 0$, we refer the reader to Deligne and Mumford's original paper [6]. The case of $n > 0$ is treated in [21], [22] and [27]. Here, we review some basic facts about the Picard group of $\overline{M}_{g,n}$ and discuss the necessary formalism for intersection theory of curves and divisors on $\overline{M}_{g,n}$.

2.1 The moduli space of n-pointed stable curves

An n-pointed curve $(C; p_1, \ldots, p_n)$ is called *stable* if C is a connected curve with at worst nodal singularities, p_i are smooth points of C, and $(C; p_1, \ldots, p_n)$ has a finite automorphism group. The last condition can also be rephrased as saying that $\omega_C(p_1 + \cdots + p_n)$ is ample, where ω_C is the dualizing line bundle on C.

The functor $\overline{\mathcal{M}}_{g,n}$ of flat families of n-pointed stable curves of arithmetic genus g is finely represented by a Deligne-Mumford stack, which we also denote $\overline{\mathcal{M}}_{g,n}$.

Given a flat family of stable curves

$$f\colon \mathcal{C} \to B$$

we consider the relative dualizing sheaf $\omega_f = \omega_{\mathcal{C}/B}$ on $\mathcal{C}$ and define the Hodge bundle of $\mathcal{C}/B$ by

$$\mathbb{E}_{\mathcal{C}/B} := f_*(\omega_f).$$

Associated to the universal family

$$\overline{\mathcal{M}}_{g,n+1} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}.$$

is the Hodge bundle $\mathbb{E} := \pi_*(\omega_\pi)$ on the moduli stack $\overline{\mathcal{M}}_{g,n}$.

Remark 1. The Hodge bundle of any flat family $\mathcal{C}/B$ is a pullback of the Hodge bundle $\mathbb{E}$ on $\overline{\mathcal{M}}_{g,n}$ under the natural morphism $B \to \overline{\mathcal{M}}_{g,n}$ induced by the family $\mathcal{C} \to B$.

Finally, the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$ exists as a projective variety and is denoted $\overline{M}_{g,n}$. The singularities of $\overline{M}_{g,n}$ are finite-quotient singularities. Therefore, every Weil divisor on $\overline{M}_{g,n}$ is $\mathbb{Q}$-Cartier:

$$\mathrm{Div}(\overline{M}_{g,n}) \otimes \mathbb{Q} = \mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}.$$

2.2 The Picard group of $\overline{M}_{g,n}$.

The Chern classes of $\mathbb{E}$ are denoted

$$\lambda_k := c_k(\mathbb{E}).$$

We reserve the symbol λ to denote the first Chern class of $\mathbb{E}$. The universal curve

$$\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

has n sections σ_i, $1 \leq i \leq n$. Recall that ψ-classes on $\overline{\mathcal{M}}_{g,n}$ are defined by

$$\psi_i = \sigma_i^*(c_1(\omega_\pi)).$$

Finally, let $\delta_{k,A}$ be the class of the divisor of curves that have a node which separates the curve into a component of genus k, marked by points $A \subset \{p_1, \dots, p_n\}$, and a component of genus $g-k$, marked by points $\{p_1, \dots, p_n\} \smallsetminus A$. When $n = 0$, we use the symbol δ_k instead of $\delta_{k,\emptyset}$. Finally, we use the symbol δ_0 to denote the boundary divisor of curves containing a non-separating node.

Arbarello and Cornalba proved [1], using purely algebro-geometric techniques, that the Picard group of $\overline{M}_{g,n}$ is generated by the classes

$$\{\lambda, \psi_i, \delta_0, \delta_{k,A}\}$$

and explicitly describe all the relations among the generators.

We will only need the following two facts. First, the Picard group of $\overline{M}_g$ is

$$\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} = \mathbb{Q}\{\lambda, \delta_0, \dots, \delta_{\lfloor g/2 \rfloor}\}.$$

Here, δ is the total class of the boundary:

$$\delta = \delta_0 + \cdots + \delta_{\lfloor g/2 \rfloor}.$$

Second, we have Mumford's formula for $\mathrm{Pic}(\overline{M}_{1,n})$:

$$\lambda = \delta_0/12. \qquad (2)$$

2.3 Curves in $\overline{M}_{g,n}$

Consider a proper family $\mathcal{X} \to C$ of curves over a one-dimensional, reduced, irreducible base C, with a reduced total space $\mathcal{X}$. Suppose that a generic fiber is a stable curve of arithmetic genus g. Then there is an open and regular $U \subset C$, such that every irreducible component of $\mathcal{X} \times_C U$ dominates U. Moreover, by the openness of the locus of at worst nodal fibers, we can choose U so that fibers of $\mathcal{X} \times_C U \to U$ are nodal and stable. Then by [17, Proposition 9.7, p. 257], $\mathcal{X} \times_C U \to U$ is a flat family of stable curves.

By the stable reduction theorem (cf. [16, Proposition 3.47]), there is a finite surjective base change $f\colon C' \to C$ and a flat family $\pi\colon \mathcal{Y} \to C'$ of stable curves of genus g such that

$$\mathcal{Y} \times_C U \cong \mathcal{X} \times_C U.$$

Since $\overline{M}_g$ is a coarse moduli space for the functor $\overline{\mathcal{M}}_g$, the family $\mathcal{Y}/C'$ induces a morphism $C' \to \overline{M}_g$.

Definition 2.1. For a proper family $\mathcal{X} \to C$ of curves as above, the *degree of* λ *on* $\mathcal{X}/C$ is the number

$$\lambda_{\mathcal{X}/C} := c_1(\mathbb{E}_{\mathcal{Y}/C'})/\deg f.$$

Without performing a stable reduction, we still have a natural rational map from C to $\overline{M}_g$:

$$j\colon C \dashrightarrow \overline{M}_g.$$

The map j is regular on an open subset $U \subset C$ and by normality of $\overline{M}_g$ extends to a regular morphism $j\colon C^\nu \to \overline{M}_g$ after the normalization of C. We remark that $\lambda_{\mathcal{X}/C}$ also equals to the intersection number

$$j_*(C^\nu) \cdot \lambda,$$

where $j_*(C^\nu)$ is the pushforward of the fundamental class of C^ν under j and λ is the $\mathbb{Q}$-Cartier divisor on $\overline{M}_g$ associated to the λ-class on the stack $\overline{\mathcal{M}}_g$.

Definition 2.2. For a proper family $\mathcal{X} \to C$ whose generic fiber is a disjoint union of stable curves, we can decompose $\mathcal{X}$ into a union of components such that every component has a connected stable generic fiber over C. We define $\lambda_{\mathcal{X}/C}$ to be the sum of degrees of λ on these components.

Note that $\lambda_{\mathcal{X}/C}$ depends only on the geometry of the family $\mathcal{X} \to C$ at a generic point of C. If the family is understood, we use the shorthand λ_C to denote the degree of λ on $\mathcal{X}/C$.

We will use the following two simple statements often.

Lemma 2.1. *Suppose $\mathcal{X} \to B$ is a flat proper family of curves of arithmetic genus g over a reduced one-dimensional base B. If a generic fiber of $\mathcal{X}$ has δ nodes and no other singularities, then the normalization $\mathcal{X}^\nu \to B$ is a proper (not necessarily flat) family, whose generic fiber is a smooth curve of genus $g - \delta$, and we have*

$$\lambda_{\mathcal{X}/B} = \lambda_{\mathcal{X}^\nu/B}.$$

Proof. After performing a stable reduction of $\mathcal{X} \to B$ and after a suitable base-change, we may assume that $\mathcal{X} \to B$ is a flat family of curves in $\overline{\mathcal{M}}_g$ with δ sections $N_1, \ldots, N_\delta$ such that $N_i(b)$ is a node of $\mathcal{X}_b$ for all $b \in B$.

Consider the normalization $\nu\colon X^\nu \to X$. The family $\mathcal{X}^\nu \to B$ has two sections P_i and R_i lying over each section N_i, for $1 \leq i \leq \delta$. Let Res be the residue map obtained by restricting to $\omega_{X/B}$ the natural sheaf homomorphism

$$\nu_*(\omega_{X^\nu/B}(P_i + R_i)) \to \oplus_{i=1}^{\delta} \nu_*(\mathcal{O}_{P_i}).$$

Then we have an exact sequence of sheaves on $\mathcal{X}$

$$0 \to \nu_*\omega_{\mathcal{X}^\nu/B} \to \omega_{\mathcal{X}/B} \xrightarrow{\text{Res}} \oplus_{i=1}^{\delta} \mathcal{O}_{N_i} \to 0.$$

The pushforward of this exact sequence to B gives

$$0 \to \mathbb{E}_{\mathcal{X}/B} \to \mathbb{E}_{\mathcal{X}^\nu/B} \to \mathcal{O}_B^{\oplus\delta} \to 0.$$

The lemma follows. $\square$

Lemma 2.2. *Let $\mathcal{X}_1/B$ and $\mathcal{X}_2/B$ be two families of stable curves in $\overline{\mathcal{M}}_{g_1,n+k}$ and $\overline{\mathcal{M}}_{g_2,m+k}$, respectively. Form a family*

$$\mathcal{X} := \mathcal{X}_1 \cup \mathcal{X}_2$$

of curves in $\overline{\mathcal{M}}_{g_1+g_2+k-1,n+m}$ by identifying the section σ_i^1 of $\mathcal{X}_1$ with the section σ_i^2 of $\mathcal{X}_2$, for every $1 \leq i \leq k$. Then

$$\lambda_{\mathcal{X}} = \lambda_{\mathcal{X}_1} + \lambda_{\mathcal{X}_2}.$$

Proof. Normalizing the family $\mathcal{X}$ and using Lemma 2.1, we reduce the lemma to the case of the disjoint union ($k = 0$). In this case the statement of the lemma follows by Definition 2.2. $\square$

2.4 Effective divisors on $\overline{M}_g$

By Proposition 2.4 of [4], the λ-class is nef on $\overline{M}_g$. Using this fact it is easy to show that a class of an effective irreducible divisor D on $\overline{M}_g$, which is not contained in the boundary, can be written as

$$[D] = a\lambda - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i \delta_i,$$

where a and b_i's are positive numbers.

Definition 2.3. The *slope of D* is defined as

$$s(D) := \frac{a}{\min\{b_i\}}.$$

Effective divisors with small slopes play especially important role. For example, Eisenbud and Harris proved [8] that $\overline{M}_g$ is of general type for $g \geq 24$ by constructing divisors with slope less than $6\frac{1}{2}$. More recently, Farkas found such a divisor for $g = 22$, thus proving that $\overline{M}_{22}$ is of general type [9].

We denote by s_g the infimum of slopes $s(D)$ over all effective irreducible divisors D, not contained in the boundary, on $\overline{M}_g$.

Harris and Morrison conjectured [14] that

$$s_g \geq 6 + \frac{12}{g+1}.$$

However, Farkas and Popa [10] showed that the divisor of genus 10 curves lying on $K3$ surfaces has slope 7, providing a counterexample to the conjecture. Thus, the problem of finding optimal lower bounds on s_g remains open. We make a contribution to this program by computing new lower bounds in the final section of this thesis. For large g our bounds appear to be the best currently known.

2.5 Bounding slopes

The standard approach to bounding slopes of effective divisors from below is by finding a class in a cone of curves $N_1(\overline{M}_g)$ which pairs non-negatively with every effective divisor. A natural choice is a class of a moving curve in $\overline{M}_g$.

Definition 2.4. An irreducible curve $C \xrightarrow{f} X$ is a *moving curve* in X if there exists a flat family $\mathcal{C} \xrightarrow{\pi} T$ and a dominant map $F\colon \mathcal{C} \to X$, whose fiber over a closed point $t \in T$ is $f\colon C \to X$.

Remark 2. Informally, deformations of a moving curve are dense in the ambient variety X.

Clearly, the class of a moving curve C intersects non-negatively all effective divisor classes. For an effective $[D] = a\lambda - \sum b_i\delta_i$ this translates into a series of inequalities:

$$
\begin{aligned}
C \cdot D &\geq 0, \\
a(C \cdot \lambda) &\geq \sum_{i=1}^{\lfloor g/2 \rfloor} b_i (C \cdot \delta_i), \\
a(C \cdot \lambda) &\geq \min\{b_i\}(\sum C \cdot \delta_i) = \min\{b_i\}(C \cdot \delta)\,.
\end{aligned}
$$

Numbers $C \cdot \lambda$ and $C \cdot \delta_i$ are basic numerical invariants of a curve $C \to \overline{M}_g$. In terms of them, we have the following inequality:

$$
s(D) \geq \frac{C \cdot \delta}{C \cdot \lambda} \geq \frac{C \cdot \delta_0}{C \cdot \lambda}.
$$

Finding moving curves in $\overline{M}_g$ and computing their pairing with the Picard group is an important problem. Speaking generally, a curve in $\overline{M}_g$ is determined by a flat one-parameter family $\mathcal{C} \to B$ of genus g stable curves, i.e., by a one-dimensional algebraic system of curves on a surface $\mathcal{C}$. From this point of view, the families in $\overline{M}_g$ for which the intersection numbers with divisors are easiest to compute are simply linear pencils of curves on surfaces. The drawback of linear pencils is that associated curves in $\overline{M}_g$ are rarely moving, since a rational curve can pass through a generic point of $\overline{M}_g$ only when the moduli space is uniruled. For $g \geq 22$, the Kodaira dimension of $\overline{M}_g$ is non-negative and linear pencils on surfaces cannot be moving curves.

In this thesis we go one step further and consider algebraic families of curves defined by imposing nodal singularities on elements of a linear system on a surface. The resulting families are *curve sections of Severi varieties* whose precise definition is given in Section 3.

3 Generalized Severi varieties of divisors and maps

3.1 Surfaces

We consider the class of smooth log surfaces (S, L) that is slightly more narrow than the one considered in [31]. Namely, we assume the following:

1. The surface S is rational and $L \cong \mathbb{P}^1$ is a smooth divisor on S.

2. The surface S contains no (-1)-curves except for, possibly, L.

3. For every effective divisor C on S

$$-(K_S + L) \cdot C \geq 1,$$

   and if the equality holds, then C is smooth.

4. If C is a curve with a triple point or a tacnode, then

$$-(K_S + L) \cdot C \geq 4.$$

Remark 3. Vakil considered [31] log surfaces satisfying conditions (1-3).

Examples of log pairs satisfying above definitions are:

1. (S, L), with $S \cong \mathbb{P}^2$ and L a line in $\mathbb{P}^2$.

2. $(S, L) \cong (\mathbb{F}_n, E)$, where $\mathbb{F}_n \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(n)\right)$ is the Hirzebruch surface with E the unique section of self-intersection $-n$, for some non-negative integer n.

3.2 Stable maps to surfaces

Consider a projective smooth surface S. Recall that a *stable map* to S is a morphism

$$f\colon C \to S$$

with a finite automorphism group and whose domain is a Deligne-Mumford semistable curve C. The notion of a stable map was introduced by Kontsevich (see [24] and [25]). The coarse moduli space $\overline{M}_{g,0}(S,\beta)$ of genus g stable maps $f\colon C \to S$, with the image of the fundamental class of C in $\beta \in H_2(S,\mathbb{Z})$, was constructed by Fulton and Pandharipande [11].

The λ-class on $\overline{M}_{g,0}(S,\beta)$ is defined via the pullback under the forgetful morphism

$$\overline{M}_{g,0}(S,\beta) \to \overline{M}_g.$$

In the case S is a smooth surface with

$$A_1(S) \cong H_2(S,\mathbb{Z}) \tag{3}$$

and D a divisor class on S, we obtain the moduli space $\overline{M}_{g,0}(S,D)$ of stable maps $f\colon C \to S$ with the image of the fundamental class of C in D. Note that a smooth rational surface satisfies condition (3).

From now on, all surfaces S satisfy conditions of Section 3.1, and, in particular, are rational. We recall the notion of quasi-stable maps as it appears in [31].

Definition 3.1. Given a smooth projective surface S, a *quasi-stable map*

$$f\colon C \to S$$

is a map from a possibly disconnected nodal curve C such that its restriction to every connected component of C is stable in the sense of Kontsevich.

The existence of a coarse moduli space for stable maps implies the existence of the coarse moduli space

$$M^{D,g}(S)$$

for quasi-stable maps $f\colon\ C \to S$ with $p_a(C) = g$ and the image cycle $f_*[C]$ in D.

Since the domain of a map is allowed to be disconnected, a forgetful morphism from $M^{D,g}(S)$ to the moduli space $\overline{M}_g$ of stable curves cannot be defined. Instead, consider a connected component M of $M^{D,g}(S)$. Its closed points correspond, for some k, to quasi-stable maps whose domain consists of k connected components of arithmetic genera $g_1, \ldots, g_k$. We can consider a forgetful morphism (which exists after possibly a finite base change that distinguishes the connected components) to the product of the moduli spaces of stable curves

$$\overline{M}_{g_1} \times \cdots \times \overline{M}_{g_k}.$$

We define the λ-class on M to be the sum of the pullbacks of the λ-classes from $\overline{M}_{g_i}$. In this way, we obtain a well-defined divisor class λ on $M^{D,g}(S)$.

3.3 The cycle morphism

Consider an arbitrary family in $M^{D,g}(S)$ of quasi-stable maps over a *reduced* base B.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & B \times S \\ & {\scriptstyle \pi} \searrow \quad \swarrow & \\ & B & \end{array}$$

Form an image cycle $f_*[\mathcal{X}]$ on $B \times S$. Since pushforward commutes with specialization to a regular point [12, Proposition 10.1],

$$(f_*[\mathcal{X}])_b = (f_b)_*[\mathcal{X}_b] \tag{4}$$

for any regular point $b \in B$. Let B^* be the open subscheme of regular points of B and $\mathcal{X}^* = \mathcal{X} \times_B B^*$. The cycle $\mathcal{D} = f_*[\mathcal{X}^*]$ is a Cartier divisor on $B^* \times S$ by smoothness of $B^* \times S$. Moreover, its fibers over B are in the class D on S by (4) and so have the same Hilbert polynomial. Therefore, $\mathcal{D}$ is a flat family of effective Cartier divisors. By the universal property of the Hilbert scheme, we have an induced regular morphism

$$\psi\colon\ B^* \to \mathrm{Hilb}_D(S).$$

Let $\tilde{B}$ be the graph of the rational map $\psi\colon\ B \dashrightarrow \mathrm{Hilb}_D(S)$.

Observation 1. *The morphism $\tilde{B} \to B$ is finite and set-theoretically one-to-one.*

Proof. Take a point $b \in B$. Consider an arbitrary punctured smooth curve $(T, 0)$ and a morphism $\iota\colon\ (T, 0) \to B$, with $\iota(0) = b$ and the generic point of T mapping to B^*. Let $f\colon\ \mathcal{C}/T \to S$ be the induced family of quasi-stable maps over T. Then the flat limit of $f_*[\mathcal{C}_t]$, for $t \to 0$, is $f_*[\mathcal{C}_0] = f_*[\mathcal{X}_b]$. This shows that the fiber of $\tilde{B} \to B$ is set-theoretically a point. Finally, $\tilde{B} \to B$ is quasi-finite and projective, hence finite. $\square$

Consider an open subscheme $\mathbb{U} = M^{D,g}(S)_{reg}$ of regular points of $M^{D,g}(S)$. Clearly, $\mathbb{U}$ is smooth. The closure $\overline{\mathbb{U}}$ in $M^{D,g}(S)$ is reduced and by the discussion above, there is a degree 1 finite base change $\tilde{\mathbb{U}} \to \overline{\mathbb{U}}$, such that there is a well-defined morphism $\psi\colon\ \tilde{\mathbb{U}} \to |D|$, sending a closed point $[f\colon\ X \to S]$ in $\tilde{\mathbb{U}}$ to a cycle $f_*[X]$ on S. The normalization of $\overline{\mathbb{U}}$ factors through $\tilde{\mathbb{U}}$. Denote the normalization by $M^{D,g}(S)^+$. There is now a well-defined morphism from $M^{D,g}(S)^+$ to $|D|$. The morphism sends a closed point of $M^{D,g}(S)^+$, lying over $[f\colon X \to S]$ in $M^{D,g}(S)$ to the image cycle $f_*[X]$. We call it the *cycle morphism* and denote

$$\psi\colon M^{D,g}(S)^+ \to |D|.$$

Definition 3.2. Suppose T is an irreducible subvariety of $M^{D,g}(S)$ intersecting the regular locus non-trivially. The *intersection dimension of* T, denoted $\operatorname{idim} T$, is a dimension of the image of $T \cap M^{D,g}(S)_{reg}$ in $|D|$ under the cycle morphism ψ.

3.4 Definitions

Let (S, L) be a projective smooth log surface. For a fixed set of *distinct* points

$$\Omega = \{p_{i,j}\}_{1 \le j \le \alpha_i} \subset L$$

we make the following definition.

Definition 3.3. Consider a divisor class D, a non-negative integer δ, and two sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ of non-negative integers such that

$$I\alpha + I\beta = D \cdot L\ .$$

We define a *generalized Severi variety of divisors*

$$V^{D,\delta}(\alpha, \beta)(\Omega)$$

to be the closure of the locus of δ-nodal elements X of $|D|$ that do not contain L, and such that for the normalization map

$$\nu : X^\nu \to X \subset S,$$

we have

$$\nu^*(L) = \sum i \cdot q_{i,j} + \sum i \cdot r_{i,j}$$

for some $|\alpha|$ points $q_{i,j}$ and $|\beta|$ points $r_{i,j}$ on X^ν satisfying further conditions

$$\nu(q_{i,j}) = p_{i,j}.$$

Remark 4. We often drop the modifier Ω from the notation, understanding that $V^{D,\delta}(\alpha, \beta)$ is a variety obtained by letting Ω to be a generic set of points.

