Exponential sums and equidistribution

1 02/13 (Kevin): Gauss sums and Kloosterman sums: Kloosterman sheaves

Setup: $\mathbb{F}_q$ is a finite field. ψ an additive character $\mathbb{F}_q \to \overline{\mathbb{Q}_\ell}^\times$ non-trivial and a multiplicative character $\chi : \mathbb{F}_q^\times \to \overline{\mathbb{Q}_\ell}^\times$.

Gauss sums are given by

$$g(\psi, \chi) = \sum_{a \in \mathbb{F}_q^\times} \psi(a)\chi(a).$$

Kloosterman sums are given by

$$\mathrm{Kl}(\psi; \chi_1, \ldots, \chi_n)(\mathbb{F}_q, a) = \sum_{x_1 \cdots x_n = a} \psi\left(\sum x_j\right) \chi_1(x_1) \cdots \chi_n(x_n)$$

with $a \in \mathbb{F}_q^\times$.

Let's discuss the Fourier transform next! It takes a function $f : \mathbb{F}_q^\times \to \overline{\mathbb{Q}_\ell}$ to another function $\hat{f} : \mathbb{F}_q^\times \to \overline{\mathbb{Q}_\ell}$, where we identify the second $\mathbb{F}_q^\times$ with $\mathrm{Hom}\left(\mathbb{F}_q^\times, \overline{\mathbb{Q}_\ell}^\times\right)$, taking χ to $\sum_{a \in \mathbb{F}_q^\times} f(a)\chi(a)$.

Recall that the convolution $f * g$ sends a to $\sum_{xy=a} f(x)g(y)$.

For example,

$f(a)$	$\hat{f}(\chi)$
$\psi(a)$	$g(\psi, \chi)$
$\sum_{x_1 \cdots x_n = a} \psi(x_1) \cdots \psi(x_n) = \mathrm{Kl}(\psi; 1, 1, \ldots, 1)$	$g(\psi, \chi)^n$
$\mathrm{Kl}(\psi; \chi_1, \ldots, \chi_n)(\mathbb{F}_q, a)$	$\prod g(\psi, \chi\chi_i)$

Let's recall the function-sheaf correspondence:

Given a sheaf $\mathcal{F}$ on $\mathbb{G}_m$over $\mathbb{F}_q$, then there is a corresponding function $\mathbb{F}_q^\times \to \overline{\mathbb{Q}_\ell}$ by viewing $\mathbb{F}_q^\times$ as $\mathbb{G}_m(\mathbb{F}_q)$. Take $a \in \mathbb{G}_m(\mathbb{F}_q)$, then map it to $\mathrm{Tr}\left(\mathrm{Fr} \,|\, \mathcal{F}_a\right)$. A Kloosterman sheaf is something on the left hand side such that the corresponding function is a Kloosterman sum.

Next time, we'll geometrize convolution, which will allow us to define a Kloosterman sheaves Kl as the n-fold convolutions of sheaves.

This time, let's just describe the case $n = 1$, which will come from an Artin-Schreier sheaf. To geometrize ψ and χ, we'll need Lang torsors.

Definition 1. Let G be a connected algebraic group over $\mathbb{F}_q$, e.g. $\mathbb{G}_a$ or $\mathbb{G}_m$. Then, consider

$$0 \to G(\mathbb{F}_q) \to G \to G \to 0,$$

where the surjective map is $x \mapsto x - \mathrm{Fr}(x)$.

Let $\rho: G(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}^\times$ be a character. Then the Lang torsor $\mathcal{L}_\rho$ is the rank one local system on G with descent data given by ρ.

Remark 2. A more concrete way to think about $\mathcal{L}_\rho$ (which is a rank one lisse sheaf, or alternatively as we have defined it in this seminar, a one-dimensional representation of $\pi_1(G)$) is as the composition $\pi_1(G) \to G(\mathbb{F}_q) \to \mathrm{GL}_1(\mathbb{C})$, where the second map is ρ and the first map is the canonical surjection: recall that the $\pi_1(G)$ is an inverse limit of automorphism groups of finite etale covers of G, and since $G \to G$ given by $x \mapsto x - \mathrm{Fr}(x)$ above is an example of a finite etale cover, this is simply the projection of an inverse limit onto one of its components (the automorphism group of $G \to G$, which is $G(\mathbb{F}_q)$).

Example 3.

(i) For $G = \mathbb{G}_a$, $\mathcal{L}_\psi$ is the Artin-Schreier sheaf.

(ii) For $G = \mathbb{G}_m$, $\mathcal{L}_\chi$ is the Kummer sheaf.

Lemma 4.

(i) For $x \in G(\mathbb{F}_{q^r})$, we have $\mathrm{Tr}(\mathrm{Fr} | \mathcal{L}_{\rho,x}) = \rho(\mathrm{Tr}\, x)$, where $\mathrm{Tr}: \mathbb{F}_{q^r} \to \mathbb{F}_q$. In particular, this geometrizes $\psi(-)$ and $\chi(-)$. Moreover, $\mathcal{L}_\psi \otimes \mathcal{L}_\chi$ geometrizes $\psi\chi$.

(ii) $\mathrm{Sw}_\infty(\mathcal{L}_\psi) = 1$ (wild ramification).

(iii) $\mathcal{L}_\chi$ is tame, and in fact Sw_0 and Sw_∞ are 0.

Next time, we'll prove the following existence theorem:

Theorem 5. There exists a local system $\mathrm{Kl}(\psi; \chi_1, \ldots, \chi_m)$ on $\mathbb{G}_m$ such that

(i) Kl has rank n.

(ii) $\mathrm{Tr}(\mathrm{Fr} | \mathrm{Kl}) = \mathrm{Kl}$, where the RHS is as we defined it earlier.

(iii) $\mathrm{Sw}_\infty(\mathrm{Kl}) = 1$ and Kl is totally wild at ∞.

(iv) $\mathrm{Sw}_0(\mathrm{Kl}) = 0$.