We also denote by $\mathcal{Y}^{D,\delta}(\alpha, \beta)$ the restriction of the universal divisor over $|D|$ to $V^{D,\delta}(\alpha, \beta)$. We call $\mathcal{Y}^{D,\delta}(\alpha, \beta)$ the *universal family.*

A definition analogous to 3.3 for the locus of stable maps with postulated tangency along L is due to Vakil. Here we will allow for a slightly more general tangency condition. Namely, let

$$\Omega = \{p_{i,j}\}_{1 \le j \le \alpha_i} \subset L$$

be a set of *not necessarily distinct* points on L.

Definition 3.4. For a given divisor class D, an integer g (possibly negative), and two sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ of non-negative integers such that

$$I\alpha + I\beta = D \cdot L \ ,$$

we define a *generalized Severi variety of maps*

$$M^{D,g}(\alpha,\beta)(\Omega)$$

as the closure inside $M^{D,g}(S)$ of the locus of quasi-stable maps $f\colon X \to S$ satisfying the following conditions:

1. The map f is birational on every irreducible component of X.

2. For some $|\alpha|$ points $q_{i,j}$ and $|\beta|$ points $r_{i,j}$

$$f^{-1}(L) = \sum i \cdot r_{ij} + \sum i \cdot q_{ij}.$$

3. Points $q_{i,j}$ verify $f(q_{i,j}) = p_{i,j}$.

Definition 3.5. We denote the degree of $V^{D,\delta}(\alpha,\beta)$ inside $|D|$ by

$$N^{D,\delta}(\alpha,\beta).$$

The following extension of Definition 3.3 will be useful in the following discussion.

Definition 3.6. We define

$$V_L^{D,\delta}(\alpha,\beta)(\Omega) := \{X \cup L :\ X \in V^{D,\delta}(\alpha,\beta)(\Omega)\} \subset |\mathcal{O}_S(D+L)|.$$

3.5 Linear sections

The main objects of study in this thesis are curve-sections of $V^{D,\delta}(\alpha,\beta)$ by hyperplanes of the special form in $|D|$. Namely, if the dimension of the linear system $|D|$ is positive, then the locus of curves passing through a fixed point $p \in S$ is at most codimension one linear subspace of $|D|$. If p is not a base point of $|D|$, which is always true for a general point $p \in S$, then the above locus is a hyperplane in $|D|$, denoted by H_p.

Definition 3.7. For a set $\Gamma = \{p_i\}_{1\leq i\leq k}$ of points in S, define

$$H_\Gamma = H_{p_1} \cap \cdots \cap H_{p_k}.$$

We will be mostly concerned with linear sections of $V^{D,\delta}(\alpha,\beta)$ cut out by linear spaces of the form H_Γ. In particular, we will study the geometry of one and two-dimensional such linear sections.

Definition 3.8. Let $\Upsilon = \dim V^{D,\delta}(\alpha,\beta)$. For a collection $\Gamma = \{p_i\}_{1\leq i\leq \Upsilon}$ of Υ general points in S, we define

$$\begin{aligned} S^{D,\delta}(\alpha,\beta)^\Gamma &:= V^{D,\delta}(\alpha,\beta) \cap H_{p_1} \cap \cdots \cap H_{p_{\Upsilon-2}}, \\ C^{D,\delta}(\alpha,\beta)^\Gamma &:= V^{D,\delta}(\alpha,\beta) \cap H_{p_1} \cap \cdots \cap H_{p_{\Upsilon-1}}, \\ N^{D,\delta}(\alpha,\beta)^\Gamma &:= V^{D,\delta}(\alpha,\beta) \cap H_{p_1} \cap \cdots \cap H_{p_\Upsilon}. \end{aligned}$$

Remark 5. Similar definitions are made for the variety $V^{D,\delta}_{con}(\alpha,\beta)$.

Remark 6. If the scheme $N^{D,\delta}(\alpha,\beta)^\Gamma$ is reduced, its cardinality is

$$\deg V^{D,\delta}(\alpha,\beta) = N^{D,\delta}(\alpha,\beta).$$

Remark 7. We occasionally drop the superscript Γ when the set Γ is understood.

4 Fundamental dimension estimates

In this section we collect important results on dimensions of generalized Severi varieties of divisors and maps. Most of these results have either appeared or were implicit in the papers of Caporaso-Harris and Vakil. We heavily rely on these in our proofs of some minor variations of dimension estimates. The most work in this section is devoted to building a bridge connecting generalized Severi varieties of divisors and maps.

We first recall a result of Vakil regarding the dimension of deformation spaces of maps. The result is a combination of Theorem 3.1 and Proposition 3.4 from [31].

Fact 1 (Vakil)**.** *Let V be an irreducible subvariety of $M^{D,g}(S)$ with $f\colon X \to S$ a map corresponding to a general point of V such that*

$$f^{-1}(L) = \sum_{i,1\le j\le \alpha_i} i\cdot q_{i,j} + \sum_{i,1\le j\le \beta_i} i\cdot r_{i,j},$$

and where $\Omega = \{f(q_{i,j})\}$ is a set of (not necessarily distinct) fixed points on L. Then the intersection dimension of V verifies

$$\operatorname{idim} V \le -(K_S + L)\cdot D + g + |\beta| - 1.$$

Moreover, the following are equivalent:

1. *The equality holds in the inequality above.*

2. *The curve X is smooth and the map f is unramified.*

3. *V is smooth at f and is a component of $M^{D,g}(\alpha,\beta)(\Omega)$.*

The following statement is a generalization of an analogous result for $S = \mathbb{P}^2$ appearing in [2, Corollary 2.7].

Proposition 4.1. *Fix a set Ω of points on a divisor $L \subset S$. Let V be a locally closed irreducible subvariety of $M^{D,g}(S)$ such that its generic point is a map $f\colon X \to S$, satisfying $L \not\subset f(X)$. Set $e := \operatorname{card}(f(X) \cap (L \setminus \Omega))$. Then*

$$\operatorname{idim} V \leq -(K_S + L) \cdot D + g + e - 1.$$

Moreover, the equality holds if and only if the curve X is smooth and the map f is unramified.

Proof. We start with the case $e = 0$. In this situation, $f^{-1}(L) = \sum i \cdot q_{i,j}$, with $f(q_{i,j})$ fixed (not necessarily distinct) points in Ω. By Fact 1,

$$\operatorname{idim} V \leq -(K_S + L) \cdot D + g - 1.$$

Moreover, if the equality holds then the curve X is smooth, the map f is unramified.

The general case follows by considering a rational map

$$V \dashrightarrow \operatorname{Div}^e(L)$$

sending a map $f\colon X \to S$ to a reduced divisor $(f(X) \cap (L \setminus \Omega))_{red}$. By the previous paragraph, a fiber of this map containing a generic point of V has intersection dimension at most

$$-(K_S + L) \cdot D + g - 1.$$

Therefore,

$$\operatorname{idim} V \leq -(K_S + L) \cdot D + e + g - 1.$$

Again, if the equality holds then X is smooth and f is unramified. □

Corollary 4.1. *In the setup of Proposition 4.1, suppose that X is smooth and*

$$\operatorname{idim} V = -(K_S + L) \cdot D + g + e - 2. \tag{5}$$

(That is the intersection dimension is one less than the maximal possible). Then for every irreducible component R of X, one of the following holds:

1. *The map f is birational on R.*

2. *The map $f\colon\ R \to f(R)$ is a degree 2 cover and geometric genera verify*

$$g(R) = g(f(R)) = 0,$$

or

$$g(R) = g(f(R)) = 1.$$

Proof. It is sufficient to prove the statement in the case X is irreducible. Suppose f is a degree m cover of $f(X)$. The divisor class of $f(X)$ is $D = mC$. By Proposition 4.1 applied to the normalization map

$$\nu\colon\ C^\nu \to C,$$

the intersection dimension of V is at most

$$-(K_S + L) \cdot C + g(C) + e - 1.$$

Comparing this to the dimension assumption (5), we obtain

$$-(K_S + L) \cdot C + g(C) + e - 1 \geq -(K_S + L) \cdot D + g + e - 2,$$

which we rewrite as

$$(m-1)(K_S+L)\cdot C+g(C)-g+1\geq 0.$$

By the positivity assumption of Section 3.1, we have $(K_S+L)\cdot C\leq -1$. Since X is a branch cover of C, we also have $g\geq g(C)$. Therefore, we must have $m=2$ and the inequalities must be equalities throughout.

We finish by noting that a smooth genus g curve can be a degree 2 cover of another genus g curve only if $g=0$ or $g=1$. ☐

Proposition 4.2. *Suppose $V\subset M^{D,g}(S)$ is a subvariety whose generic point is a quasi-stable map $f\colon X\to S$ such that*

$$f^{-1}(L)=\sum i\cdot q_{i,j}+\sum i\cdot r_{i,j},$$

and points $p_{i,j}=f(q_{i,j})$ are fixed on L. Then

$$\operatorname{idim} V\leq -(K_S+L)\cdot D+g+|\beta|-1.$$

Moreover, if $p_{i,j}$ are distinct points then the equality holds if and only if the image $f(X)$ has at most nodal singularities and is smooth along L.

Proof. The dimension estimate follows from Proposition 4.1 after noticing that

$$|\beta|\geq \operatorname{card}(f(X)\cap(L\smallsetminus\Omega)).$$

If the equality holds then $f(X)$ must be unibranch along $L\smallsetminus\Omega$.

It remains to show that $f(X)$ is nodal. By Fact 1, we know that X is smooth and f is unramified. The deformations of f are governed by the twisted normal bundle

$$N_f(-L) := \mathcal{O}_X(X, K_X - f^*(K_S + L)).$$

To finish we recall that, by the Caporaso and Harris' proof of Proposition 2.2 in [2], the image $f(X)$ has to be nodal if for any 3 distinct points on X there is a section of the line bundle $N_f(-L)$ that vanishes at some two of them and does not vanish at the third. For example, this is satisfied when any 3 points impose independent conditions on the sections of the line bundle $N_f(-L)$.

We have

$$\deg N_f(-L) = -(K_S + L) \cdot f(X) + 2g - 2.$$

By Condition (4) imposed on (S, L) in Section 3.1,

$$-(K_S + L) \cdot f(X) \geq 4$$

if $f(X)$ has either a triple point or a tacnode. The independence of 3 point conditions follows by Riemann-Roch if X is irreducible. If X is reducible, we can take a section of $N_f(-L)$ that is identically zero on the components containing some two of the points, and does not vanish at the third point. $\square$

Corollary 4.2. *In the setup of Proposition 4.2, suppose* $\Omega = \{p_{i,j}\}$ *is a set of distinct points. Then every irreducible component of* $M^{D,g}(\alpha, \beta)(\Omega)$ *of intersection dimension*

$$\Upsilon = -(K_S + L) \cdot D + g + |\beta| - 1$$

is a strict transform of an irreducible component of $V^{D,\delta}(\alpha, \beta)(\Omega)$ *(of dimension* Υ*) under the cycle morphism*

$$\psi\colon\ M^{D,g}(S) \to |D|.$$

Corollary 4.2 is important as it shows that a generic point of any component of $M^{D,g}(\alpha, \beta)(\Omega)$ can be thought of as a nodal divisor on the surface S.

5 Degenerations and enumerative results

We recall next a result of Vakil generalizing an analogous result of Caporaso and Harris [2, Theorem 1.2]. Although it is not stated explicitly in [31], its analog for spaces of maps is an immediate consequence of Theorem 5.1 and Propositions 6.1, 6.4 in [31]. The version for spaces of divisors that we use follows from the result of Vakil and Corollary 4.2.

Fact 2. *For a general $q \in L$, we have the following equality of cycles*

$$\begin{aligned} V^{D,\delta}(\alpha,\beta)(\Omega) \cap H_q = &\sum_k kV^{D,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{q\}) \\ &+ \sum I^{\beta'-\beta}\binom{\beta'}{\beta} V_L^{D-L,\delta'}(\alpha',\beta')(\Omega'), \end{aligned} \tag{6}$$

where the second sum is taken over all triples (δ',α',β') satisfying[2]

$$|\beta'-\beta| + \delta - \delta' = D\cdot L - L^2,$$

and over all sets of points $\Omega' = \{p'_{i,j}\}_{1\le j\le \alpha'_i} \subset \Omega$.

We refer to components of $H_q := V^{D,\delta}(\alpha,\beta)(\Omega)\cap H_q$ appearing on the first line of Equation (6) as *Type I components.* A generic point of a Type I component does not contain the line L. The components of H_q appearing on the second line of Equation (6) are called *Type II components* and parameterize curves containing the line L.

An immediate corollary of Fact 2 is that the degrees $N^{D,\delta}(\alpha,\beta)$ of the Severi

[2]It is implicit in Equation (6) that only varieties $V_L^{D-L,\delta'}(\alpha',\beta')(\Omega')$ with $\alpha'\le\alpha, \beta'\ge\beta$ and $\delta'\le\delta$ appear with a non-zero coefficient.

varieties $V^{D,\delta}(\alpha,\beta)$ satisfy the recursion

$$\begin{aligned} N^{D,\delta}(\alpha,\beta) = &\sum_k kN^{D,\delta}(\alpha+e_k,\beta-e_k) \\ &+ \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}N^{D-L,\delta'}(\alpha',\beta'), \end{aligned} \tag{7}$$

where the second sum is taken over all triples (δ',α',β') satisfying

$$|\beta'-\beta|+\delta-\delta' = D\cdot L - L^2.$$

This is stated as Theorem 6.7 of [31] for the generalized Severi varieties of maps. Again, this formula is motivated by and is a direct generalization of [2, Theorem 1.1] enumerating nodal plane curves.

5.1 Statement of the main theorem

Set $g = p_a(D) - \delta$. Consider the strict transform of $V^{D,\delta}(\alpha,\beta)$ inside the moduli space of quasi-stable maps $M^{D,g}(S)$. By Corollary 4.2, it is precisely $M^{D,g}(\alpha,\beta)$. The strict transform comes equipped with the cycle morphism

$$\psi\colon\ M^{D,g}(\alpha,\beta)\to V^{D,\delta}(\alpha,\beta),$$

sending a stable map to its image cycle. Let

$$j\colon\ V^{D,\delta}(\alpha,\beta) \dashrightarrow M^{D,g}(\alpha,\beta)$$

be the rational map defined as the inverse of ψ. Then j is defined on the dense open U of $V^{D,\delta}(\alpha,\beta)$ whose complement has codimension at least 2. Therefore, a general curve section $C^{D,\delta}(\alpha,\beta)$ lies in U and $j_*(C^{D,\delta}(\alpha,\beta))$ is a well-defined curve in $M^{D,g}(\alpha,\beta)$.

Definition 5.1. The *degree of λ-class on $C^{D,\delta}(\alpha,\beta)$* is

$$L^{D,\delta}(\alpha,\beta) := j_*(C^{D,\delta}(\alpha,\beta)) \cdot \lambda,$$

where λ-class on $M^{D,g}(S)$ is defined in Section 3.2.

We now describe our strategy for calculating numbers $L^{D,\delta}(\alpha,\beta)$. Let $\Upsilon = \dim V^{D,\delta}(\alpha,\beta)$. Take

$$\Gamma = \{p_1, \ldots, p_\Upsilon\}$$

to be a set of Υ general points on S, and take $q \in L$ general. Intersecting both sides of the linear equivalence relation

$$V^{D,\delta}(\alpha,\beta) \cap H_{p_{\Upsilon-1}} \sim V^{D,\delta}(\alpha,\beta) \cap H_q$$

with $H_{\{p_1,\ldots,p_{\Upsilon-2}\}}$ and using a rational equivalence of cycles (6) from Fact 2, we obtain a linear equivalence relation

$$\begin{aligned} C^{D,\delta}(\alpha,\beta)^\Gamma \sim & \sum_k kC^{D,\delta}(\alpha+e_k,\beta-e_k)(\Omega\cup\{q\}) \\ & + \sum I^{\beta'-\beta}\binom{\beta'}{\beta} C_L^{D-L,\delta'}(\alpha',\beta')(\Omega'), \end{aligned} \tag{8}$$

of Weil divisors on

$$S^{D,\delta}(\alpha,\beta)^\Gamma = V^{D,\delta}(\alpha,\beta) \cap H_{\{p_1,\ldots,p_{\Upsilon-2}\}}.$$

The divisors on the right hand-side of (8) are separated into *Type I* and *Type II* components as in Fact 2.

We would like to use the linear equivalence (8) to obtain a recurrence relation among numbers $L^{D,\delta}(\alpha,\beta)$ paralleling that of Equation (7). Unfortunately, the rational map

$$j\colon\ S^{D,\delta}(\alpha,\beta) \dashrightarrow M^{D,g}(\alpha,\beta)$$

is not defined at the finite number of points along $S^{D,\delta}(\alpha,\beta)\cap H_q$.

In Section 6, we analyze the point of indeterminacy of the moduli map j. In Section 11, we analyze the exceptional divisors arising in the regularization of the map j and compute the degree of the λ-class on these exceptional divisors, obtaining a proof of the following theorem, which is the principal result of this thesis.

Theorem 1 (Main Theorem)**.** *The numbers $L^{D,\delta}(\alpha,\beta)$ satisfy the recursion*

$$\begin{aligned} L^{D,\delta}(\alpha,\beta) = & \sum_k k L^{D,\delta}(\alpha+e_k,\beta-e_k) \\ & + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta} L^{D-L,\delta'}(\alpha',\beta') \\ & + \frac{1}{12}\sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}\cdot\Big(\sum_k(\beta'_k-\beta_k)(k^2-1)\Big)\cdot N^{D-L,\delta'}(\alpha',\beta')\ ; \end{aligned}$$

where the second sum is taken over all triples (δ',α',β') satisfying

$$|\beta'-\beta| + \delta - \delta' = D\cdot L - L^2\ ,$$

and the third sum is taken over all triples (δ',α',β') satisfying

$$|\beta'-\beta| + \delta - \delta' = D\cdot L - L^2 - 1.$$

6 Relationship between maps and divisors

6.1 Nodal reduction

In our analysis of subvarieties of $V^{D,\delta}(\alpha,\beta)$ we will often need to understand the stable limits of one-parameter families inside $V^{D,\delta}(\alpha,\beta)$, especially, if the central fiber of the family is a curve containing L as an irreducible component. We use the approach developed in Section 3 of [2] in what follows.

Consider a curve $C = Y \cup aL$ belonging to $V^{D,\delta}(\alpha,\beta)$, where Y is a divisor non-containing L and a is a non-negative integer. Let T be an arc in $V^{D,\delta}(\alpha,\beta)$ that passes through $[C]$ and whose generic point corresponds to a nodal element in $V^{D,\delta}(\alpha,\beta)$. Let $\mathcal{Y}_T = \mathcal{Y}^{D,\delta}(\alpha,\beta) \times_{V^{D,\delta}(\alpha,\beta)} T$ be the restriction of the universal family.

It is a standard result that, after possibly a finite base change $T' \to T$, we may perform a nodal reduction of $\mathcal{Y}_T \to T$ to obtain a flat family $\mathcal{C} \to T'$ of generically smooth genus g curves satisfying the following conditions:

1. The total space $\mathcal{C}/T'$ is a smooth relatively minimal surface and there is a map
$$\eta \colon \mathcal{C} \to S.$$

2. The central fiber C_0 is nodal and decomposes as
$$C_0 = \overline{Y} \cup Z \cup P,$$
where $\overline{Y}$ is a union of irreducible components of C_0 mapping to Y with a positive degree, the curve Z is a union of components of C_0 contracted by η and not mapping to L, and P is a union of components mapping to L.

3. The family $\mathcal{C} \to T'$ has a multisection $F = \sum Q_{i,j}$ such that $\eta(Q_{i,j}) = p_{i,j}$,

and a multisection $V = \sum R_{i,j}$ such that

$$\eta^*(L) = \sum_{i,j} i \cdot Q_{i,j} + \sum_{i,j} i \cdot R_{i,j} + \mathcal{P},$$

where $\mathcal{P}$ is a divisor supported on P. Moreover, F and V meet C_0 in $|\alpha|$ and $|\beta|$ distinct points, respectively.

$$\begin{array}{ccc} C_0 \subset \mathcal{C} & \xrightarrow{\eta} & S \\ \downarrow \quad \downarrow & & \\ [C] \in T' & & \end{array}$$

We also introduce a decomposition

$$P = \sum_{i=1}^{c_1} P_i' + \sum_{i=1}^{c_2} P_i''$$

of P into connected components such that P_i''s are not contracted by η and P_i'''s are contracted by η. Note, immediately, that by degree considerations

$$c_1 \leq a. \tag{9}$$

Observation 2. *Every point of $\eta(Z)$ is a non-nodal singularity of C.*

Observation 3. *Every connected component P_i'' has to meet either V or F.*

We let the union of connected components of P'' meeting V and F be P_v'' and P_f'', respectively. The number of connected components of P_v'' and P_f'' is denoted by v and f.

6.2 Exceptional locus of the cycle map

Consider a log surface (S, L) from our list in Section 3.1, and $\Omega = \{p_{i,j}\}_{1 \leq j \leq \alpha_i} \subset L$ a general set of $|\alpha|$ points on L. Let $V^{D,\delta}(\alpha, \beta)(\Omega)$ be a generalized Severi variety of divisors on the surface S defined with respect to the divisor L. Geometric genus of a general curve in $V^{D,\delta}(\alpha, \beta)$ is

$$g = p_a(D) - \delta.$$

By Fact 1,

$$\dim V^{D,\delta}(\alpha, \beta) = \Upsilon := -(K_S + L) \cdot D + |\beta| + g - 1.$$

By Corollary 4.2, the strict transform of $V^{D,\delta}(\alpha, \beta)$ inside $M^{D,g}(S)$, the moduli space of quasi-stable maps, is $M^{D,g}(\alpha, \beta)$. The strict transform comes equipped with the cycle morphism

$$\psi \colon\ M^{D,g}(\alpha, \beta) \to V^{D,\delta}(\alpha, \beta),$$

sending a stable map to its image cycle.

We analyze the exceptional locus $\mathbf{Exc} := \mathrm{Exc}(\psi)$ consisting of positive dimensional fibers of ψ. Since $\mathbf{Exc}$ is a proper subvariety of $M^{D,g}(\alpha, \beta)$, we have $\dim \psi(\mathbf{Exc}) \leq \Upsilon - 2$. In other words, ψ is at most finite-to-one in codimension one on the Severi variety of divisors.

Remark 8. This does not preclude curves in codimension one on $V^{D,\delta}(\alpha, \beta)$ from degenerating to curves with singularities worse than nodes, but it does mean that for any such degeneration, the stable limit is defined uniquely up to a finite set. A good example is a variety of cuspidal plane curves inside the discriminant locus $V^{d,1}$ of degree d plane curves.

Observation 4. *Consider a quasi-stable map $f \colon\ X \to S$. If the deformations of f preserving the dual graph of X and preserving the image cycle of every irreducible component of X vary in a family of positive dimension then one of the following* *holds:*

1. *There is an irreducible component of* X *that maps multiple-to-one onto its image.*

2. *There is an irreducible component of* X *that is contracted by* f *and that either has a positive genus or has genus* 0 *and meets the rest of the curve in at least* 4 *points.*

Fix a generic point $q \in L$. We would like to describe all irreducible components W of $\psi(\mathbf{Exc}) \cap H_q$ of dimension $\Upsilon - 2$. Note that a generic point of W lies in the base locus of the system of divisors $\{H_q\}_{q \in L}$ and hence corresponds to a curve $[C]$ containing the line L.

6.3 Nodal reduction analysis

We write $C = Y \cup aL$. For any 1-parameter family T in $V^{D,\delta}(\alpha, \beta)$ with the center at $[C]$ we consider a nodal reduction of the family as described in Section 6.1. Since $[C] \in \psi(\mathbf{Exc})$ we must obtain a positive dimensional family of stable limits as T varies over all possible arcs.

Let $\eta\colon \overline{Y} \to Y$ be the map from Section 6.1. Set $b = \mathrm{card}(\overline{Y} \cap \eta^{-1}(L \setminus \Omega))$ and $e = \mathrm{card}(Y \cap (L \setminus \Omega))$. Naturally, $b \geq e$ and the equality holds if only if η is one-to-one over $L \setminus \Omega$.

By Proposition 4.1 applied to η, we have

$$\Upsilon - 2 = \dim W \leq -(K_S + L) \cdot Y + g(\overline{Y}) + e - 1. \tag{10}$$

Equivalently,

$$g(\overline{Y}) \geq g - e + |\beta| + 2a - 2 \ . \tag{11}$$

Let $p = \sum_{i=1}^{c_1} p_a(P'_i)$ be the sum of arithmetic genera of connected components of P'. We have the following series of inequalities:

$$\begin{aligned}
g &= p_a(\overline{Y}) + \left[p_a(Z) + \mathrm{card}(\overline{Y} \cap Z) - 1\right] + \left[p_a(P) + \mathrm{card}(P \cap \overline{Y}) - 1\right] \\
&\geq p_a(\overline{Y}) + p_a(P) + \mathrm{card}(P \cap \overline{Y}) - 1 & (12) \\
&\geq g(\overline{Y}) + p_a(P) + \mathrm{card}(P \cap \overline{Y}) - 1. & (13)
\end{aligned}$$

So far, we have only used the fact that $p_a(Z) + \mathrm{card}(\overline{Y} \cap Z) - 1 \geq 0$ and $p_a(\overline{Y}) \geq g(\overline{Y})$. Therefore, equalities will hold in (12)-(13) if and only if $\overline{Y}$ is smooth and Z is empty. We continue the series:

$$\begin{aligned}
g &\geq g(\overline{Y}) + p_a(P) + \mathrm{card}(P \cap \overline{Y}) - 1 \\
&\geq g(\overline{Y}) + (p + 1 - c_1 - c_2) + \mathrm{card}(P \cap \overline{Y}) - 1 & (14) \\
&= g(\overline{Y}) + (p - c_1 - v - f) + \mathrm{card}(P' \cap \overline{Y}) + \mathrm{card}(P'' \cap \overline{Y}) \\
&\geq g(\overline{Y}) + (p - a - v - f) + \mathrm{card}(P' \cap \overline{Y}) + \mathrm{card}(P'' \cap \overline{Y}) & (15) \\
&= g(\overline{Y}) + (p - a - v) + \mathrm{card}(P''_v \cap \overline{Y}) + \mathrm{card}(P' \cap \overline{Y}) + \mathrm{card}(V \cap \overline{Y}) \\
&- \mathrm{card}(V \cap \overline{Y}) + \mathrm{card}(P''_f \cap \overline{Y}) - f \\
&\geq g(\overline{Y}) + (p - a - v) + b - \mathrm{card}(V \cap \overline{Y}) & (16) \\
&\geq g - e + |\beta| + 2a - 2 + (p - a - v) + b - \mathrm{card}(V \cap \overline{Y}) & (17) \\
&= g + (a - 2) + p + (b - e) + \left(|\beta| - v - \mathrm{card}(V \cap \overline{Y})\right).
\end{aligned}$$

We used the fact that P has $c_1 + c_2$ connected components and arithmetic genera of components of P'' are at least 0 in (14). Hence, the equality holds if and only if arithmetic genera of all connected components of P' are 0. We used $c_1 \leq a$ in (15). The equality holds if and only if every component of P' maps with degree 1 onto L. In (16), we used $\mathrm{card}(P''_f \cap \overline{Y}) \geq f$. The equality holds only if every connected component of P''_f intersects $\overline{Y}$ in exactly one point. Finally, the inequality (17) follows from (11).

As a result, we obtain the inequality

$$(a-2)+p+(b-e)+(|\beta|-v-V\cap\overline{Y})\leq 0. \tag{18}$$

Note, however, that

$$|\beta|=\operatorname{card}(V\cap C_0)\geq v+\operatorname{card}(V\cap\overline{Y}), \tag{19}$$

with equality only if V meets each P''_v once and does not meet P'.

By assumption, the quasi-stable map

$$\eta\colon\ \overline{Y}\to Y\subset S$$

varies in a positive dimensional family preserving the image Y and the dual graph of $\overline{Y}$, and so Observation 4 applies.

We begin to draw consequences from the inequality (18). First of all, we must have $a\leq 2$.

If $a=2$, then $p=0$ and all inequalities above have to be equalities. In particular, this implies that Z is empty and both connected components of P' are trees of rational curves. Since $b=e$, the unique non-contracted irreducible component of each P'_i (for $i=1,2$) is connected with $\overline{Y}$ by chains of rational curves. We conclude that the central fiber C_0 of the nodal reduction looks like the curve on the left in Figure 1. In particular, there are finitely many possible stable limits of 1-parameter families with the center at $[C]$, a contradiction.

Consider now the case of $a=1$. If Z is not empty, then inequality (12) is

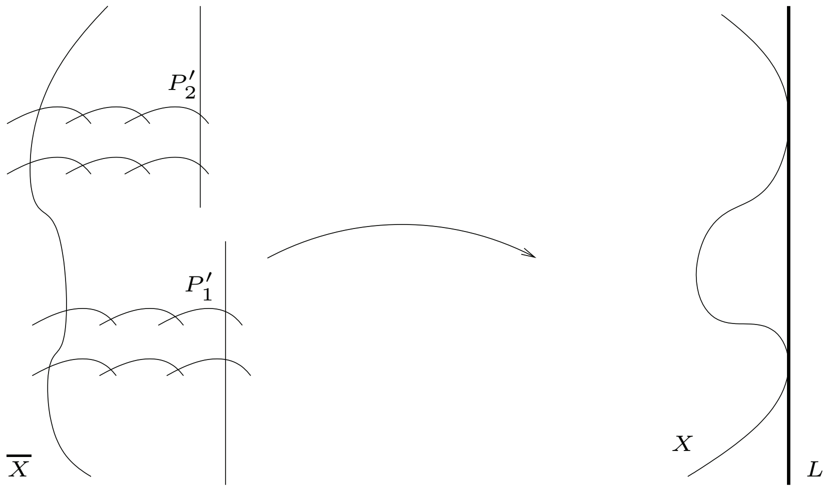

Figure 1:

strict and from the resulting strict inequality in (18) we obtain

$$0 \leq p + (b - e) + (|\beta| - v - V \cap \overline{Y}) \leq 0,$$

which implies that $b = e$ and all inequalities except (12) are equalities. In particular, Y is unibranch at every point of L and equality holds in (10). Hence by Proposition 4.2, the curve Y is nodal and is a generic element of a generalized Severi variety of divisors. Necessarily, $\eta(Z)$ is a set of at worst nodal points of $Y \setminus L$, a contradiction.

Suppose now that $a = 1$ and Z is empty. Then we obtain

$$0 \leq p + (b - e) + (|\beta| - v - V \cap \overline{Y}) \leq 1. \tag{20}$$

First, suppose that $p = 0$. We have the following possibilities:

Case 1: All inequalities in the series are equalities and

$$(b - e) + (|\beta| - v - V \cap \overline{Y}) = 1.$$

Since equality in (10) holds, the curve $\overline{Y}$ is smooth and the map η is birational on its every component by Proposition 4.1. Connected components of P'' and P' are trees of rational curves. Since $b - e \leq 1$, the stabilization of C_0 can have at most one contracted rational component meeting the rest of the curve in 3 points and none meeting the rest of the curve in 4 points. Hence $\eta\colon C_0 \to C$ does not move in a positive dimensional family by Observation 4.

This case also provides examples of $C = Y \cup L$ with Y singular along L and with $\psi^{-1}([C])$ finite. The first possibility occurs when the unique component of P''_v meets $\overline{Y}$ in two points and meets one of the sections $R_{i,j}$. In this case, P' meets $\overline{Y}$ in points which all map to different points on L.

The other possibility is that P'' is empty and P' meets $\overline{Y}$ in a set of points among which there are two that map to the same point on L. The first possibility is presented in Figure 2, where the curve on the left is a central fiber of the nodal reduction, the curve in the center is its stabilization. The curve on the right is the planar image. The second possibility is presented in Figure 3.

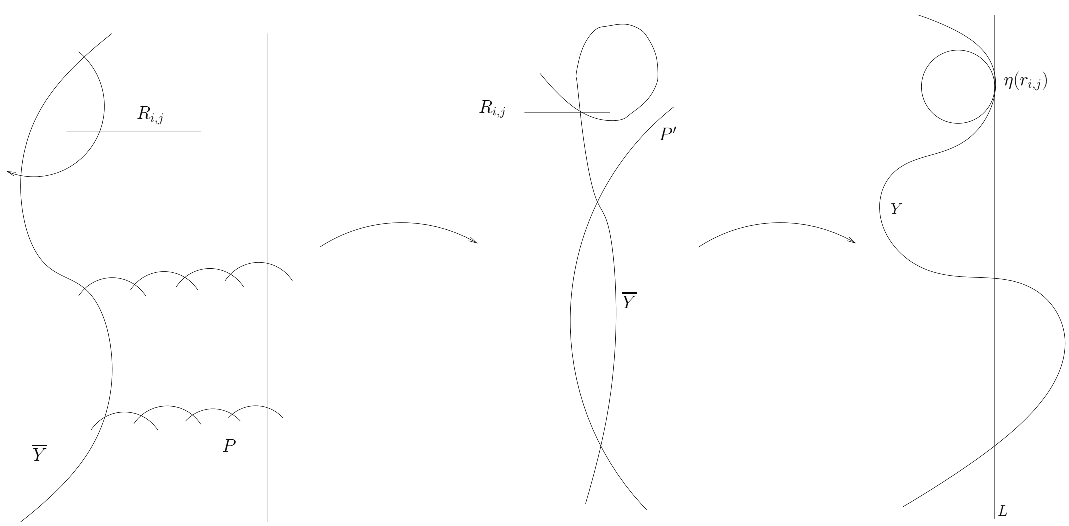

Figure 2:

Case 2: Suppose one of the inequalities in the series is strict and all others are

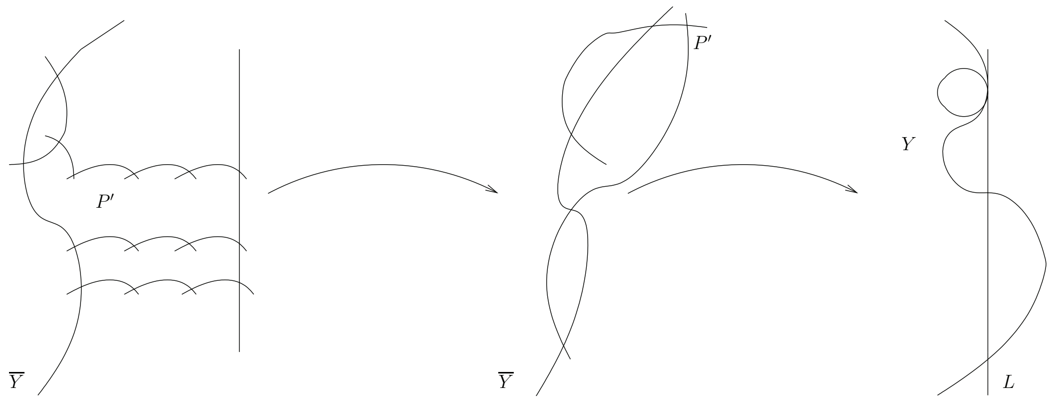

Figure 3:

equalities. In this case

$$(b - e) + (|\beta| - v - V \cap \overline{Y}) = 0.$$

If (10) holds then analysis of *Case 1* applies. Suppose that in the inequality (10), the left-hand side is exactly 1 less than the right-hand side. The curve $\overline{Y}$ is smooth, since equality holds in (13). Therefore, by Corollary 4.1 the map $\eta \colon \overline{Y} \to Y$ is birational on every component with a possible exception that there is either an elliptic component which is a degree 2 unramified cover of its image or a rational component which is a degree 2 cover of its image. Note, however, that such degree 2 covers do not vary in moduli, a contradiction.

At last, suppose $p = 1$. Then all inequalities in the series are equalities and

$$(b - e) + (|\beta| - v - V \cap \overline{Y}) = 0.$$

Since $[C] \in \psi(\mathbf{Exc})$ and Z is empty, there is either a component of positive genus, or a rational subcurve of P' meeting $\overline{Y}$ in 4 points, contracted by η. The latter is impossible, since $b = e$.

In the former case, we must have $p = 1$, meaning that there is exactly one contracted positive dimensional component and it is elliptic. Also $b = e$, meaning that Y is unibranch along $L \smallsetminus \Omega$. We conclude that P' has a unique irreducible rational component mapping onto L isomorphically and is connected to $\overline{Y}$ by chains of curves. Moreover, all these chains have arithmetics genus 0, except one that has arithmetic genus 1. Since equlity holds in (16), we also deduce that Y is unibranch at every point of Ω it passes through. By Proposition 4.2, the curve Y is nodal and is a generic element of a generalized Severi variety of divisors $V^{D-L,\delta'}(\alpha', \beta')$. From

$$p_a(D-L) - \delta' + |\beta' - \beta| - 1 = p_a(C_0) = g = p_a(D) - \delta,$$

and Equality (1), we obtain

$$|\beta' - \beta| + \delta - \delta' = D \cdot L - L^2 - 1. \tag{21}$$

The curve described in the last case corresponds to the point of indeterminacy of the moduli map.

We summarize this section in the following result.

Theorem 2. *Fix a generic point $q \in L$. Set $\Upsilon = \dim V^{D,\delta}(\alpha, \beta)$. Let **Exc** be the exceptional locus of the cycle map*

$$\psi\colon\ M^{D,g}(\alpha, \beta)(\Omega) \to V^{D,\delta}(\alpha, \beta)(\Omega).$$

(Part 1) The only dimension $\Upsilon - 2$ components of $\psi(\textbf{Exc}) \cap H_q$ are varieties

$$V_L^{D-L,\delta'}(\alpha', \beta')(\Omega'),$$

where

$$|\beta' - \beta| + \delta' - \delta = D \cdot L - L^2 - 1,$$

and where $\Omega' \subset \Omega$.

(Part 2) At a generic point $C = Y \cup L$ *of a component* $V_L^{D-L,\delta'}(\alpha', \beta')$ *of Part 1, the fiber* $\psi^{-1}([C])$ *consists, set-theoretically, of stable maps*

$$f\colon X \to C \subset S,$$

such that, for $k = |\beta' - \beta|$,

(a) $X \cong \overline{Y} \cup E \cup P$ *is a* $(k+1)$*-nodal union of* $\overline{Y} \cong Y^\nu$ *(the normalization of* Y*), an elliptic curve* $(E; e_1, e_2) \in \overline{M}_{1,2}$ *and* $P \cong \mathbb{P}^1$.

(b) E *meets* $\overline{Y}$ *at* e_1 *and* P *at* e_2.

(c) $\overline{Y}$ *and* P *meet in* $k-1$ *nodes.*

(d) $f\colon P \to L$ *is an isomorphism.*

(e) The $k+1$ *nodes of* X *map to some* k *distinct tacnodes of* C*, such that* E *is contracted by* f *to a single tacnode of* C.

Remark 9. To lighten the language, we introduce several terms relevant to the situation of Theorem 2. We keep the notations of the theorem and of Section 6.3.

First of all, observe that by the analysis above in the case when f contracts a component of arithmetic genus 1, the $|\beta|$ out of $|\beta'|$ tacnodes of C are the limits of free points of tangency in the nearby fibers, since they correspond to points where the multi-section V meets $\overline{Y}$. We call these *"old"* tacnodes. The remaining $k = |\beta' - \beta|$ tacnodes of C are the images of $k+1$ nodes of X, as described in Part 2 of the theorem. Call these tacnodes *"new"*. There is a distinguished tacnode among "new", the image of the contracted elliptic curve E. It will be termed *the tacnode responsible for indeterminacy*. This (rather long) name is self-explanatory – indeterminacy of the moduli map j is due to the variation in moduli of the elliptic curve E contracted to the tacnode.

We finish by noting that, because of monodromy, we cannot in general distinguish which k nodes are "new" and which tacnode among "new" is responsible for indeterminacy on the curve C, globally. However, an assignment of k "new" tacnodes and a tacnode responsible for indeterminacy defines an analytic component of $V^{D,\delta}(\alpha, \beta)$ around the point of indeterminacy.

The curve with a tacnode responsible for indeterminacy and the nodal reduction corresponding to a generic arc with the center at the point of indeterminacy is depicted in Figure 4.

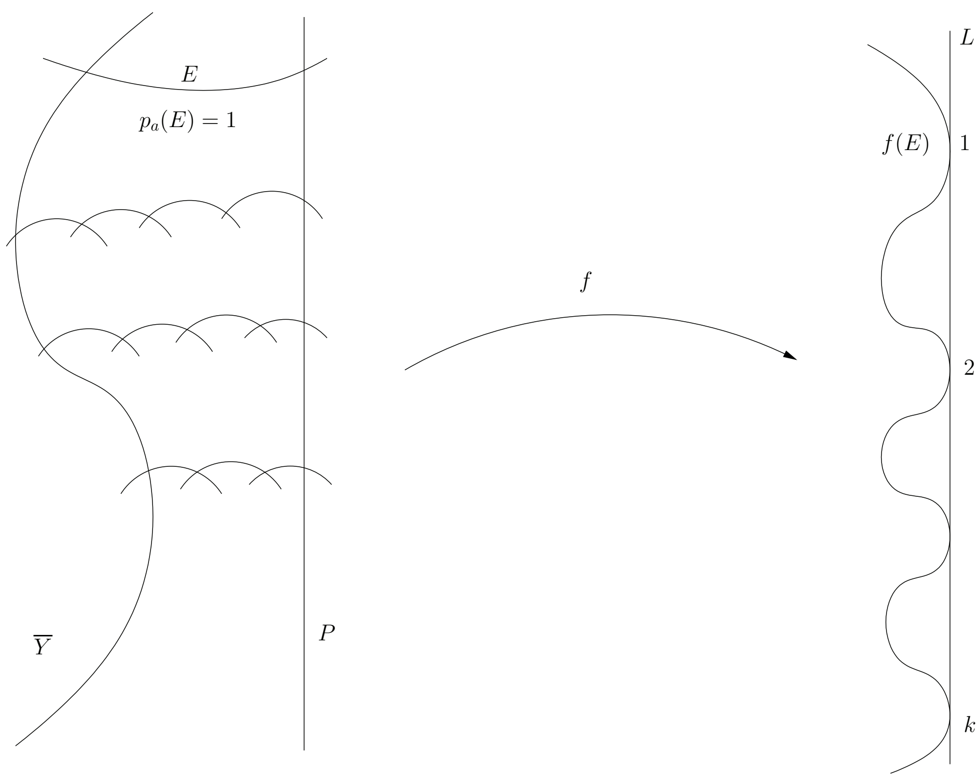

Figure 4: Point of indeterminacy

7 Deformation of maps with tangency conditions

In this section, we recall the deformation-theoretic results appearing in [31] that will be essential for the description of the local geometry of the universal family

$$\mathcal{Y}^{D,\delta}(\alpha,\beta) \to S^{D,\delta}(\alpha,\beta).$$

We start with a simple observation. Let $f\colon X \to S$ be a local embedding of a reduced projective (possibly singular) curve X into a smooth surface S. Consider deformations of $f\colon X \to S$ preserving the target. Denote the resulting versal deformation space by $\mathrm{Def}(f)$.

Proposition 7.1. *Suppose that* $\mathrm{Def}(f)$ *is smooth. Then there is a natural morphism*

$$\psi\colon \ \mathrm{Def}(f) \to \mathrm{Hilb}(S),$$

sending a deformation of f *to a defomation of its image subscheme. Moreover,* ψ *is an isomorphism onto the image locally at the point* $[f]$.

Proof. For the first part, it is enough to show that any deformation of f over an Artin ring induces a deformation of the image subscheme. This is standard and we omit the proof.

For the second part, suppose that on an affine open $\operatorname{Spec} R$ in S, the image subscheme is a union of the images of affines $X_1, \ldots, X_k$ on X embedded by f into S. Suppose that X_i is given by an equation $f_i = 0$, so that the image subscheme is given by the equation

$$f_1 f_2 \cdots f_k = 0.$$

A first-order deformation of f locally on X_i is given by an element $g_i \in R/(f_i)$ so that the deformation of X_i is given by the equation

$$f_i + \epsilon g_i = 0.$$

The induced first-order deformation of the image subscheme is defined by the equation

$$(f_1 + \epsilon g_1)(f_2 + \epsilon g_2) \cdots (f_k + \epsilon g_k) = f_1 \cdots f_k + \epsilon(g_1 f_2 \cdots f_k + \cdots + f_1 \cdots f_{k-1} g_k).$$

This deformation is trivial if and only if $g_1 f_2 \cdots f_k + \cdots + f_1 \cdots f_{k-1} g_k$ is divisible by $f_1 \cdots f_k$. This happens if and only if $g_i = 0$ in $R/(f_i)$.

We conclude that the differential of ψ is injective, proving the second part. $\square$

Consider the variety $V_L^{D-L,\delta}(\alpha, \beta)$ and a generic point $X = Y \cup L$ of one of its irreducible components. The curve Y is a generic point of one of the irreducible components of $V^{D-L,\delta}(\alpha, \beta)$. By definition, Y has only nodal singularities and is smooth along L. Let $\{q_{i,j}\}_{1 \leq j \leq \alpha_i}$ and $\{r_{i,j}\}_{1 \leq j \leq \beta_i}$ be the points on the normalization

$$\nu \colon\ Y^\nu \to Y$$

satisfying

$$\nu^*(L) = \sum i \cdot q_{i,j} + \sum i \cdot r_{i,j}\ ,$$

and such that points $p_{i,j} = \nu(q_{i,j})$ are fixed. Denote $\Omega = \{p_{i,j}\}$ to be the set of $|\alpha|$ fixed points of contact of Y with L. The $|\beta|$ free points of contact are $s_{i,j} = \nu(r_{i,j})$, $1 \leq j \leq \beta_i$.

We would like to study deformations of X satisfying certain tangency conditions. To setup our deformation problem, let

$$\Omega^* = \{p_{i,j}\}_{1 \leq j \leq \alpha_i^*} \supset \Omega$$

be an enlargement of Ω by a generic set of points. Here, $\alpha_i^* \geq \alpha_i$ for all i. Also, for every i, pick a subset $\{r_{i,j}^*\}_{1 \leq j \leq \beta_i^*}$ of the set $\{r_{i,j}\}$, for some $\beta_i^* \leq \beta_i$. Denote $s_{i,j}^* = \nu(r_{i,j}^*)$. Then $\{s_{i,j}^*\}$ is a subset of free points of contact of Y with L. For notational convenience, order the remaining points $s_{i,j}$ arbitrarily into a sequence

$$\{t_1, \ldots, t_\gamma\}.$$

Let the multiplicity of intersection of Y with L at these points be given by numbers $\{m_1, \ldots, m_\gamma\}$, so that the point t_k is a tacnode of order $m_k - 1$ on the curve X.

Consider the partial normalization $\tilde{X}$ of X which normalizes all singularities of X except tacnodes $t_1, \ldots, t_\gamma$:

$$f\colon\ \tilde{X} \to X.$$

Clearly, the decomposition of $\tilde{X}$ into irreducible components is $Y^\nu \cup P$, where P is isomorphic to L via f and where two components meet in the tacnodes $t_1, \ldots, t_\gamma$. Finally, introduce points $q_{i,j} = f^{-1}(p_{i,j})$ for all i and $\alpha_i + 1 \leq j \leq \alpha_i^*$.

7.1 Relaxed Severi varieties revisited

Keep the notations of the previous section. Let $\Upsilon := \dim V_L^{D-L,\delta}(\alpha, \beta)$. Fix Υ generic points $\Gamma := \{z_1, \ldots, z_\Upsilon\}$ on Y. Consider the following list of conditions:

1. Deformations of points $q_{i,j}$ remain points of tangency of order i with the divisor L at the fixed points $p_{i,j}$.

2. Deformations of points $r_{i,j}^*$ remain (unspecified) points of tangency of order i with the divisor L. This is equivalent to the condition that deformations of $s_{i,j}^* = \nu(r_{i,j}^*)$ remain tacnodes of X.

3. Deformations of $\tilde{X}$ pass through fixed points $z_1, \ldots, z_\Upsilon$.

We consider two subschemes of Def(f). The first is

$$\mathrm{Def}\left(f;\Omega^*;\{s^*_{i,j}\}\right)$$

comprised of the deformations of $f\colon\ \tilde{X}\to S$ that satisfy Conditions (1-2) above. The second is

$$\mathrm{Def}\left(f;\Omega^*;\{s^*_{i,j}\}\right)^\Gamma$$

comprised of the deformations satisfying all of the above conditions.

Remark 10. Note that we do not impose any requirements on the deformations of tacnodes t_i of the curve $\tilde{X}$.

Remark 11. The image of Def $\left(f;\Omega^*;\{s^*_{i,j}\}\right)$ in $|D|$, under the morphism constructed in Proposition 7.1, is a *relaxed local Severi variety* of Proposition 4.8 in [2].

By Lemma 6.5 of [31], which is a generalization of Lemma 4.9 in [2] we know that the scheme Def $\left(f;\Omega^*;\{s^*_{i,j}\}\right)^\Gamma$ is smooth at the point $[f\colon\ \tilde{X}\to S]$ and has dimension

$$\sum_i^\gamma (m_i-1)+1.$$

Being a local embedding into S at every point of $\tilde{X}$, the map f induces a natural classifying morphism

$$\phi\colon\ \mathrm{Def}(f)\to\prod_{i=1}^\gamma \mathrm{Def}(\tilde{X},t_i)$$

from the versal deformation space of f into the product of the versal deformation spaces of the tacnodes t_i of X.

At this point we refer the reader to the beginning of Section 9 for basic definitions regarding the deformation space of a tacnode. Shortly, we have

$$\mathrm{Def}(\tilde{X}, t_i) = \mathrm{Def}(y(y + x^{m_i}) = 0) \cong \operatorname{Spec} \mathbb{C}[a_{i,2}, \ldots, a_{i,2m_i}],$$

where $\{a_{i,2m_i} = 0\}$ is a distinguished hyperplane in $\mathrm{Def}(\tilde{X}, t_i)$ parameterizing deformations vanishing at t_i. The subvariety of $\mathrm{Def}(y(y+x^{m_i}) = 0)$ parameterizing deformations that preserve m nodes is denoted Δ_m.

Finally, we state the key result needed to describe the local geometry of $S^{D,\delta}(\alpha, \beta)$. It appears as Lemma 6.6 in [31], which in turn is a generalization of the discussion following Lemma 4.9 in [2].

Fact 3. *The following is true for the morphism*

$$\phi\colon\ \mathrm{Def}\left(f; \Omega^*; \{s^*_{i,j}\}\right)^{\Gamma} \to \prod_{i=1}^{\gamma} \mathrm{Def}(\tilde{X}, t_i).$$

1. *ϕ is an immersion.*
2. *The image $W := Im(\phi)$ contains $\Delta_{\mathfrak{m}}$ and is smooth of dimension*

   $$\sum (m_i - 1) + 1$$

   at the origin.
3. *The tangent space to W at the origin is not contained in the union of hyperplanes $a_{i,2m_i} = 0$.*

8 Local geometry of the family $\mathcal{Y}^{D,\delta}(\alpha,\beta)$

Let $\Upsilon = \dim V^{D,\delta}(\alpha,\beta)$. In the following discussion,

$$\Gamma = \{p_1, \ldots, p_{\Upsilon-2}\}$$

is a set of $\Upsilon - 2$ general points on S.

We take $S^{D,\delta}(\alpha,\beta)$ to be the surface-section of the generalized Severi variety of divisors $V^{D,\delta}(\alpha,\beta)$ formed by intersecting with the linear subspace H_Γ. Given a generic $q \in L$, we consider a special curve-section

$$H_q := S^{D,\delta}(\alpha,\beta) \cap H_q$$

that breaks into a union of Type I and Type II components. In this chapter, we describe the geometry of the universal family

$$\mathcal{Y}^{D,\delta}(\alpha,\beta) \to S^{D,\delta}(\alpha,\beta)$$

locally around the points of H_q.

As in Section 6.2, consider the cycle morphism

$$\psi\colon\ M^{D,g}(\alpha,\beta) \to S^{D,\delta}(\alpha,\beta),$$

from the strict transform of $S^{D,\delta}(\alpha,\beta)$ in $M^{D,g}(S)$.

By Theorem 2 in Section 6, the only points on $S^{D,\delta}(\alpha,\beta) \cap H_q$ with positive dimensional fibers are exactly points of the varieties

$$V_L^{D-L,\delta'}(\alpha',\beta') \cap H_\Gamma,$$

where $|\beta' - \beta| + \delta - \delta' = D \cdot L - L^2 - 1$.

Since

$$\operatorname{card}(\Gamma) = \dim V^{D,\delta}(\alpha, \beta) - 2 = \dim V_L^{D-L,\delta'}(\alpha', \beta'),$$

there are finitely many such points. The moduli map

$$j\colon\ S^{D,\delta}(\alpha, \beta) \dashrightarrow M^{D,\delta}(\alpha, \beta).$$

is undefined at these points even after the normalization of the domain. From now on, we call them *points of indeterminacy.*

The only singularities, besides nodes, that appear on the curves corresponding to generic points of Type II components and on the curves corresponding to the points of indeterminacy are higher-order tacnodes. Therefore the description of the local geometry of $\mathcal{Y}^{D,\delta}(\alpha, \beta)$ invariably invokes versal deformation spaces of (higher-order) tacnodes. We point the reader to the beginning of Section 9 where the necessary definitions are recalled.

8.1 Geometry at a generic point of Type II component

First, we recall the description of the geometry of $S^{D,\delta}(\alpha, \beta)$ at a generic point of a Type II component of $H_q = S^{D,\delta}(\alpha, \beta) \cap H_q$ as presented in [31].

Suppose that

$$X = Y \cup L \in C_L^{D-L,\delta'}(\alpha', \beta')$$

is such that Y is a generic point of

$$C^{D-L,\delta'}(\alpha', \beta') = V^{D-L,\delta'}(\alpha', \beta') \cap H_\Gamma,$$

where

$$|\beta' - \beta| + \delta - \delta' = D \cdot L - L^2.$$

The curve X has $|\alpha'| + |\beta'|$ tacnodes, at the points of contact of Y with L, of which $|\alpha'|$ are fixed and belong to the set $\Omega \subset L$. Among the free $|\beta'|$ points of contact, exactly $|\beta|$ are limits of free points of tangency in the nearby fibers. The rest of the tacnodes correspond to *new* points of tangency of Y with L. Set $\beta'' := \beta' - \beta$ and $\gamma := |\beta''|$.

The surface $S^{D,\delta}(\alpha,\beta)$ has several branches. First of all, there is a choice of $|\beta|$ points

$$\{s^*_{i,j}\}_{1\le j\le \beta_i}$$

that are the limits of free points of tangency in the nearby fibers. The remaining $\gamma = |\beta' - \beta|$ tacnodes of X are designated as *new* and denoted

$$\{t_1, \dots, t_\gamma\}.$$

Suppose that t_i is a tacnode of order $m_i - 1$. Consider the partial normalization

$$f\colon\ \tilde{X} \to X$$

of X preserving tacnodes t_i and normalizing all other singularities. Then the deformation space (defined in Section 7.1)

$$\mathrm{Def}(f;\Omega;\{s^*_{i,j}\})^\Gamma$$

maps, by Proposition 7.1, to the Severi variety $V^{D,\delta''}(\alpha,\beta)(\Omega) \cap H_\Gamma$, where $\delta'' = \delta - (I\beta'' - \gamma)$. Its image is a smooth analytic component of $V^{D,\delta''}(\alpha,\beta) \cap H_\Gamma$.

Note that

$$S^{D,\delta}(\alpha,\beta) \subset (V^{D,\delta''}(\alpha,\beta) \cap H_\Gamma).$$

In the neighborhood of $[f\colon\ \tilde{X} \to X]$, the preimage of the variety $S^{D,\delta}(\alpha,\beta)$ is the closure of the deformations of f such that every tacnode t_i, for $1 \le i \le \gamma$, deforms to $m_i - 1$ nodes.

To say it differently, consider the map

$$\phi\colon \ \mathrm{Def}(f;\Omega;\{s^*_{i,j}\})^\Gamma \to \prod_{i=1}^{\gamma} \mathrm{Def}(\tilde{X}, t_i).$$

By Fact 3, its image W contains $\Delta_{\mathfrak{m}}$ and is smooth of dimension

$$\sum_{i=1}^{\gamma}(m_i - 1) + 1.$$

The component of the variety $S^{D,\delta}(\alpha,\beta)$ inside $\mathrm{Def}(f;\Omega;\{s^*_{i,j}\})^\Gamma$ is isomorphic to the preimage under ϕ of the variety

$$\Gamma_{\mathfrak{m}} := \overline{(W \cap \Delta_{\mathfrak{m}-\mathfrak{1}}) \smallsetminus \Delta_{\mathfrak{m}}}\ . \tag{22}$$

There are $\binom{\beta'}{\beta}$ choice of points $\{s^*_{i,j}\}_{1\le j\le \beta_i}$ among points $\{s_{i,j}\}_{1\le j\le \beta'_i}$. Therefore, there are $\binom{\beta'}{\beta}$ analytic components of $S^{D,\delta}(\alpha,\beta)$ at a generic point X of a Type II component $C^{D-L,\delta'}(\alpha',\beta')$ of H_q. Each component maps smoothly, with one-dimensional fibers, onto the variety $\Gamma_{\mathfrak{m}}$.

8.2 Geometry at a point of indeterminacy

Suppose $X = Y \cup L$ is a point of indeterminacy on

$$S^{D,\delta}(\alpha,\beta) = V^{D,\delta}(\alpha,\beta) \cap H_\Gamma.$$

Then Y is a generic point of

$$V^{D-L,\delta'}(\alpha',\beta') \cap H_\Gamma,$$

where

$$|\beta' - \beta| + \delta - \delta' = D \cdot L - L^2 - 1.$$

Note that there are precisely

$$\binom{\alpha}{\alpha'} N^{D-L,\delta'}(\alpha', \beta') \tag{23}$$

points of indeterminacy corresponding to the triple $(\delta', \alpha', \beta')$.

As in the previous section, we note that $S^{D,\delta}(\alpha, \beta)$ has several analytic components in the neighborhood of X. Each component is specified by the choice of $|\beta|$ tacnodes

$$\{s^*_{i,j}\}_{1 \leq j \leq \beta_i}$$

of X that are the limits of free points of tangency in the nearby fibers. There are $\binom{\beta'}{\beta}$ such components.

Choose one of them. Set $\beta'' = \beta' - \beta$. Let $t_1, \ldots, t_\gamma$ be the remaining $\gamma := |\beta''|$ "new" tacnodes of X, of orders $m_1 - 1, \ldots, m_\gamma - 1$, respectively. Consider the partial normalization

$$f\colon\ \tilde{X} \to X$$

of X preserving only the tacnodes t_i. Then the deformation space

$$\mathrm{Def}(f; \Omega; \{s^*_{i,j}\})^\Gamma$$

maps isomorphically onto its image in the Severi variety

$$V^{D,\delta''}(\alpha, \beta),$$

where $\delta'' = \delta - (I\beta'' - \gamma) + 1$. We identify $\mathrm{Def}(f; \Omega; \{s^*_{i,j}\})^\Gamma$ with its image.

For every $1 \leq j \leq \gamma$, consider the closure of the locus of deformations of f

such that every tacnode t_i deforms to $m_i - 1 - \delta_{i,j}$ nodes. The union of these loci inside

$$\operatorname{Def}(f;\Omega;\{s^*_{i,j}\})^\Gamma$$

is an analytic component of the Severi variety $S^{D,\delta}(\alpha,\beta)$.

To say it differently, consider the morphism

$$\phi\colon\ \operatorname{Def}(f;\Omega;\{s^*_{i,j}\})^\Gamma \to \prod_{i=1}^{\gamma} \operatorname{Def}(\tilde{X}, t_i).$$

By Fact 3, the map ϕ is a local isomorphism. Its image W contains $\Delta_{\mathfrak{m}}$ and is smooth of dimension

$$\sum_{i=1}^{\gamma}(m_i - 1) + 1.$$

Finally, the component of the variety $S^{D,\delta}(\alpha,\beta)$ in $\operatorname{Def}(f;\Omega;\{s^*_{i,j}\})^\Gamma$ is isomorphic to the variety

$$S_{\mathfrak{m}} := \bigcup_{i=1}^{\gamma} S_i,$$

where

$$S_i := \overline{W \cap \Delta_{\mathfrak{m}-\mathbb{1}-e_i} \smallsetminus \Delta_{\mathfrak{m}}}\ .$$

9 Geometry of the deformation space of a tacnode

9.1 Versal deformation space of a planar curve singularity

Suppose C is defined by the equation $g(x,y) = 0$ in $\mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[x,y]$ and $p = (0,0)$ is the origin. A planar curve singularity of analytic type (C,p) is a curve singularity whose complete local ring is

$$\hat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x,y]]/g(x,y).$$

We will often use terms "singularity (C,p)" and "singularity $g(x,y) = 0$" interchangeably when referring to a singularity of analytic type (C,p).

If C is reduced, we can take the underlying affine space of the Tjurina algebra

$$\mathbb{C}[x,y]/(g, \partial g/\partial x, \partial g/\partial y)$$

to be the *miniversal deformation space* of the singularity (C,p), and denote it $\operatorname{Def}(C,p)$. In particular, the versal[3] deformation space is smooth for reduced planar curves singularities.

9.2 Double points

Recall that a multiplicity 2 planar curve singularity is analytically isomorphic to

$$y^2 - x^n = 0$$

for some $n \geq 2$. If $n = 2m$ is even, then we have the $(m-1)^{\text{th}}$-order tacnode that is also isomorphic to

$$y(y + x^m) = 0.$$

[3]From now on, we will use the term "versal" even when referring to a miniversal deformation.

If $n = 2m + 1$ is odd, then we have a (higher-order) cusp. In either case, a quick computation of the Tjurina algebra shows that the versal deformation space is the affine space $\mathbb{C}^{n-1}$.

9.3 Tacnodes

The versal deformation space of the $(m-1)^{\text{th}}$-order tacnode

$$y(y + x^m) = 0$$

is $T = \operatorname{Spec} \mathbb{C}[a_2, \ldots, a_{2m}]$. The versal family $\mathcal{Y}$ over T is defined by the equation

$$\Psi_a(x, y) := y^2 + (x^m + a_2 x^{m-2} + \cdots + a_m)y + a_{m+1}x^{m-1} + \cdots + a_{2m} = 0.$$

inside $T \times \operatorname{Spec} \mathbb{C}[x, y]$. We will also make use of the quasi-homogeneous polynomial

$$\Psi_a(x, y, z) := y^2 + (x^m + a_2 x^{m-2} z^2 + \cdots + a_m z^m)y + a_{m+1}x^{m-1}z^{m+1} + \cdots + a_{2m}z^{2m}.$$

We can and will regard the polynomial

$$\Psi(a_2, \ldots, a_{2m})(x, y) := \Psi_a(x, y)$$

as a quadratic polynomial in y. The discriminant of $\Psi_a(x, y)$ is then defined by

$$\delta_a(x) = \delta(\Psi_a(x, y)) := (x^m + a_2 x^{m-2} + \cdots + a_m)^2 - 4(a_{m+1}x^{m-1} + \cdots + a_{2m}).$$

Let D_{2m} be the affine space of monic polynomials of degree $2m$ with a trivial

x^{2m-1} coefficient

$$D_{2m} = \{x^{2m} + d_2 x^{2m-2} + \cdots + d_{2m-1}x + d_{2m}\}.$$

Any polynomial in D_{2m} can be represented as a sum of the square of a monic polynomial of degree m and a polynomial of degree $m-1$. More precisely, we have the following result.

Lemma 9.1. *The morphism $\delta\colon T \to D$ sending a point $a \in T$ to the discriminant of the polynomial $\Psi_a(x, y)$ is an isomorphism.*

Proof. It is easy to see that δ is bijective on the closed points and that its Jacobian is an upper-triangular matrix with non-zero complex numbers along the diagonal. □

Remark 12. Under the isomorphism δ, the locus $\{a_2 = 1\}$ in T is mapped to the locus $\{d_2 = 2\}$ in D_{2m}.

Definition 9.1. A *combinatorial type* of a polynomial $\mathfrak{P}$ of degree d with r roots is a sequence $(m_1, \ldots, m_r)$ of the roots' multiplicities.

The deformation space T has a natural geometric stratification given by the combinatorial type of the discriminant of $\Psi_a(x, y)$.

Definition 9.2. We denote the stratum of deformations whose discriminant has the polynomial type $(m_1, \ldots, m_r)$ by

$$\Delta^0\{m_1, \ldots, m_r\},$$

and denote its closure by $\Delta\{m_1, \ldots, m_r\}$.

Remark 13. For $a \in \Delta^0\{m_1, \ldots, m_r\}$, the singularities of the fiber $\mathcal{Y}_a$, defined by the equation

$$\Psi_a(x, y) = 0,$$

are double points $y^2 = x^{m_i}$. Therefore, $\Delta^0\{m_1, \ldots, m_r\}$ is an equisingular stratum.

We also define

$$\Delta_r := \Delta\{\underbrace{2, 2, \ldots, 2}_{r}, \underbrace{1, \ldots, 1}_{2m-2r}\}$$

to be the closure of the locus of r-nodal deformations. For example, the locus of reducible deformations Δ_m is given by the equations

$$a_{m+1} = \cdots = a_{2m} = 0.$$

We will distinguish the hyperplane H inside T defined by $a_{2m} = 0$. Under the identification of the tangent space $\mathbb{T}_0 T$ with the space $\mathrm{Def}_1(y(y + x^m) = 0)$ of the first-order deformations of the tacnode, $\mathbb{T}_0 H$ corresponds to the first-order deformations of $y(y + x^m) = 0$ vanishing at $(0, 0) \in \mathbb{C}^2$.

9.4 Multiple tacnodes

Let $\mathfrak{m} = (m_1, \ldots, m_n)$ be a sequence of positive integers. For $1 \leq i \leq n$, let

$$T(i) \cong \operatorname{Spec} \mathbb{C}[a_{i,2}, \ldots, a_{i,2m_i}]$$

be the versal deformation space of the tacnode $y(y + x^{m_i}) = 0$. We set

$$T := \prod_{i=1}^{n} T(i) \cong \mathbb{C}^{2(\sum_i m_i) - n}$$

to be the product of these deformation spaces.

By analogy with a single tacnode case, we define the following loci inside T:

$$\Delta_{\mathfrak{m}} := \prod_{i=1}^{n} \Delta_{i,m_i}\,,$$
$$\Delta_{\mathfrak{m}-\mathbb{1}} := \prod_{i=1}^{n} \Delta_{i,m_i-1}\,,$$
$$\Delta_{\mathfrak{m}-\mathbb{1}-e_j} := \prod_{i=1}^{n} \Delta_{i,m_i-1-\delta_{i,j}}, \quad 1 \le j \le n.$$

9.5 Alterations

We begin the study of T by making the base change

$$\pi_1\colon\ T' = \operatorname{Spec}\mathbb{C}[b_2, b_3, \ldots, b_{2m}] \to T$$

defined by $a_i = b_i^i$.

Consider the finite group

$$\mu := \mu_2 \times \mu_3 \times \cdots \times \mu_{2m},$$

where μ_r is the cyclic group of r^{th} roots of unity. Then, under the natural action of μ on T', we have

$$T = T'//\mu.$$

Next, we consider the ordinary blow-up of T' at the origin

$$\pi_2\colon\ \operatorname{Bl}_0 T' \to T'.$$

Denote the exceptional divisor of π_2 by E. We denote the composition of π_2 and π_1 by π:

$$\pi = \pi_2 \circ \pi_1\colon\ T'' \to T.$$

Denote by $\mathcal{Y}'' \subset T'' \times \operatorname{Spec} \mathbb{C}[x,y]$ the pullback of $\mathcal{Y}$ to T''. Let I_E be the ideal sheaf of E on T'', and consider the ideal sheaf

$$\mathcal{I} := ((I_E, x)^m, y)$$

on $T'' \times \operatorname{Spec} \mathbb{C}[x,y]$. The last birational modification that we introduce is the blow-up of the ideal sheaf $\mathcal{I}$

$$\chi\colon\ \mathcal{X} := Bl_{\mathcal{I}}(T'' \times \operatorname{Spec} \mathbb{C}[x,y]) \to T'' \times \operatorname{Spec} \mathbb{C}[x,y].$$

9.6 Family of surfaces

First, note that

$$\mathcal{X} = Bl_{\mathcal{I}}(T'' \times \operatorname{Spec} \mathbb{C}[x,y]) \to T''$$

is a family of surfaces. A fiber of $\mathcal{X}$ over a point in $T'' \setminus E$ is the affine plane $\operatorname{Spec} \mathbb{C}[x,y]$, and a fiber over a point in the exceptional divisor E is a union of

$$\mathrm{Bl}_{(x^m,y)} \operatorname{Spec} \mathbb{C}[x,y]$$

and

$$\mathbb{P}(1,1,m) = \operatorname{Proj} \mathbb{C}[x,y,z].$$

Here, z stands for a local generator of I_E and $\mathbb{C}[x,y,z]$ is graded with $\deg x = \deg z = 1$ and $\deg y = m$. The weighted projective space $\mathbb{P}(1,1,m)$ is the projective cone over the rational normal curve of degree m. We denote by $\mathcal{O}(1)$ the restriction to $\mathbb{P}(1,1,m)$ of the line bundle $\mathcal{O}_{\mathbb{P}^{m+1}}(1)$. It generates the Picard group of $\mathbb{P}(1,1,m)$.[4]

[4]But not the class group of Weil divisors.

Finally, we consider a family of curves

$$\mathcal{Z} := \mathrm{Bl}_{\mathcal{I}}\,\mathcal{Y}''.$$

The morphism from $\mathcal{Z}$ to T'' is denoted F. The family of curves

$$F\colon\ \mathcal{Z} \to T''.$$

will be the main object of study in the next sections. All spaces and morphisms that we have introduced fit into the following commutative diagram with Cartesian squares.

$$\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{\ \chi\ } & T'' \times \mathbb{C}^2 & \longrightarrow & T \times \mathbb{C}^2 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{Z} & \xrightarrow{\ \chi\ } & \mathcal{Y}'' & \longrightarrow & \mathcal{Y} \\
& \overset{F}{\searrow} & \downarrow & & \downarrow \\
& & T'' & \xrightarrow{\ \pi\ } & T
\end{array}$$

9.7 Geometry of $\mathcal{Z}$

The family $F\colon\ \mathcal{Z} \to T''$ can be seen as a first step towards the stable reduction of the versal family $\mathcal{Y} \to T$. The following paragraphs make this statement more precise.

Let $\overline{\{b_i = 0\}}$ be the strict transform of the locus $\{b_i = 0\}$ under the blow-up π_2. Consider a usual open cover of the blow-up T'' by the affine charts

$$D_{(b_i)}T'' := T'' \setminus \overline{\{b_i = 0\}}.$$

We have

$$D_{(b_i)}T'' \cong \operatorname{Spec}\mathbb{C}\left[\frac{b_2}{b_i}, \frac{b_3}{b_i}, \ldots, b_i, \ldots, \frac{b_{2m}}{b_i}\right].$$

Then, over $D_{(b_i)}T''$, the family $\mathcal{Y}''$ has equation

$$\Psi\left(\left(\frac{b_2}{b_i}\right)^2, \ldots, \left(\frac{b_{2m}}{b_i}\right)^{2m}\right)(x, y, b_i) = 0. \tag{24}$$

The restriction $E_{(b_i)}$ of the exceptional divisor E to $D_{(b_i)}T''$ is given by the equation $b_i = 0$. The ideal $\mathcal{I}$ restricts to the ideal $((b_i, x)^m, y)$ on the variety

$$D_{(b_i)}T'' \times \operatorname{Spec} \mathbb{C}[x, y] \cong \operatorname{Spec} \mathbb{C}\left[\frac{b_2}{b_i}, \frac{b_3}{b_i}, \ldots, b_i, \ldots, \frac{b_{2m}}{b_i}\right] \times \operatorname{Spec} \mathbb{C}[x, y].$$

Consider the complement V_i of the strict transform of $E_{(b_i)} \times \operatorname{Spec} \mathbb{C}[x, y]$ in the blow-up

$$\mathcal{X} \times_{T''} D_{(b_i)}T'' = \operatorname{Bl}_{\mathcal{I}}\left(D_{(b_i)}T'' \times \operatorname{Spec} \mathbb{C}[x, y]\right).$$

It is easy to see that

$$V \cong \operatorname{Spec} \mathbb{C}\left[\frac{b_2}{b_i}, \frac{b_3}{b_i}, \ldots, b_i, \ldots, \frac{b_{2m}}{b_i}\right] \times \operatorname{Spec} \mathbb{C}\left[\frac{x}{b_i}, \frac{y}{b_i^m}\right],$$

and the restriction of the family $\mathcal{Z}$ to V_i is given by the equation

$$\Psi\left(\left(\frac{b_2}{b_i}\right)^2, \ldots, \left(\frac{b_{2m}}{b_i}\right)^{2m}\right)\left(\frac{x}{b_i}, \frac{y}{b_i^m}\right) = 0. \tag{25}$$

It follows from Equation (24) that the restriction of $\mathcal{Z}$ to $E_{(b_i)}$

$$\mathcal{Z}_E := \mathcal{Z} \times_{T''} E_{(b_i)}$$

has two irreducible components $\mathcal{S}$ and $\mathcal{T}$. The variety $\mathcal{S}$ is a strict transform of $\mathcal{Y}'' \times_{T''} E_{(b_i)}$. It is a trivial family over $E_{(b_i)}$ defined by the equation

$$\frac{y}{x^m}\left(\frac{y}{x^m} + 1\right) = 0$$

inside $E_{(b_i)} \times \mathrm{Bl}_{(x^m,y)} \operatorname{Spec} \mathbb{C}[x,y]$. The scheme $\mathcal{T}$ is the exceptional divisor of the blow-up $\chi : \mathcal{Z} \to \mathcal{Y}''$. It is defined by the equation

$$\Psi\left(\left(\frac{b_2}{b_i}\right)^2, \dots, \left(\frac{b_{2m}}{b_i}\right)^{2m}\right)(x,y,z) = 0$$

inside $E_{(b_i)} \times \mathbb{P}(1,1,m)$.

In words, over $E_{(b_i)}$, the family $\mathcal{Z}_E$ has two components. The first component, $\mathcal{S}$, is a trivial family whose fibers are isomorphic to the normalization of the curve $y(y+x^m) = 0$. The second component, $\mathcal{T}$, is a family of divisors, not passing through the vertex, in the linear series $|\mathcal{O}(2)|$ on $\mathbb{P}(1,1,m)$. Following [18], we designate fibers $\mathcal{T}_p$, of the family $\mathcal{T} \to E$, as *tails*. Note that all such tails are hyperelliptic curves of arithmetic genus $m-1$ with 2 distinguished marked points defined by $z = 0$. The marked points are the points of intersection with $\mathcal{S}$ and are exchanged by the hyperelliptic involution $y \mapsto -y$.

A fiber of $\mathcal{Z}$ over a point $p \in E$ is a nodal union of $\mathcal{S}_p$ and $\mathcal{T}_p$ along two nodes. We call the nodes $\mathcal{S}_p \cap \mathcal{T}_p$ *connecting nodes*. The only singularities of $\mathcal{Z}_p$ for $p \in E$ are connecting nodes and the singularities on the tail $\mathcal{T}_p$. We observe that singularities appearing on the tail are planar double points $y^2 - x^n = 0$ with $n < 2m$. Hence, after an alteration of the versal deformation space and an appropriate blow-up of the versal family, we arrived at the family of curves with milder singularities.

Definition 9.3. The locus $\Delta\{m_1, \dots, m_k\}$ inside T'" is a strict transform of $\pi^{-1}(\Delta\{m_1, \dots, m_k\})$ under the blow-up $\pi_2 \colon T'' \to T'$.

The locus $\Delta\{m_1, \dots, m_k\}$ can still be thought as an equisingular locus, but with a twist. Namely, a point $p \in T''$ belongs to $\Delta\{m_1, \dots, m_k\}$ if and only if the fiber of the open family $(\mathcal{Z} \smallsetminus \mathcal{S}) \to T''$ at the point p has double point singularities of type $y^2 - x^{m_i} = 0$, for $1 \le i \le k$. By the paragraph above, if $p \in E$, then the singularities lie on the tail $\mathcal{T}_p$.

We note for the future use the following result implied by by Equation (25).

Observation 5. *Locally on the target, the family*

$$(\mathcal{Z} \smallsetminus \mathcal{S}) \to T''$$

is a product of a smooth curve and the family $(\mathcal{T} \smallsetminus (\mathcal{T} \cap \mathcal{S})) \to E$.

Proof. Equation (25) does not depend on b_i. Therefore, $\operatorname{Spec} \mathbb{C}[b_i]$ is a requisite curve. □

In particular, the same statement is true for the strict transform in T'' of any equisingular locus in T. Locally on T'', the locus $\Delta\{m_1, \ldots, m_k\}$ is a product of $\Delta\{m_1, \ldots, m_k\}_E$ and a smooth curve.

9.8 Weighted blow-ups

Note that all blow-ups in the previous section were made with μ-invariant centers. Hence, the action of μ extends to T'' and $\mathcal{Z}$ making the morphism F equivariant. For every $2 \le i \le 2m$, we consider the quotient of

$$F\colon\ \mathcal{Z} \to T''$$

by the natural action of the subgroup

$$\tilde{\mu}_i := \mu_2 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_{2m}.$$

We set $\mathcal{Z}_i := \mathcal{Z}/\!/\tilde{\mu}_i$ and $T_i := T''/\!/\tilde{\mu}_i$. The quotient of $F\colon\ Z \to T''$ is a morphism

$$F_i\colon\ \mathcal{Z}_i \to T_i.$$

We define $\mathcal{T}_i$ to be the quotient $\mathcal{T}//\tilde{\mu}_i$ of the tails component. Note that the morphism

$$f_i\colon\ T_i \to T$$

is nothing else than the weighted blow-up of T with weights $(2, 3, \dots, 2m)$, followed by the base change $a_i = b_i^i$. By abuse of notation, we also use E to denote the exceptional divisor of every f_i. If W is a subvariety of T, we denote by W_E its exceptional divisor in T_i under f_i.

We define the locus $\Delta\{m_1, \dots, m_k\}$ on T_i either as the quotient of the locus $\Delta\{m_1, \dots, m_k\}$ on T'' or as the strict transform of the locus $\Delta\{m_1, \dots, m_k\}$ on T under the alteration f_i. The two definitions are equivalent. The ambiguity of notation will not cause trouble, as we will always specify which f_i we are considering.

Note that the action of $\tilde{\mu}_i$ is free on $D_{(b_i)}T''$. The quotient, denoted $D_{(b_i)}T_i$, is isomorphic to $\operatorname{Spec}\mathbb{C}[c_2, \dots, b_i, \dots, c_{2m}]$ and $f_i\colon\ D_{(b_i)}T_i \to T$ is given by

$$a_i \mapsto b_i^i,$$
$$a_j \mapsto c_j b_i^j.$$

The equation of $E_{(b_i)} := E \cap D_{(b_i)}T_i$ is $b_i = 0$. Over $E_{(b_i)}$, the equation of $\mathcal{T}_i$ inside $E_{(b_i)} \times \mathbb{P}(1, 1, m)$ is

$$\Psi(c_2, \dots, c_{i-1}, 1, c_{i+1}, \dots, c_{2m})(x, y, z) = 0.$$

Henceforth, in our discussion we will occasionally identify the base of the family $\mathcal{T}_i \to E_{(b_i)}$ with the affine space

$$\operatorname{Spec}[c_2, \dots, \hat{c}_i, \dots, c_{2m}]$$

of quasi-homogeneous polynomials $\Psi(c_2, \ldots, c_{2m})(x, y, z)$ with $c_i = 1$, or, equivalently, in-homogeneous polynomials $\Psi(c_2, \ldots, c_{2m})(x, y)$ with $c_i = 1$.

9.9 Alteration f_2

We will be mostly concerned with the alteration $f_2\colon T_2 \to T$ and the family $F_2\colon \mathcal{Z}_2 \to T_2$. The following result explains our choice.

Lemma 9.2. *Given any point $p \in E_{(b_2)}$, denote by $\mathcal{D}(p)$ the product of the deformation spaces of the singularities of the fiber $(\mathcal{T}_2)_p$. Then the family*

$$\mathcal{T}_2 \to E_{(b_2)}$$

induces a smooth morphism from an analytic neighborhood of p in $E_{(b_2)}$ to an analytic neighborhood of the origin in $\mathcal{D}(p)$.

Proof. Suppose $p = (\lambda_3, \ldots, \lambda_{2m}) \in \Delta^0\{m_1, \ldots, m_k\}_E$. Then the polynomial

$$P := \delta(\Psi(1, \lambda_3, \ldots, \lambda_{2m})(x, y))$$

factors as

$$\prod_{i=1}^{k}(x - t_i)^{m_i},$$

where t_i satisfy $\sum_i t_i m_i = 0$, and has coefficient 2 at x^{2m-2}. The equation of $(\mathcal{T}_2)_p$ is

$$\Psi(1, \lambda_3, \ldots, \lambda_{2m})(x, y, z) = 0$$

and so the singularities of $(\mathcal{T}_2)_p$ are double points $y^2 - x^{m_i} = 0$. Hence,

$$\mathcal{D}(p) \cong \mathbb{C}^{\sum(m_i - 1)},$$

with equisingular locus being the origin.

By Lemma 9.1, the first-order deformations of p correspond to the first-order deformations of P of the form $P + \epsilon Q$, with Q being an arbitrary polynomial of degree $2m-3$. Equisingular deformations are precisely those satisfying

$$\prod_{i=1}^{k}(x-t_i)^{m_i-1} \,|\, Q(x).$$

This divisibility condition imposes $\sum_{i=1}^{k}(m_i-1)$ independent linear conditions on the coefficients of Q. Therefore, the map on the tangent spaces

$$\mathbb{T}_p E_{(b_2)} \to \mathbb{T}_0 \mathcal{D}(p)$$

is surjective. The lemma follows by the Jacobian criterion of smoothness (cf. [17, Proposition 10.4]). □

9.10 Tangent cones

Recall that the hyperplane H in T is defined by the equation $a_{2m} = 0$.

Lemma 9.3. *Suppose W is a subvariety of T that is smooth at the origin, has dimension m, contains Δ_m, and whose tangent space is not contained in the hyperplane H. Then the exceptional divisor of W in T'' is given by equations*

$$a_{m+i} = b_{m+i}^{m+i} = 0, \;\; 1 \le i \le m-1.$$

Here $[b_2 : \ldots : b_{2m}]$ are homogeneous coordinates on $E \subset T''$.

Proof. The tangent space of W in T is a linear space of dimension m not contained in the hyperplane $a_{2m} = 0$, but containing $\{a_{m+1} = \cdots = a_{2m-1} = a_{2m} = 0\}$.

Therefore, the initial ideal of W satisfies

$$\mathbf{in}(I(W)) = \ker[\ \mathbf{0}_{m-1} \Big| \mathbb{I} \Big| \vec{t}\ ],$$

where $\mathbf{0}_{m-1}$ is the zero $(m-1)\times(m-1)$ matrix, $\mathbb{I}$ is a row-permutation of the identity matrix $\mathbb{I}_{m-1}$, and $\vec{t} = [t_1, \ldots, t_{m-1}]^T$ is a column of complex numbers. Equivalently,

$$\mathbf{in}(I(W)) = (a_{m+\sigma(i)} + t_i a_{2m}),\ 1 \le i \le m-1,\ t_i \in \mathbb{C},\ \sigma \in \mathfrak{S}_{m-1}.$$

Therefore, under the substitution $a_i = b_i^i$, the initial ideal of $\pi_1^{-1}(W)$ in T' becomes

$$\{b_{m+i}^{m+i} = 0,\ 1 \le i \le m-1\}.$$

Proof is finished. □

Starting with the simple Lemma 9.3, we draw important consequences regarding various geometric strata inside T. Consider a point $a \in \Delta^0\{2m_1, \ldots, 2m_n\}$. Then we have

$$\delta(\Psi_a(x,y)) = (x-t_1)^{2m_1}\cdots(x-t_n)^{2m_n}$$

where $\sum_{i=1}^n m_i t_i = 0$. We set

$$T(i) = \operatorname{Spec} \mathbb{C}[c_{i,j}]_{2\le j\le 2m_i}$$

to be the versal deformation space of the m_i^{th}-order tacnode $(x,y) = (t_i, -t_i^{m_i}/2)$ in the fiber $\mathcal{Y}_a$. Then we have a natural map

$$\Phi : (T,a) \to \prod_{i=1}^n T(i),$$

defined in a neighborhood of the point $a \in T$.

Lemma 9.4. *Consider the linear subspace*

$$W := \{a_{m+1} = \cdots = a_{2m-1} = 0\}$$

of T and a point $a \in W$. Then $d\Phi(\mathbb{T}_a W)$ is not contained in the union of hyperplanes $c_{i,2m_i} = 0$.

Remark 14. We can reformulate the corollary as the statement that for all i

$$(\mathbb{T}_a W) \cap (d\Phi)^{-1}(\{c_{i,2m_i} = 0\}) = \mathbb{T}_a \Delta_m.$$

Proof. The statement follows from the geometric interpretation of the hyperplane $(d\Phi)^{-1}(\{c_{i,2m_i} = 0\})$ as the tangent space to deformations of $\mathcal{Y}_a$ vanishing at the point $(x, y) = (t_i, -t_i^{m_i}/2)$. Therefore to prove the lemma, we need to exhibit a first-order deformation of $\Psi_a(x, y) = 0$ in W that does not vanish at the above points. The deformation satisfying this condition is

$$\Psi_a(x, y) + \epsilon.$$

□

Lemma 9.5. *Consider $W_E := \{b_{m+1} = \cdots = b_{2m-1} = 0\} \subset E$. The following statements hold*

1. *$W_E \cap (\Delta_{m-1})_E$ is analytically irreducible at every point of $(\Delta_m)_E$.*

2. *$W_E \cap (\Delta_{m-2})_E$ is analytically irreducible at all points of strata*

$$\Delta^0\{2m_1, \ldots, 2m_n\}_E$$

where $(m_1, \ldots, m_n)$ *is an arbitrary* n*-tuple with* $n \geq 3$*.*

Proof. We prove the second statement as the proof of the first one is similar and easier. By Observation 5, it is enough to prove that, for $n \geq 3$, the variety $W \cap \Delta_{m-2}$ is analytically irreducible at all points of strata

$$\Delta^0\{2m_1, \ldots, 2m_n\}$$

inside $T \setminus \{0\}$.

Working on $T \setminus \{0\}$, we consider a point $a \in \Delta\{2m_1, \ldots, 2m_n\}$ and the induced map

$$\Phi : (T, a) \to \prod_{i=1}^{n} T(i).$$

We observe that

$$\Delta_{m-2} = \bigcup_{i,j=1}^{n} \Phi^{-1}\left(\prod_{k=1}^{n} \Delta_{m_k - \delta_{i,k} - \delta_{j,k}}\right).$$

Since $n \geq 3$, for some k we have

$$m_k - \delta_{i,k} - \delta_{j,k} = m_k.$$

Therefore, by Lemma 9.4, we have

$$\mathbb{T}_a \Phi^{-1}\left(\prod_{k=1}^{n} \Delta_{m_k - \delta_{i,k} - \delta_{j,k}}\right) \cap \mathbb{T}_a W = \mathbb{T}_a \Delta_m.$$

Hence, Δ_m is, locally at a, a smooth component of the intersection

$$\Phi^{-1}\left(\prod_{k=1}^{n} \Delta_{m_k - -\delta_{k,i} - \delta_{k,j}}\right) \cap W.$$

This finishes the proof. ∎

9.11 Geometry of Δ_{m-1}

We recall the following algebraic fact whose variant was proved in Lemma 2.12 of [3].

Fact 4. *Suppose λ is a non-zero number. For every positive integer m, there is a polynomial $P_m(x) = x^m + \alpha_2 x^{m-2} + \cdots + \alpha_m$ such that $P_m(x)^2 - \lambda$ has $m-1$ double roots. Moreover, $\alpha_{2k-1} = 0$ for all k , $\alpha_2 \neq 0$, and $P_m(x)$ is unique up to scaling $\alpha_{2k} \mapsto \xi^k \alpha_{2k}$, where ξ is an m^{th} root of unity.*

We now reprove Lemma 4.1 of [2].

Lemma 9.6. *For any m-dimensional smooth variety W, containing Δ_m, whose tangent space is not contained in the hyperplane $a_{2m} = 0$, we have*

$$W \cap \Delta_{m-1} = \Delta_m \cup \Gamma$$

where Γ is a smooth curve tangent to Δ_m with order m at the origin.

Proof. Recall that the morphism $f\colon T_2 \to T$ is the weighted blow-up of T followed by the base change of order 2. Consider the weighted projective tangent cones of W and Δ_{m-1}, denoted respectively W_E and $(\Delta_{m-1})_E$, inside the exceptional divisor E. Lemma 4 implies that away from the locus $b_2 = 0$, the intersection of W_E and $(\Delta_{m-1})_E$ is a single point G, at least set-theoretically.

From now on, we work on $E_{(b_2)} = \operatorname{Spec} \mathbb{C}[c_3, \ldots, c_{2m}]$ inside T_2. By Lemma 9.3,

$$W_E = \{c_{m+1} = \cdots = c_{2m-1} = 0\} \subset E_{(b_2)}$$

and hence

$$\mathcal{T}_2 \times_E W_E$$

is given by the equation

$$y^2 + (x^m + x^{m-2}z^2 + \cdots + c_m z^m)y + c_{2m}z^{2m} = 0$$

inside $\mathcal{E}_{(b_2)} \times \mathbb{P}(1,1,m)$. Suppose coordinates of G in $E_{(b_2)}$ are $(\lambda_3, \ldots, \lambda_{2m})$, where $\lambda_{2m} \neq 0$. We then have

$$\delta_G = \delta(\Psi_G(x,y)) = P^2(x) - 4\lambda_{2m} = Q^2(x)S(x)$$

where $P(x) = x^m + x^{m-2} + \sum_{i=3}^{m} \lambda_i x^{m-i}$, the polynomial $Q(x)$ is monic of degree $m-1$, with distinct roots, and $S(x)$ is a quadric. We will proceed now to show that intersection $W_E \cap (\Delta_{m-1})_E$ is transverse at G.

We use the isomorphism δ from Lemma 9.1. First, observe that the tangent space to Δ_{m-1} at δ_G is given by polynomials of degree $2m-3$ divisible by Q. The tangent space to $\delta(W_E)$ at δ_G consists of the first-order deformations

$$(P(x) + \epsilon R(x))^2 - 4\lambda_{2m} - 4\epsilon\lambda'_{2m} = \delta_G + \epsilon(2P(x)R(x) - 4\lambda'_{2m}),$$

where $R(x)$ is a polynomial of degree $m-3$.

To prove that the two tangent spaces intersect transversely, we need to show that

$$2P(x)R(x) - 4\lambda'_{2m}$$

is divisible by Q only if $R = 0$ and $\lambda'_{2m} = 0$. This is straightforward. Suppose $P(x)R(x) - 2\lambda'_{2m}$ is divisible by Q. Then

$$P^2R^2 - 4\lambda'_{2m}PR + 4(\lambda'_{2m})^2 \equiv 0 \bmod Q^2$$

$$4\left(\lambda_{2m}R^2 - \lambda'_{2m}PR + (\lambda'_{2m})^2\right) \equiv 0 \bmod Q^2$$

Observing that the left-hand side of the equality above has degree less than $2m-2 = \deg Q^2(x)$, we conclude that

$$\lambda_{2m}R^2 - \lambda'_{2m}PR + (\lambda'_{2m})^2 = 0.$$

This implies that $R = 0$ and $\lambda'_{2m} = 0$.

Denote the strict transform of Γ under f_2 by $\overline{\Gamma}$. We have proved that $G = \overline{\Gamma} \cap E$ is a smooth point of the exceptional divisor $E \subset T_2$. Therefore $\overline{\Gamma}$ is a smooth curve intersecting E transversely at G.

Let $L := \{a_2 = 0\}$ and $H := \{a_{2m} = 0\}$ be the coordinate hyperplanes in T. Then

$$f^*L \cdot \overline{\Gamma} = (\overline{L} + 2E) \cdot \overline{\Gamma} = 2E \cdot \overline{\Gamma} = 2.$$

Using the projection formula and the fact that $\overline{\Gamma}$ is a double cover of Γ, we deduce that $L \cdot \Gamma = 1$. This necessarily implies that Γ is smooth near the origin in T. Similarly, equalities

$$f^*H \cdot \overline{\Gamma} = (\overline{H} + 2mE) \cdot \ \overline{\Gamma} = 2mE \cdot \overline{\Gamma} = 2m$$

imply that $H \cdot \Gamma = m$. We finish the proof by observing that $\Delta_m = W \cap H$.

□

It is enlightening to think of the family

$$\mathcal{Z}_{\overline{\Gamma}} := \mathcal{Z}_2 \times_{T_2} \overline{\Gamma}$$

as a stable reduction of the family

$$\mathcal{Y}_{\Gamma} := \mathcal{Y} \times_T \Gamma.$$

The central fiber of $\mathcal{Z}_{\overline{\Gamma}}$ is the union of the normalization of $\mathcal{Y}_0$ and the $(m-1)$-nodal hyperelliptic tail $(\mathcal{T}_2)_G$, while the generic fiber is an $(m-1)$-nodal curve, by construction. By Observation 5, $\mathcal{Z}_{\overline{\Gamma}} \smallsetminus \mathcal{S}_0$ is locally a product of $\mathcal{T}_0$ and a smooth

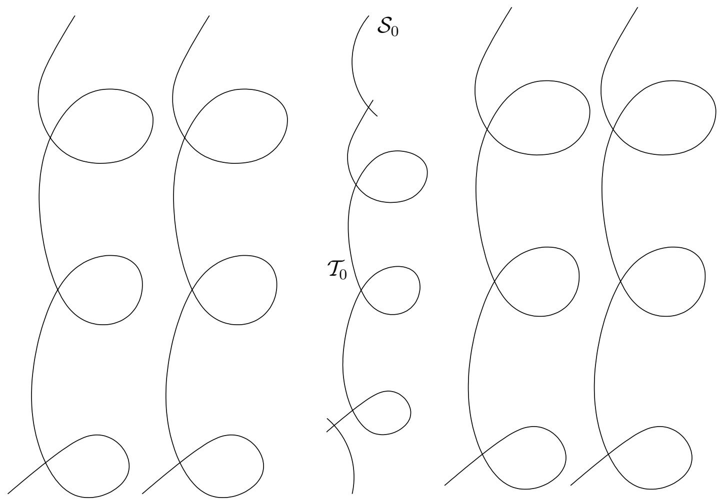

Figure 5: The family $\mathcal{Z}$ over the curve $\overline{\Gamma}$.

curve. It follows that the normalization of $\mathcal{Z}_{\overline{\Gamma}}$ is a family of generically smooth curves with the central fiber being the union of the normalization of $y(y+x^m)=0$ and the rational curve attached at the points lying over the tacnode of $\mathcal{Y}_0$. By the same observation, the total space of the normalized family is smooth. It follows that any family of stable curves of genus g that globalizes $(\mathcal{Z}_{\overline{\Gamma}})^\nu$ intersects the boundary δ in $\overline{\mathcal{M}}_g$ with multiplicity 2 at the point $G \in \overline{\Gamma}$. As $\overline{\Gamma}$ is a degree 2 cover of Γ, ramified at G, we conclude that any family of stable curves of genus g that globalizes $(\mathcal{Y}_\Gamma)^\nu$ intersects the boundary of $\overline{\mathcal{M}}_g$ with multiplicity 1 at the origin. We state this more precisely as the following result

Lemma 9.7. *The central fiber of the family $(\mathcal{Y}_\Gamma)^\nu \to \Gamma$ has a single node (lying over the tacnode in the central fiber of $\mathcal{Y}_\Gamma$). Moreover, the total space of the family is smooth at this node.*

Proof. Consider a relative canonical model $\hat{\mathcal{Z}}$ of the normalization $\mathcal{Z}^\nu_{\overline{\Gamma}}$. It is obtained by a contraction of a (-2)-curve $\mathcal{T}_0^\nu$ in $\mathcal{Z}_{\overline{\Gamma}}^\nu$. Consider also a flat family

$$\hat{\mathcal{Y}} := (\mathcal{Y}_\Gamma)^\nu \times_\Gamma \overline{\Gamma}.$$

By construction, there exists a morphism $\mathcal{Z}_{\overline{\Gamma}}^\nu \to \hat{\mathcal{Y}}$. It descends to a morphism $\hat{\mathcal{Z}} \to \hat{\mathcal{Y}}$. Both $\hat{\mathcal{Z}}$ and $\hat{\mathcal{Y}}$ are flat families of curves over $\overline{\Gamma}$. Moreover, they are isomorphic over a punctured curve $\overline{\Gamma} \setminus 0$ and fibers over 0 are nodal. It follows that $\hat{\mathcal{Z}} \cong \hat{\mathcal{Y}}$. Since $\hat{\mathcal{Z}}$ has an A_1-singularity, we conclude that $\mathcal{Y}_\Gamma^\nu$ is smooth. □

9.12 Geometry of Δ_{m-2}

Let W be as above. Then we have

$$W \cap \Delta_{m-2} = \Delta_m \cup S,$$

where S is residual to Δ_m in the intersection. We would like to understand the geometry of S and the restriction of the miniversal family to S. We do this by passing to T'' and T_2.

If S_E is the exceptional divisor of S in T'', then we have

$$W_E \cap (\Delta_{m-2})_E = (\Delta_m)_E \cup S_E.$$

We study the geometry of the family $\mathcal{T}_S := \mathcal{T} \times_E S_E$ of tails over S_E. First of all, we have the following result that follows from Lemma 9.5.

Lemma 9.8.

$$S_E \cap (\Delta_m)_E = \bigcup_{i=1}^{\lfloor m/2 \rfloor} \Delta\{2i, 2(m-i)\}_E. \tag{26}$$

From the above discussion, we know that a generic fiber of $\mathcal{T}_S \to S_E$ is an

$(m-2)$-nodal hyperelliptic curve of arithmetic genus $m-1$ in the linear series $|\mathcal{O}(2)|$ on $\mathbb{P}(1,1,m)$ and passing through points $[x:y:z]=[1:0:\pm1]$. Hence, the normalization of a generic fiber is an elliptic curve, with two distinguished points. It follows that there is an induced moduli map from the normalization S_E^ν to the coarse moduli space $\overline{\mathcal{M}}_{1,2}$. The map is defined, generically, by sending a point $p \in (S_E)^\nu$ to the normalization of the tail $(\mathcal{T}_S)_p$, marked at the points $[1:0:\pm1]$ of attachment to $\mathcal{S}$. The first order of business is to understand which fibers of $\mathcal{T}_S$ are degenerate, i.e., have geometric genus 0 or less.

Proposition 9.1. *The only degenerate fibers of $\mathcal{T}_S \to S_E$ are fibers over the points*

$$\Delta\{2i, 2(m-i)\}_E \quad \textit{and} \quad (\Delta_{m-1})_E \cap S_E.$$

Proof. The δ-invariant of the singularity $y^2 = x^a$ is $\lfloor a/2 \rfloor$. Therefore the geometric genus of a curve in a stratum

$$\Delta^0\{a_1, \ldots, a_k\}_E \subset E$$

is $m - 1 - \sum_i \lfloor a_i/2 \rfloor$.

By Lemma 9.8, we have

$$S_E \cap (\Delta_m)_E = \bigcup_{i=1}^{\lfloor m/2 \rfloor} \Delta\{2i, 2(m-i)\}_E.$$

On the complement of $(\Delta_m)_E$, we have $c_{2m} \neq 0$. It is therefore enough to analyze the possible combinatorial types of polynomials with equation

$$\Psi_c(x,y) = y^2 + P(x)y + 1 = 0,$$

where

$$P(x) = (x^m + c_2 x^{m-2} + \cdots + c_m).$$

Suppose $\delta(\Psi_c(x, y))$ has combinatorial type $(m_1, \ldots, m_k)$. Then

$$P(x)^2 - 4 = (x - t_1)^{m_1} \cdots (x - t_k)^{m_k}.$$

Differentiating, we obtain

$$2P(x)P'(x) = (x - t_1)^{m_1 - 1} \cdots (x - t_k)^{m_k - 1} Q(x),$$

for some polynomial $Q(X)$. Hence $P'(x)$ is a multiple of

$$(x - t_1)^{m_1 - 1} \cdots (x - t_k)^{m_k - 1}.$$

We conclude that

$$m - 1 \geq \sum_i m_i - k = 2m - k,$$

or, equivalently, $k \geq m + 1$.

Since $\sum_i m_i = 2m$, we can have

$$\sum \lfloor \frac{m_i}{2} \rfloor \geq m - 1$$

only if there are at most two odd numbers among $(m_1, \ldots, m_n)$. As we are interested in points outside Δ_m, there should be precisely two odd numbers, and the only case in which this occurs, under the additional constraint $k \geq m+1$, is when

$$(m_1, \ldots, m_n) = (\underbrace{2, \ldots, 2}_{m-1}, 1, 1).$$

The corresponding point on E is the point

$$G = (\Delta_{m-1})_E \cap S_E,$$

which is unique by the proof of Lemma 9.6. □

We would like now to understand the geometry of $\mathcal{T}_S \to S_E$ around the points $(\Delta_{m-1})_E \cap S_E$ and $W_E \cap S_E$. We have the following

Lemma 9.9. *Let $G = (\Delta_{m-1})_E \cap S_E$. Then S_E has $m-1$ smooth branches around G.*

Proof. We work on $E_{(b_2)} \subset T_2$. The de-homogenization of the equation of $\mathcal{T}_G$ is

$$\Psi_G(x, y) = y^2 + P(x)y + \lambda_{2m} = (x - t_1)^2 \cdots (x - t_{m-1})^2 S(x),$$

where $P(x) = x^m + x^{m-2} + \sum_{i=3}^{m} \lambda_i x^{m-i}$ is a polynomial of degree m and $S(x)$ is a monic quadric.

The tangent cone to $(\Delta_{m-1})_E$ at δ_G is a union of linear spaces of polynomials vanishing at the subset of $m-2$ nodes out of $t_1, \ldots, t_{m-1}$. These linear spaces correspond to deformations preserving all nodes except one. We conclude that $(\Delta_{m-2})_E$ has $m-1$ smooth branches around G. To establish the lemma, it remains to show that W_E intersects all branches transversely.

As we have seen in the proof of the Lemma 9.6, the first order deformations of δ_G in W_E are of the form

$$\Psi_G(x) + \epsilon(2P(x)R(x) - 4\lambda'_{2m})$$

where $R(x)$ is a polynomial of degree $m-3$.

It remains to observe that given a subset $\{t_{i_1}, \ldots, t_{i_{m-2}}\}$ of $\{t_1, \ldots, t_{m-1}\}$, there is a unique, up to scaling, pair $(R(x), \lambda'_{2m})$ such that

$$(x - t_{i_1})(x - t_{i_2}) \cdots (x - t_{i_{m-2}}) \mid 2P(x)R(x) - 4\lambda'_{2m}.$$

□

At last, we discuss the geometry of $\mathcal{T}_S \to S_E$ around a point p of

$$\Delta\{2i, 2(m-i)\}_E.$$

The singularities of the fiber $(\mathcal{T}_S)_p$ are tacnodes $y(y+x^i) = 0$ and $y(y+x^{m-i}) = 0$. We denote $T(0) := \mathrm{Def}(y(y + x^i) = 0)$ and $T(1) := \mathrm{Def}(y(y + x^{m-i}) = 0)$ to be the deformation spaces of these singularities.

By Lemma 9.2, the family $\mathcal{T}_2 \to E$ induces an isomorphism of a neighborhood of p in E with a neighborhood of the origin in $T(0) \times T(1)$. Let $H_0 \subset T(0)$ and $H_1 \subset T(1)$ be the distinguished hyperplanes (see Section 9 for the definition). By Lemma 9.4, locally at p, the variety W_E is identified with a smooth $m-1$ dimensional subvariety V of $T(0) \times T(1)$, and by Lemma 9.4, the tangent space of V does not lie in the union of the preimages of H_0 and H_1.

Therefore S_E is identified with the curve $\Gamma_{i,m-i}$ residual to $\Delta_i \times \Delta_{m-i}$ in the intersection

$$V \cap (\Delta_{i-1} \times \Delta_{m-i-1}) = (\Delta_i \times \Delta_{m-i}) \cup \Gamma_{i,m-i}.$$

This curve is an analog of a curve studied in the Lemma 9.6. It will be studied in the next section.

10 Multiple tacnodes case

10.1 Notations

In this chapter, we study the geometry of the product of the deformation spaces of n tacnodes. Let $\mathfrak{m} = (m_1, \ldots, m_n)$ be a sequence of positive integers. We define

$$M := \prod_{i=1}^{n} m_i \quad \text{and} \quad m := \sum_{i=1}^{n} (m_i - 1) + 1.$$

For $1 \leq i \leq n$, let

$$T(i) \cong \operatorname{Spec} \mathbb{C}[a_{i,2}, \ldots, a_{i,2m_i}]$$

be the versal deformation space of the $(m_i - 1)^{\text{th}}$-order tacnode $y(y + x^{m_i}) = 0$ with the miniversal family over it given by the equation

$$y^2 + (x^{m_i} + a_{i,2}x^{m_i-2} + \cdots + a_{i,m_i})y + a_{i,m+1}x^{m-1} + \cdots + a_{i,2m_i} = 0$$

inside $T(i) \times \operatorname{Spec} \mathbb{C}[x, y]$. We set

$$T = \prod_{i=1}^{n} T(i) = \operatorname{Spec} \mathbb{C}[a_{i,j}]_{1 \leq i \leq n, 2 \leq j \leq 2m_i}.$$

Inside T, we denote the hyperplane $\{a_{i,2m_i} = 0\}$ by H_i. We refer the reader for the definitions of loci $\Delta_{\mathfrak{m}}$, $\Delta_{\mathfrak{m}-\mathbb{1}}$ and $\Delta_{\mathfrak{m}-\mathbb{1}-e_i}$ to Section 9.4.

We denote by $\tilde{\mathcal{Y}}_i$ the pullback to T of the versal family $\mathcal{Y}_i \to T(i)$ via the projection morphism $T \to T(i)$. Set

$$\mathcal{Y} := \bigsqcup_{i=1}^{n} \tilde{\mathcal{Y}}_i$$

to be the disjoint union of the pullbacks of versal families. Clearly, there is a natural morphism $\mathcal{Y} \to T$.

10.2 Alterations of defomation spaces

In analogy with the case of a single tacnode, we first make the base change

$$a_{i,j} = b_{i,j}^{j\frac{M}{m_i}}$$

to arrive at the space $T' = \operatorname{Spec} \mathbb{C}[b_{i,j}]_{i,j}$.

Next, we consider the blow-up of T' at the origin

$$T'' = \operatorname{Bl}_0 T',$$

and denote by $E \subset T''$ the exceptional divisor of the blow-up. In analogy with the single tacnode case, we consider the quotients of T'' by the groups

$$\tilde{\mu}_{a,b} = \prod_{(i,j)\neq(a,b)} \mu_{j\frac{M}{m_i}}.$$

Finally, we denote the quotient $T''//\tilde{\mu}_{a,b}$ by $T_{a,b}$. We single out the quotient $T_{1,2}$ and denote the morphism from $T_{1,2}$ to T by f. By abuse of notation, the exceptional divisor of the morphism

$$f\colon\ T_{1,2} \to T$$

is also denoted E.

We finish this section with the fact generalizing Lemma 9.3.

Lemma 10.1. *Consider any subvariety $W \subset T$, of dimension $m = \sum_{i=1}^{n}(m_i - 1) + 1$, that is smooth at the origin, contains $\Delta_{\mathfrak{m}}$, and whose tangent space is not contained in the union of hyperplanes H_i. Then the exceptional divisor W_E of W*

in T'' is defined inside E by the equations

$$b_{i,j}^{j\frac{M}{m_i}} = 0, \quad 1 \leq i \leq n,\ m_i + 1 \leq j \leq 2m_i - 1; \tag{27}$$

$$b_{i,2m_i}^{2M} = \alpha_i b_{1,2m_1}^{2M}, \quad 2 \leq i \leq n; \tag{28}$$

where α_i are some non-zero complex numbers. Here, $b_{i,j}$ are homogeneous coordinates on E induced by coordinates $b_{i,j}$ on T'.

Proof. The proof is a straightforward generalization of the proof of Lemma 9.3 of Section 9. □

10.3 Geometry of $\Delta_{\mathfrak{m}-\mathfrak{1}}$

Lemma 10.2. *For any $W \subset T$, as in Lemma 10.1, we have*

$$W \cap \Delta_{\mathfrak{m}-\mathfrak{1}} = \Delta_{\mathfrak{m}} \cup \Gamma_{\mathfrak{m}},$$

where $\Gamma_{\mathfrak{m}}$ is a curve intersecting $\Delta_{\mathfrak{m}}$ with multiplicity M at the origin.

Proof. First, we would like to understand how W_E and $(\Delta_{\mathfrak{m}-\mathfrak{1}})_E$ intersect inside $E \subset T''$. By Lemma 10.1, any point of the intersection has homogeneous coordinates $[\lambda_{i,j}]$ satisfying Equations (27)-(28). Moreover, if the point is outside $(\Delta_{\mathfrak{m}})_E$, then by Lemma 4 we have $\lambda_{i,2} \neq 0$.

Take the quotient $T_{1,2}$ of T''. Consider the complement inside $T_{1,2}$ of the strict transform of $\{a_{1,2} = 0\}$ under the morphism f defined in the previous section. The complement is an open affine

$$U = \operatorname{Spec} \mathbb{C}[c_{i,j}]_{i,j},$$

with the morphism $f\colon U \to T$ given by

$$a_{1,2} \mapsto c_{1,2}^{\frac{2M}{m_1}}$$
$$a_{i,j} \mapsto c_{i,j} c_{1,2}^{j\frac{M}{m_i}}, \quad (i,j) \neq (1,2).$$

The equation of $E_U := E \cap U$ is $c_{1,2} = 0$, so that

$$E_U = \operatorname{Spec} \mathbb{C}[c_{i,j}]_{(i,j)\neq(1,2)}.$$

It follows from Equations (27)-(28) that the tangent cone W_E in $E_U \cup U$ is cut out by the equations

$$c_{i,j} = 0, \quad 1 \leq i \leq n, \ m_i + 1 \leq j \leq 2m_i - 1; \tag{29}$$
$$c_{i,2m_i} = \alpha_i c_{1,2m_1}, \quad 2 \leq i \leq n; \tag{30}$$

We conclude that, set theoretically, $W_E \cap (\Delta_{\mathfrak{m}-\mathfrak{1}})_E \cap U$ consists of points whose coordinates $(\lambda_{i,j})_{(i,j)\neq(1,2)}$ on U satisfy the following conditions:

1. λ_{ij}'s satisfy Equations (29)-(30).
2. $\lambda_{1,2m_1} \neq 0$. This simply means that a point is not in $\Delta_{\mathfrak{m}}$.
3. The polynomial

   $$(x^{m_1} + x^{m_1-2} + \lambda_{1,3}x^{m_1-3} \cdots + \lambda_{1,m_1})^2 - 4\lambda_{1,2m_1}$$

   has $m_1 - 1$ double roots.
4. For each $2 \leq i \leq n$, the polynomial

   $$(x^{m_1} + \lambda_{i,2}x^{m_1-2} + \cdots + \lambda_{i,m_i})^2 - 4\lambda_{i,2m_i}$$

has $m_i - 1$ double roots.

The condition (3) together with Lemma 4 uniquely determines numbers $\lambda_{1,j}$, for $3 \leq j \leq 2m_1$. Then Equation (30) determines all numbers $\lambda_{i,2m_i}$, for $2 \leq i \leq n$. Finally, we again invoke Lemma 4 and condition (4) to conclude that each of the sequences $(\lambda_{i,2}, \lambda_{i,3}, \ldots, \lambda_{i,2m_i})$ is determined up to a scaling by an m_i^{th} root of unity.

Summarizing, $W_E \cap (\Delta_{\mathfrak{m}-\mathbb{1}})_E \cap U$ consists of $m_2 \cdots m_n$ points. The tangent cone argument, analogous to that of Lemma 9.3, shows that the intersection is transverse at these points.

Denote by $\overline{\Gamma_{\mathfrak{m}}}$ the strict transform of $\Gamma_{\mathfrak{m}}$ under f. We conclude that

$$\overline{\Gamma_{\mathfrak{m}}} \cdot E = m_2 \cdots m_n.$$

Observe that

$$f^* \Delta_{\mathfrak{m}} = \overline{\Delta_{\mathfrak{m}}} + 2m_1 m_2 \cdots m_n E.$$

An application of the projection formula now gives

$$\begin{aligned}
\deg(f)\,(\Gamma_{\mathfrak{m}} \cdot \Delta_{\mathfrak{m}}) &= f(\overline{\Gamma_{\mathfrak{m}}}) \cdot \Delta_{\mathfrak{m}} \\
&= \overline{\Gamma_{\mathfrak{m}}} \cdot f^* \Delta_{\mathfrak{m}} \\
&= \overline{\Gamma_{\mathfrak{m}}} \cdot (\overline{\Delta_{\mathfrak{m}}} + 2m_1 m_2 \cdots m_n E) \\
&= 2m_1 m_2 \cdots m_n (\overline{\Gamma_{\mathfrak{m}}} \cdot E) \\
&= 2m_1 (m_2 \cdots m_n)^2
\end{aligned}$$

Since $\deg(f) = 2m_2 \cdots m_n$, we establish that

$$\Gamma_{\mathfrak{m}} \cdot \Delta_{\mathfrak{m}} = \prod_{i=1}^{n} m_i = M. \tag{31}$$

□

We observe that a subvariety $W \subset T$ satisfying the conditions of Lemma 10.1 maps smoothly under the projection

$$\mathrm{proj}_k \colon T \to T(k)$$

onto a subvariety $W_k := \mathrm{proj}_k(W)$ of $T(k)$ satisfying the conditions of Lemma 9.6. Therefore, we can define the curve Γ_k inside $T(k)$ to be the residual to Δ_{k,m_k} in the intersection

$$W_k \cap \Delta_{k,m_k-1}.$$

We arrive at the following conclusion.

Lemma 10.3. *The projection*

$$\mathrm{proj}_k \colon \Gamma_{\mathfrak{m}} \to \Gamma_k$$

has degree $\prod_{i \neq k} m_i$.

Proof. Clearly, $(\mathrm{proj}_k)_*[\Gamma_{\mathfrak{m}}] = \deg(\mathrm{proj}_k)[\Gamma_k]$. Note that $(\mathrm{proj}_k)^*\Delta_{i,m_i} = \Delta_{\mathfrak{m}}$. Now using $\Gamma_{\mathfrak{m}} \cdot \Delta_{\mathfrak{m}} = M$, proved in Lemma 10.2, and applying the projection formula, we obtain the needed result. □

10.4 Alterations of versal families

Let $\mathcal{Y} \subset T \times \bigsqcup_{i=1}^{n} \operatorname{Spec} \mathbb{C}[x, y]$ be the disjoint union of the pullbacks to T of the miniversal families $\mathcal{Y}_i \to T(i)$. Then $\mathcal{Y}$ is given by the equations

$$\Psi(x, y) = y^2 + (x^{m_i} + a_{i,2}x^{m_i - 2} + \cdots + a_{i,m_i})y + a_{i,m+1}x^{m-1} + \cdots + a_{i,2m_i} = 0$$

inside $\mathcal{T} \times \bigsqcup_{i=1}^{n} \operatorname{Spec} \mathbb{C}[x, y]$. Let $\mathcal{Y}''$ be the pullback of $\mathcal{Y}$ to T'':

$$\mathcal{Y}'' \subset T'' \times \bigsqcup_{i=1}^{n} \operatorname{Spec} \mathbb{C}[x, y].$$

Define the ideal sheaf $\mathcal{I} = \left(\left(I_E^{M/m_i}, x\right)^{m_i}, y\right)$ on the i^{th} copy of $\mathcal{T}'' \times \operatorname{Spec} \mathbb{C}[x, y]$. Set

$$\mathcal{Z} := \operatorname{Bl}_{\mathcal{I}} \mathcal{Y}''$$

and

$$F\colon\ \mathcal{Z} \to T''.$$

As in Section 9,

$$\mathcal{X} := \operatorname{Bl}_{\mathcal{I}} \left(T'' \times \bigsqcup_{i=1}^{n} \operatorname{Spec} \mathbb{C}[x, y] \right) \to T''$$

is a family of surfaces with a fiber over a point in $T'' \smallsetminus E$ being a disjoint union of n copies of the affine plane, and a fiber over a point in the exceptional divisor E being the disjoint union of n reducible surfaces with components $\operatorname{Bl}_{(x^{m_i}, y)} \operatorname{Spec} \mathbb{C}[x, y]$ and $\mathbb{P}(1, \frac{M}{m_i}, M) = \mathbb{P}(1, 1, m_i) = \operatorname{Proj} \mathbb{C}[x, y, z]$, for $1 \leq i \leq n$. Here, z stands for a local generator of I_E, and $\mathbb{C}[x, y, z]$ is graded with $\deg x = \frac{M}{m_i}$, $\deg y = M$ and $\deg z = 1$.

Consider a distinguished open affine $D_{(b_{1,2})}T'' = \operatorname{Spec}\mathbb{C}[c_{i,j}]_{i,j}$ in T'', where

$$\begin{aligned} c_{1,2} &= b_{1,2}, \\ c_{i,j} &= \frac{b_{i,j}}{b_{1,2}},\ (i,j) \neq (1,2). \end{aligned}$$

The exceptional divisor E on $D_{(b_{1,2})}T''$ is given by $b_{1,2} = 0$ and the restriction of $\mathcal{Z}$ to E is

$$\mathcal{Z}_E := \mathcal{Z} \times_{T''} E = \mathcal{S} \cup \mathcal{T};$$

where $\mathcal{S}$ is

$$\bigsqcup_{i=1}^{n} \left\{ \frac{y}{x^{m_i}} \left(\frac{y}{x^{m_i}} + 1 \right) = 0 \right\} \subset E \times \bigsqcup_{i=1}^{n} \mathrm{Bl}_{(x^{m_i},y)} \operatorname{Spec}\mathbb{C}[x,y]$$

and $\mathcal{T} = \bigsqcup_{i=1}^{n} \mathcal{T}(i)$; where $\mathcal{T}(1)$ is given by

$$\left\{ \Psi\left(1, c_{1,3}^{3M/m_1}, \ldots, c_{1,2m_1}^{2M}\right)(x, y, z^{M/m_1}) \right\} \subset E \times \mathbb{P}(1, M/m_1, M),$$

and for $2 \leq i \leq n$, $\mathcal{T}(i)$ is given by

$$\left\{ \Psi\left(c_{i,2}^{2M/m_i}, c_{i,3}^{3M/m_i}, \ldots, c_{1,2m_i}^{2M}\right)(x, y, z^{M/m_i}) \right\} \subset E \times \mathbb{P}(1, M/m_i, M).$$

We have a result analogous to Observation 5. The proof is identical.

Observation 6. *Locally on the target, the family*

$$(\mathcal{Z} \setminus \mathcal{S}) \to T''$$

is a product of a smooth curve and the family $(\mathcal{T} \setminus (\mathcal{T} \cap \mathcal{S})) \to E$.

10.5 The geometry of $\Delta_{\mathfrak{m}-\mathbb{1}-e_i}$

We consider the intersection of a subvariety $W \subset T$, satisfying the conditions of Lemma 10.1, with $\Delta_{\mathfrak{m}-\mathbb{1}-e_1}$ inside T. We have

$$W \cap \Delta_{\mathfrak{m}-\mathbb{1}-e_1} = \Delta_{\mathfrak{m}} \cup \Theta,$$

where Θ is residual to $\Delta_{\mathfrak{m}}$ in the intersection. Generalizing the arguments of Section 9, we study the geometry of Θ in what follows.

We denote the strict transform of Θ in T'' by $\overline{\Theta}$ and the exceptional divisor of $\overline{\Theta}$ in T'' by Θ_E.

Lemma 10.4. *We have*

$$\Theta_E \cap (\Delta_{\mathfrak{m}})_E = \bigcup_{a=1}^{\lfloor m_1/2 \rfloor} (\Delta\{2a, 2(m_1 - a)\} \times 0 \times \cdots \times 0)_E.$$

Proof. This follows from the analogous statement for the single tacnode, proved in Lemma 9.8. □

Remark 15. It follows that Θ_E is a curve in E. Hence Θ has pure dimension 2.

We now work on the quotient $T_{1,2}$ of T'' by $\tilde{\mu}_{(1,2)}$. Recall that there is an alteration $f\colon T_{1,2} \to T''$ and $\mathcal{Z}_{1,2} = \mathcal{Z}//\tilde{\mu}_{(1,2)}$ is a family of curves over $T_{1,2}$.

The quotient by $\tilde{\mu}_{(1,2)}$ of the distinguished open affine $D_{(b_{1,2})}T''$, say V, is isomorphic to $\mathrm{Spec}[c_{i,j}]_{i,j}$. The map from V to T is given by

$$a_{1,2} \mapsto c_{1,2}^{\frac{2M}{m_1}},$$
$$a_{i,j} \mapsto (c_{1,2})^{j\frac{M}{m_i}} c_{i,j},\ (i,j) \neq (1,2).$$

The quotient $\mathcal{T}(1)_{1,2}$ of $\mathcal{T}(1)$ is given by the equation

$$\Psi(1, c_{1,3}, \ldots, c_{1,2m_1})(x, y, z) = 0. \tag{32}$$

inside $V \times \mathbb{P}(1, 1, m_1)$ and, for $2 \leq i \leq n$, the quotient $\mathcal{T}(i)_{1,2}$ of $\mathcal{T}(i)$ is given by the equation

$$\Psi(c_{i,2}, c_{i,3}, \ldots, c_{i,2m_i})(x, y, z) = 0. \tag{33}$$

inside $V \times \mathbb{P}(1, 1, m_i)$. The total family of tails is

$$\mathcal{T}_{1,2} := \bigsqcup \mathcal{T}(i)_{1,2}.$$

Note that at any point $p \in (\Delta\{2a, 2(m_1-a)\} \times 0 \times \cdots \times 0)_E$, the fiber $(\mathcal{T}(1)_{1,2})_p$ has two tacnodes: of order a and $m_1 - a$, respectively. The fiber $(\mathcal{T}(i)_{1,2})_p$ has a tacnode $y(y+x_i^m) = 0$. We denote the deformation spaces of tacnodes on $(\mathcal{T}(1)_{1,2})_p$ by $D(0) = \mathrm{Def}(y(y + x^a) = 0)$ and $D(1) = \mathrm{Def}(y(y + x^{m_1-a}) = 0)$. For $2 \leq i \leq n$, we also denote the deformation space of the tacnode on $(\mathcal{T}(i)_{1,2})_p$ by $D(i)$. By Equation (33), $D(i)$ is naturally isomorphic to $\mathrm{Spec}\, \mathbb{C}[c_{i,j}]_j$, for $2 \leq i \leq n$. The isomorphism is induced by the family $\mathcal{T}(i)_{1,2} \to \mathrm{Spec}\, \mathbb{C}[c_{i,j}]_j$.

By Lemma 9.2, the family $\mathcal{T}(1)_{1,2}$ induces an isomorphism between a neighborhood of the point p in $\mathrm{Spec}\, \mathbb{C}[c_{1,j}]_{3 \leq j \leq 2m_1}$ and a neighborhood of the origin in $D(0) \times D(1)$. We conclude that at the point $p \in (\Delta\{2a, 2(m_1 - a)\} \times 0 \times \cdots \times 0)_E$, the family

$$\mathcal{T}_{1,2} \to E_{(b_{1,2})}$$

induces a local analytic isomorphism

$$\Phi : E_{(b_{1,2})} \to \prod_{i=0}^{n} D(i),$$

such that the map from $E_{(b_{1,2})}$ to the product of the versal deformation spaces of the singularities of $(\mathcal{T}_{1,2})_p$. By a slight reformulation of Lemma 9.4, the image $\Phi(W_E)$ satisfies the conditions of Lemma 10.1.

We can now state the result that we will use in the next section.

Lemma 10.5. *Under the isomorphism Φ, the exceptional divisor Θ_E of Θ is identified with the curve $\Gamma_{a,m_1-a,m_2,\ldots,m_n}$, of Lemma 10.2, inside*

$$\prod_{i=0}^{n} D(i).$$

11 Proof of the Main Theorem

Recall from Section 5.1 that $S^{D,\delta}(\alpha,\beta) := S^{D,\delta}(\alpha,\beta)^{\Gamma}$ is the surface section of the generalized Severi variety of divisors $V^{D,\delta}(\alpha,\beta)$ formed by intersecting with the linear space $H_{\{p_1,\ldots,p_{\Upsilon-2}\}}$. Here, $\Upsilon = \dim V^{D,\delta}(\alpha,\beta)$ and $\Gamma = \{p_1,\ldots,p_{\Upsilon}\}$ is the set of Υ generic points on S.

For a generic $q \in L$, we considered the special curve section

$$H_q := S^{D,\delta}(\alpha,\beta) \cap H_q$$

and established a linear equivalence of Cartier divisors on $S^{D,\delta}(\alpha,\beta)$:

$$C^{D,\delta}(\alpha,\beta)^{\Gamma} = S^{D,\delta}(\alpha,\beta) \cap H_{p_{\Upsilon-1}} \sim S^{D,\delta}(\alpha,\beta) \cap H_q. \tag{34}$$

By the previous discussion, the special curve section H_q breaks into a union of Type I and Type II components, so that we obtain a linear equivalence of Weil divisors on $S^{D,\delta}(\alpha,\beta)$:

$$H_q \sim \sum_k kC^{D,\delta}(\alpha+e_k,\beta-e_k) + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}C_L^{D-L,\delta'}(\alpha',\beta'). \tag{35}$$

In this section we prove Main Theorem 1. To do this we define a blow-up at every point of indeterminacy on $S^{D,\delta}(\alpha,\beta)$ that regularizes the moduli map $j\colon S^{D,\delta}(\alpha,\beta) \dashrightarrow M^{D,g}(S)$. The composition of the blow-ups at all points of indeterminacy is denoted

$$\pi\colon \overline{S^{D,\delta}(\alpha,\beta)} \to S^{D,\delta}(\alpha,\beta).$$

Next, we compute the pullback of the Cartier divisor H_q under π. Namely, we say with what multiplicity the exceptional divisors of π appear in the pullback

$\pi^*(H_q)$. Finally, we compute the degree of the λ-class on the exceptional divisors of the blow-up π. We emphasize that, since blow-ups are made at disjoint centers, all these can be done locally at each point of indeterminacy.

Summing over all points of indeterminacy, we obtain the equality between the degree $L^{D,\delta}(\alpha,\beta)$ of λ on $C^{D,\delta}(\alpha,\beta)$ and the degree of λ on $\pi^*(H_q)$.

Write

$$\pi^*(H_q) = \overline{H_q} + F_\pi,$$

where $\overline{H_q}$ is the strict transform of H_q and F_π is a linear combination of the exceptional divisors of π. Then by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}\overline{H_q}\cdot\lambda &= \lambda\cdot\left(\sum_k kC^{D,\delta}(\alpha+e_k,\beta-e_k) + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}C_L^{D-L,\delta'}(\alpha',\beta')\right)\\ &= \sum_k kL^{D,\delta}(\alpha+e_k,\beta-e_k) + \sum I^{\beta'-\beta}\binom{\alpha}{\alpha'}\binom{\beta'}{\beta}L^{D-L,\delta'}(\alpha',\beta').\end{aligned}$$

It remains to compute $F_\pi\cdot\lambda$. We do this in the next two sections.

11.1 Points of indeterminacy

Consider a point of indeterminacy

$$[X]\in C_L^{D-L,\delta'}(\alpha',\beta')^\Gamma \cap H_{p_{\Upsilon-1}},$$

where $|\beta'-\beta|+\delta-\delta' = D\cdot L - L^2 - 1$. By Theorem 2, all points of indeterminacy are of this form.

As described in Section 8.2, the surface $S^{D,\delta}(\alpha,\beta)$ has several analytic components at $[X]$. Each component is determined by the choice of $|\beta|$ tacnodes

$$\{s^*_{i,j}\}_{1\le j\le\beta_i}$$

of X that are the limits of free points of tangency in the nearby fibers. The remaining $n := |\beta' - \beta|$ tacnodes $t_1, \dots t_n$ are designated "new". Suppose t_i is an order $(m_i - 1)$ tacnode, given by the equation $y(y + x^{m_i}) = 0$, and $T(i) := \mathrm{Def}(X, t_i)$ is its versal deformation space with $\mathcal{Y}_i \to T(i)$ being the versal family.

Consider the analytic component S of $S^{D,\delta}(\alpha, \beta)$ that lies in the image of the local relaxed Severi variety $\mathrm{Def}(f; \Omega; \{s^*_{i,j}\})^\Gamma$ defined in Section 8.2. By Fact 3, the classifying map

$$\phi\colon\ S \to \prod_{i=1}^{n} T(i)$$

maps S locally isomorphically to the locus

$$S_{\mathfrak{m}} := \bigcup_{i=1}^{n} S_i,$$

where

$$S_i := \overline{W \cap \Delta_{\mathfrak{m}-\mathbb{1}-e_i} \setminus \Delta_{\mathfrak{m}}}.$$

We work now on the analytic branch $\Theta \cong S_1$ of S. In words, Θ is the closure of the deformations of X such that t_1 deforms to $m_1 - 2$ nodes in the nearby fibers, and t_i deforms to $m_i - 1$ nodes, for $i \geq 2$. Note that under the isomorphism ϕ, the Cartier divisor H_q on S is the pullback of the Cartier divisor $\Delta_{\mathfrak{m}}$ on $\mathrm{Def}(f; \Omega; \{s^*_{i,j}\})^\Gamma$.

Let $\mathcal{Y}$ be the restriction to Θ of the family $\mathcal{Y}^{D,\delta}(\alpha, \beta)$. In the neighborhood of the point t_i on X, the family $\mathcal{Y}$ is the pullback of the family $\mathcal{Y}_i \to T(i)$ under the projection $\Theta \to T(i)$.

11.2 The computation

Recall, that we have constructed the alteration

$$f\colon\ T_{1,2} \to \prod_{i=1}^{n} T(i),$$

with the exceptional divisor $E \subset T_{1,2}$, and the family $\mathcal{Z}_{1,2} \to T_{1,2}$.

Let $f\colon\ \overline{\Theta} \to \Theta$, be the induced alteration of Θ. Denote the exceptional divisor by Θ_E, and the restriction of the family $\mathcal{Z}_{1,2}$ to $\overline{\Theta}$ by $\mathcal{Z}_{\Theta}$.

Proposition 11.1. *The family*

$$\mathcal{Z}_{\Theta} \to \overline{\Theta}$$

has no points of indeterminacy along Θ_E.

Proof. Recall that, by Observation 6 of Section 10, locally on the target the open family

$$(\mathcal{Z}_{1,2} \smallsetminus \mathcal{S}_{1,2}) \to T_{1,2}$$

is a product of a smooth curve and the family $(\mathcal{T}_{1,2} \smallsetminus (\mathcal{T}_{1,2} \cap \mathcal{S}_{1,2})) \to E$. It follows that, locally, the surface $\overline{\Theta}$ is a product of a smooth curve and Θ_E. Since points of indeterminacy occur only in codimension 2, they cannot occur on locally trivial families of curves.

∎

The alteration f has degree

$$2m_2 \cdots m_n.$$

Also, by the construction of f

$$f^*(\Delta_{\mathfrak{m}} \cap \Theta) = \overline{\Delta_{\mathfrak{m}}} + 2m_1 m_2 \cdots m_n \Theta_E.$$

The coefficient of Θ_E is $2m_1m_2\cdots m_n$ because $\Delta_{\mathfrak{m}}$ is given by the equation

$$a_{i,2m_i}=0$$

(for any i) and the pullback of $a_{i,2m_i}=0$ to $T_{1,2}$ vanishes to the order $2m_1m_2\cdots m_n$ along E. Therefore, the contribution of S to $F_\pi\cdot\lambda$ is

$$m_1(\Theta_E\cdot\lambda).$$

It remains to compute $\Theta_E\cdot\lambda$. Recall that the family $\mathcal{Z}_{1,2}$ over Θ_E is a quotient of the union of two components. One component, $\mathcal{S}$, is an isotrivial family, and hence does not contribute to $\Theta_E\cdot\lambda$. The other component is the family of tails $\mathcal{T}$, restricted to Θ_E. Its quotient is $\mathcal{T}_{1,2}$.

The family $\mathcal{T}(1)_{1,2}\to\Theta_E$ is a family of hyperelliptic 2-pointed curves of arithmetic genus m_1-1, with generically m_1-2 nodes. Consider the family

$$\mathfrak{T}:=\mathcal{T}_{1,2}\times_{\Theta_E}(\Theta_E)^\nu\to(\Theta_E)^\nu.$$

It induces a regular map from $(\Theta_E)^\nu$ to $\overline{\mathcal{M}}_{1,2}$. By Proposition 9.1, the image curve intersects the boundary δ_0 in $\overline{\mathcal{M}}_{1,2}$ at the points of $(\Theta_E)^\nu$ lying over

$$\Theta_E\cap(\Delta_{\mathfrak{m}-\mathfrak{1}})_E$$

and

$$\Theta_E\cap(\Delta_{\mathfrak{m}})_E.$$

At the points of $\Theta_E\cap(\Delta_{\mathfrak{m}-\mathfrak{1}})_E$, of which there are exactly $m_2\ldots m_n$ by the proof of Lemma 10.2, the curve Θ_E has m_1-1 smooth branches. Each branch intersects the boundary in $\overline{\mathcal{M}}_{1,2}$ transversely.

To calculate the multiplicity with which $(\Theta_E)^\nu$ intersects the boundary at the points $\{p_1, \dots, p_\kappa\}$ lying over the point

$$p \in (\Delta\{2a, 2(m_1 - a)\} \times 0 \times \cdots \times 0)_E,$$

we need to understand the geometry of the family $\mathfrak{T}$ around the tacnodes in the fibers $\mathfrak{T}_{p_i}$.

By Lemma 10.5, the curve Θ_E is identified, in a neighborhood of p, with the curve

$$\Gamma_{a,m_1-a,m_2,\dots,m_n}$$

inside $\prod_{i=0}^n D(i)$. Therefore, by Corollary 10.3, the curve $(\Theta_E)^\nu$ maps with degree $(m_1 - a)m_2 \dots m_n$ onto the curve Γ_a inside the deformation space $D(0)$ of the tacnode $y(y + x^a) = 0$.

Let r_i, for $1 \leq i \leq \kappa$, be the ramification index of the map

$$(\Theta_E)^\nu \to \Gamma_a$$

at the point p_i. Then by Lemma 9.7 the surface $\mathfrak{T}$ has an A_{r_i-1}-singularity at the $(a-1)^{\text{th}}$-order tacnode in the fiber $\mathfrak{T}_{p_i}$. Hence, this $(a-1)^{\text{th}}$-order tacnode contributes r_i to the intersection multiplicity of $(S_E)^\nu$ with δ_0 at the point p_i.

Remembering that

$$\sum_{i=1}^{\kappa} r_i = (m_1 - a)m_2 \dots m_n,$$

and applying the same argument to the other tacnode, we conclude that the total intersection number of $(\Theta_E)^\nu$ with the boundary δ_0 at points $\{p_1, \dots, p_\kappa\}$ lying over p is

$$(m_1 - a)m_2 \cdots m_n + am_2 \cdots m_n = \prod_{i=1}^{n} m_i.$$

Noting that $\Theta_E \cap (\Delta_{\mathfrak{m}})_E$ has exactly $m_1 - 1$ points, we finally calculate the contributions of $\Theta_E \cap (\Delta_{\mathfrak{m}-\mathfrak{1}})_E$ and $\Theta_E \cap (\Delta_{\mathfrak{m}})_E$ to the intersection number of $(\Theta_E)^\nu$ with the boundary divisor δ_0 in $\overline{\mathcal{M}}_{1,2}$ is

$$\begin{aligned}(\Theta_E)^\nu \cdot \delta_0 &= (m_1 - 1)(m_1 m_2 \cdots m_n) + (m_2 \cdots m_n)(m_1 - 1)\\ &= (m_1^2 - 1)(m_2 \cdots m_n).\end{aligned}$$

It follows from Mumford's formula (2) that

$$(\Theta_E)^\nu \cdot \lambda = (m_1^2 - 1)(m_2 \cdots m_n)/12. \tag{36}$$

Performing the same calculation for the surfaces $\Theta \cong S_i$, residual to $\Delta_{\mathfrak{m}}$ in the intersection $\Delta_{\mathfrak{m}-\mathfrak{1}-e_i} \cap W$, we arrive at the contribution to $F_\pi \cdot \lambda$ of the exceptional divisor lying over the the analytic component S of $S^{D,\delta}(\alpha, \beta)$. The contribution is

$$\frac{1}{12}\left(\prod_{i=1}^{n} m_i\right)\left(\sum_{i=1}^{n}(m_i^2 - 1)\right). \tag{37}$$

By Section 8.2, at a point of indeterminacy

$$[X] \in C_L^{D-L,\delta'}(\alpha', \beta')^\Gamma \cap H_{p\Upsilon_{-1}},$$

there are $\binom{\beta'}{\beta}(\beta'_k - \beta_k)$ analytic components such that the tacnode of order $(m_k - 1)$ is responsible for indeterminacy of the moduli map, for every positive integer k.

Recalling that there are

$$\binom{\alpha}{\alpha'} N^{D-L,\delta'}(\alpha', \beta')$$

points of indeterminacy corresponding to the triple $(\delta', \alpha', \beta')$ with

$$|\beta' - \beta| + \delta - \delta' = D \cdot L - L^2 - 1,$$

we conclude that

$$F_\pi \cdot \lambda = \frac{1}{12} \sum I^{\beta' - \beta} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} \cdot \left(\sum_k (\beta'_k - \beta_k)(k^2 - 1) \right) \cdot N^{D-L, \delta'}(\alpha', \beta').$$

The proof of the Main Theorem is finished.

12 Enumerative geometry of plane curves

Let $S = \mathbb{P}^2$ and $|D| = |\mathcal{O}_{\mathbb{P}^2}(d)|$. For $g = \binom{d-1}{2} - \delta$, consider the Severi variety $V_{d,g} := V_{con}^{D,\delta}$ of irreducible plane curves of degree d and geometric genus g. Let $\pi\colon \mathcal{C}_{d,g} \to V_{d,g}$ be the universal family and $\eta\colon \mathcal{C}_{d,g} \to \mathbb{P}^2$ be the natural morphism from the universal family to $\mathbb{P}^2$. Let $\omega := \omega_\pi$ be the relative dualizing sheaf of π and let $D = \eta^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Define three standard classes on $V_{d,g}$ by

$$A = \pi_*(D^2), \qquad B = \pi_*(\omega \cdot D), \qquad C = \pi_*(\omega^2).$$

Define Δ to be the closure of the locus of $(\delta+1)$-nodal curves with at most two irreducible components.

Diaz and Harris have computed [7] a great number of geometrically meaningful divisors on $V_{d,g}$ in terms of A, B, C and Δ. In particular, recall the definitions of the following divisors:

1. CU – the closure of the locus of curves with a cusp and $\delta - 1$ nodes,
2. TN – the closure of the locus of curves with a tacnode and $\delta - 2$ nodes,
3. TR – the closure of the locus of curves with a triple point and $\delta - 3$ nodes,
4. TL – the locus of curves in $V_{d,g}$ tangent to a given line L.

In [7] the following formulas are established:

$$CU = 3A + 3B + C - \Delta, \tag{38}$$

$$TN = (3(d-3) + 2g - 2)A + (d-9)B - \frac{5}{2}C + \frac{3}{2}\Delta, \tag{39}$$

$$TR = \left(\frac{d^2 - 6d + 8}{2} - g + 1\right)A - \frac{d-6}{2}B + \frac{2}{3}C - \frac{1}{3}\Delta, \tag{40}$$

$$TL = A + B. \tag{41}$$

Let now $C^{d,\delta}$ be the curve section of the Severi variety $V_{d,g}$ as in Definition 3.8. Recall that $\dim V_{d,g} = 3d + g - 1$ and so $C^{d,\delta}$ is the variety of curves in $V_{d,g}$ passing through $3d + g - 2$ general points in the plane. Define numbers

$$N^{d,\delta} := C^{d,\delta} \cdot A,$$

$$B^{d,\delta} := C^{d,\delta} \cdot B.$$

We observe that $N^{d,\delta} = N^{D,\delta}_{con}$ – the degree of $V_{d,g}$, and so can be computed by the recursion (7) of Section 5. From Equation (41),

$$N^{d,\delta} + B^{d,\delta} = C^{d,\delta} \cdot TL.$$

Observe that

$$C^{d,\delta} \cdot TL = \deg V^{D,\delta}_{con}(0, (d - 2, 1)) = N^{D,\delta}_{con}(0, (d - 2, 1)),$$

and so can be computed by the recursion (7).

The numbers $C^{d,\delta} \cdot C$ seem to be not known previously. By Mumford's formula,

$$C = 12\lambda - \Delta$$

and hence the intersection $C^{d,\delta} \cdot C$ is computed in terms of $L^{d,\delta} := C^{d,\delta} \cdot \lambda$ and intersections of $C^{d,\delta}$ with the boundary divisors Δ.

Numbers $L^{d,\delta}$ are computed recursively using Main Theorem 1 and so intersection of $C^{d,\delta}$ with all the standard divisor classes are computable. Thus we find solutions to a large class of codimension one enumerative problems on $V_{d,g}$. For example, the intersection number of $C^{d,\delta}$ with the divisor CU is exactly the number of irreducible plane curves of degree d with a cusp, $\delta - 1$ nodes and passing through $3d + g - 2$ general points.

Similarly, we can compute the number of irreducible plane curves of degree d with a tacnode or a triple point, and nodes as the only other singularities, passing through an appropriate number of general points in $\mathbb{P}^2$.

Example 1. We collect some numerical invariants of curve sections of Severi varieties of *irreducible* genus 2 plane curves that were obtained using the recursion of Theorem 1.

In Table 1, every row corresponds to a curve section $C^{d,\delta}$ of the Severi variety $V_{con}^{D,\delta}$ of irreducible degree d plane curves of genus 2. In particular, we always have $\delta = \binom{d-1}{2} - 2$. In calculations, we have used that

$$C^{d,\delta} \cdot \delta_0 = (\delta + 1) N^{d,\delta+1}.$$

The intersections with the class δ_1 were obtained from other Severi degrees using some simple combinatorics.

Table 1: Some genus 2 invariants

d	δ	λ	δ_0	δ_1
4	1	45	$450 = 2 \times 225$	0
5	4	43,596	$435,960 = 5 \times 87192$	0
6	8	51735474	$516,917,160 = 9 \times 57435240$	218790

Note that, as expected, the following relation holds in every case:

$$\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1.$$

Example 2. A simple induction argument involving the recursion of Theorem 1 shows that

$$L^{d,1} = \frac{3}{2}(d-1)(d-2)(d-3)(d+1),$$
$$L^{d,2} = \frac{1}{4}(9d^6 - 63d^5 + 66d^4 + 333d^3 - 553d^2 - 480d + 828).$$

Similarly, we have

$$B^{d,1} = 3(d-1)(2d^2 - 5d + 1),$$
$$N^{d,2} = \frac{1}{2}(9d^4 - 36d^3 + 12d^2 + 81d - 33).$$

Using Equation (38) and the equality $\Delta \cdot C^{d,1} = 2N^{d,2}$, we recover the number of cuspidal curves in $C^{d,1}$:

$$C^{d,\delta} \cdot CU = 3N^{d,1} + 3B^{d,1} + 12L^{d,1} - 4N^{d,2} = 12(d-1)(d-2).$$

The right-hand side is the classical formula for the degree of the cuspidal locus in the variety $\mathbb{P}^{\binom{d+2}{2}-1}$ of degree d plane curves.

The numerical computations suggest that, for a fixed δ, the function $L^{d,\delta}$ is a polynomial of degree $2\delta + 2$ in d, with the leading term $\frac{3^\delta}{2\delta!}$. This should be compared with the Göttsche's conjecture [13] that states that $N^{d,\delta}$ is a polynomial of degree 2δ in d. The conjecture was proved for $\delta \leq 8$ by Kleiman and Piene in [20].

In [19, Theorem (1.2)], Kleiman and Piene enumerate curves with certain singularities in a sufficiently ample linear series on an arbitrary surface. In particular, their formulas compute the number of degree d plane curves with a single triple point and δ nodes, where $\delta \leq 3$, passing through an appropriate number of general points. The postulated number is a polynomial of degree $2(\delta + 1)$ in d. Our methods allow us to compute the number of plane curves with a triple point and an arbitrary number of nodes. Unfortunately, we cannot produce closed form formulas that work for every δ. We ran computations, with a help of a computer, for $d \leq 13$ and $\delta \leq 3$, and our numbers agree with those of [19].

In a different direction, for $g = 0, 1, 2, 3$, the recursions for the codimension one "characteristic" numbers of plane curves were given by Vakil [30]. We note that our computations agree with those of Vakil, presented in the table on page 19 of loc. cit.

13 Slopes of effective divisors on $\overline{M}_g$

We keep the notations of the previous section. Also, we refer the reader to Section 2.4 for the background information on the cone of effective divisors of $\overline{M}_g$ and for necessary definitions.

Define a *slope* of a curve $C \subset \overline{M}_g$ by

$$s(C) := \frac{C \cdot \delta}{C \cdot \lambda} \ .$$

If C is a moving curve in $\overline{M}_g$, then the slope of C gives a lower bound on the slope s_g of effective divisors on $\overline{M}_g$ as described in Section 2.5.

For $\delta = \binom{d-1}{2} - g$, consider the curve section $C^{d,\delta}$ of the Severi variety $V_{d,g}$. By the theorem of Harris [15], $V_{d,g}$ is irreducible. It follows that $C^{d,\delta}$ is also irreducible.

When the Brill-Noether number $\rho(g, 2, d) = 3d - 2g - 6$ is non-negative, there is a dominant rational map

$$j\colon\ V_{d,g} \dashrightarrow \overline{M}_g.$$

Therefore the image of $C^{d,\delta}$ is a moving curve inside $\overline{M}_g$. Recall that

$$L^{d,\delta} = j_*(C^{d,\delta}) \cdot \lambda.$$

The fraction

$$\frac{C^{d,\delta} \cdot \delta}{L^{d,\delta}}$$

provides a lower bound on the slopes of effective divisors on $\overline{M}_g$.

Table 2: Lower bounds on the slope of effective divisors on $\overline{M}_g$

g	d	δ	$3d-2g-6$	$s_g \geq$
2	4	0	1	10
3	4	0	0	9
4	5	2	1	8.45
5	5	1	-1	8.16
6	6	4	0	7.76
7	7	8	1	7.37
8	7	7	-1	7.27
9	8	12	0	6.93
10	9	18	1	6.62
11	9	17	-1	6.57
12	10	24	0	6.29
13	11	32	1	6.03
14	11	31	-1	6.00
15	12	40	0	5.76
16	13	50	1	5.53
17	13	49	-1	5.52
18	14	60	0	5.31
19	15	72	1	5.12
20	15	71	-1	5.11
21	16	84	0	4.93

We can slightly relax condition $3d-2g-6 \geq 0$ by considering curves $C^{d,\delta}$ with $3d-2g-6=-1$. Eisenbud and Harris proved [8] that the image of $V_{d,g}$ is an irreducible Brill-Noether divisor inside $\overline{M}_g$ of slope $6+\frac{12}{g+1}$. Therefore, by a standard argument (cf. [14]), whenever $3d - 2g - 6 \geq -1$, we have a bound

$$s_g \geq \min\{6 + \frac{12}{g+1}, s(C^{d,\delta})\}.$$

Using the recursion of Main Theorem 1, we computed the slopes of curves $C^{d,\delta}$ for $d \leq 16$ with a help of a computer. The resulting lower bounds on s_g for $g \leq 21$ are presented in Table 2.

Note that the bounds for $g = 2$ and $g = 3$ are sharp. The bounds for $g = 4$ and $g = 5$ are better than those given in [14], but are still not sharp. We remark that there are examples of moving curves in $\overline{M}_g$, providing sharp lower bounds for small g. For $g \leq 6$ see, for example, [5]. Finally, even though we have nothing to say about the asymptotic behavior of the bounds produced by curves $C^{d,\delta}$, it would not be surprising if these bounds approached 0, as g goes to ∞. Thus the question remains open whether a positive lower bound, which is independent of g, exists.

Bibliography

[1] Enrico Arbarello and Maurizio Cornalba. Calculating cohomology groups of moduli spaces of curves via algebraic geometry. *Inst. Hautes Études Sci. Publ. Math.*, (88):97–127 (1999), 1998.

[2] Lucia Caporaso and Joe Harris. Counting plane curves of any genus. *Invent. Math.*, 131(2):345–392, 1998.

[3] Lucia Caporaso and Joe Harris. Parameter spaces for curves on surfaces and enumeration of rational curves. *Compositio Mathematica*, 113(2):155–208, 1998.

[4] Maurizio Cornalba and Joe Harris. Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. *Ann. Sci. École Norm. Sup. (4)*, 21(3):455–475, 1988.

[5] I. Coskun, J. Harris, and J. Starr. The effective cone of the Kontsevich moduli space. To appear in the *Canadian Mathematical Bulletin*. Available online at `http://www.math.sunysb.edu/~jstarr/papers/index.html`.

[6] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[7] Steven Diaz and Joe Harris. Geometry of the Severi variety. *Trans. Amer. Math. Soc.*, 309(1):1–34, 1988.

[8] David Eisenbud and Joe Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23. *Invent. Math.*, 90(2):359–387, 1987.

[9] Gavril Farkas. $\overline{M}_{22}$ is of general type. Available online at `http://www-irm.mathematik.hu-berlin.de/~farkas/`.

[10] Gavril Farkas and Mihnea Popa. Effective divisors on $\overline{M}_g$, curves on $K3$ surfaces, and the slope conjecture. *J. Algebraic Geom.*, 14(2):241–267, 2005.

[11] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[12] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[13] Lothar Göttsche. A conjectural generating function for numbers of curves on surfaces. *Comm. Math. Phys.*, 196(3):523–533, 1998.

[14] J. Harris and I. Morrison. Slopes of effective divisors on the moduli space of stable curves. *Invent. Math.*, 99(2):321–355, 1990.

[15] Joe Harris. On the Severi problem. *Invent. Math.*, 84(3):445–461, 1986.

[16] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[17] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[18] Brendan Hassett. Local stable reduction of plane curve singularities. *J. Reine Angew. Math.*, 520:169–194, 2000.

[19] Steven Kleiman and Ragni Piene. Enumerating singular curves on surfaces. In *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, volume 241 of *Contemp. Math.*, pages 209–238. Amer. Math. Soc., Providence, RI, 1999.

[20] Steven L. Kleiman and Ragni Piene. Node polynomials for families: methods and applications. *Math. Nachr.*, 271:69–90, 2004.

[21] Finn F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2):161–199, 1983.

[22] Finn F. Knudsen. The projectivity of the moduli space of stable curves. III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic 0. *Math. Scand.*, 52(2):200–212, 1983.

[23] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[24] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.

[25] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[26] David Mumford. *Lectures on curves on an algebraic surface*. With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[27] David Mumford. Stability of projective varieties. *Enseignement Math. (2)*, 23(1-2):39–110, 1977.

[28] Ziv Ran. Families of plane curves and their limits: Enriques' conjecture and beyond. *Ann. of Math. (2)*, 130(1):121–157, 1989.

[29] Ilya Tyomkin. On Severi varieties on Hirzebruch surfaces, 2006. math.AG/0607.807.

[30] Ravi Vakil. Enumerative geometry of plane curves of low genus, 1998. math.AG/9803007.

[31] Ravi Vakil. Counting curves on rational surfaces. *Manuscripta Math.*, 102(1):53–84, 2000.