Deligne's equidistribution theorem

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Abstract

Deligne's equidistribution theorem first appeared in [Del80, Theorem 3.5.3]. It was expanded upon by Katz in several works. Here we follow Katz's presentation of this theorem in [Kat70, Chapters 1 – 3] very closely.

1 Background

Last time, Matthew explained how Deligne's work which bounded Kloosterman sums implied the equidistribution of the angles of Gauss sums with respect to a fixed prime p, as $p \to \infty$. For instance, recall the Kloosterman sum

$$\mathrm{Kl}(p,a) = \sum_{xy \equiv a \pmod p} \psi(x+y),$$

where $\psi \colon \mathbb{F}_p \to \mathbb{C}^*$ is the standard additive character $\psi(x) = e^{2\pi i x/p}$. By the Riemann Hypothesis for curves, we have $|\mathrm{Kl}(p,a)| \leq 2\sqrt{p}$. Deligne's work generalized this to Kloosterman sums in more variables.

Deligne's equidistribution theorem can be used to address a more refined question: how are the Kloosterman sums themselves distributed? In the simpler case above, $\mathrm{Kl}(p,a)$ is real, and thus it can be associated to an angle $\theta(p,a)$ such that

$$-\mathrm{Kl}(p,a) = 2\sqrt{p}\cos\theta(p,a).$$

If we fix a and let p vary, one might expect a Sato-Tate distribution for θ (i.e. $(2/\pi)\sin^2 d\theta$), but we apparently have basically no progress towards this question even today. So instead, we will fix p and let a vary, and hope to prove something about equidistribution as p gets sufficiently large.

Deligne's equidistribution gives a general statement that, under suitable conditions, the image of the Frobenius elements in the representation of the fundamental group associated to a lisse sheaf are equidistributed. What are they equidistributed in? Well, the entire image of the fundamental group is the monodromy group of the lisse sheaf in question, but by the Weil conjectures these Frobenius elements should have bounded eigenvalues. Thus they land in some compact subgroup. Thus equidistribution applies to the Haar measure on a maximal compact subgroup of the monodromy group.

The goal of this talk is to explain the statement and proof of Deligne's equidistribution theorem. The following talks will construct Kloosterman sheaves and compute their monodromy groups, so that the equidistribution theorem applied to them implies the equidistribution of Kloosterman angles.

2 Statement of the theorem

We state the main theorem here and explain what it means. It is a function field statement. Let q be a power of a prime p. We begin with a smooth geometrically connected curve $U/\mathbb{F}_q$, which is the complement of a finite set of points in a proper curve $C/\mathbb{F}_q$. Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}_l}$-sheaf on U that is pure of weight 0. (We fix an isomorphism $\overline{\mathbb{Q}_l} \cong \mathbb{C}$.) This means that eigenvalues of the Frobenius at all points of U in the representation associated to $\mathcal{F}$ all have absolute value 1. Fixing a geometric base point $\bar{x}$ in U (which we will ignore for convenience), we define the étale fundamental groups

$$\pi_1^a := \pi_1(U), \qquad \pi_1^g = \pi_1(U_{\overline{\mathbb{F}_q}}).$$

Then we define the geometric monodromy group G_{geom} to be the Zariski closure of the image of π_1^g in $\mathrm{Aut}_{\overline{\mathbb{Q}_l}(\mathcal{F}_{\bar{x}})} \cong \mathrm{GL}(n, \overline{\mathbb{Q}_l})$. We define the arithmetic monodromy group G_{arith} similarly.

For simplicity, we denote $G = G_{\text{geom}}$; in the statement of our theorem we will assume that it equals G_{arith} anyways[1].

A fundamental result of Deligne gives that the purity of $\mathcal{F}$ implies that the identity component $G^0(\overline{\mathbb{Q}_l})$ is semisimple. On the other hand, $G(\mathbb{C})$ is a complex semisimple Lie group. We now recall the following facts from representation theory.

- (Cartan's theorem) There exists a maximal compact subgroup of $G(\mathbb{C})$; call it K.
- (Cartan's theorem) All compact subgroups of $G(\mathbb{C})$ are conjugate to a subgroup of K.
- (Weyl's unitarian trick) Finite-dimensional $\overline{\mathbb{Q}_l}$-representations of G are equivalent to finite-dimensional holomorphic representations of $G(\mathbb{C})$ as a complex Lie group, which are equivalent to finite-dimensional continuous representations of K.
- (Peter-Weyl) Traces of finite-dimensional continuous representations of K separate K-conjugacy classes.

For each point $u \in U$, the semisimplification of the image of the Frobenius conjugacy class F_u in $G(\mathbb{C})$ lies in a compact subgroup of $G(\mathbb{C})$, as it has eigenvalues of magnitude 1 (since $\mathcal{F}$ is pure of weight 0). That is, $\rho(F_u)^{\text{ss}}$ is conjugate to an element of K. This gives a well-defined element $\theta(u) \in K^\natural$ (the set of conjugacy classes of K) – here we use Peter-Weyl.

The Haar measure on K descends to one on $K^\natural$. Concretely, given a continuous function f on $K^\natural$, we have

$$\int_{K^\natural} f d\mu^\natural := \int_K f d\mu_{\text{Haar}},$$

where we extend f to a central function on K.

Example 2.1. If $G = \mathrm{SL}(2)$, then $K = \mathrm{SU}(2)$ and we can choose $K^\natural$ to be represented by matrices $\begin{pmatrix} e^{i\theta(u)} & 0 \\ 0 & e^{-i\theta(u)} \end{pmatrix}$, thus identifying it with the interval $[0, \pi]$. The measure $\mu^\natural$ is the Sato-Tate measure $\frac{2}{\pi}\sin^2\theta d\theta$.

Theorem 2.2 (Deligne's equidistribution theorem). *Let $\mathcal{F}$ be a lisse sheaf on a smooth geometrically connected curve $U/\mathbb{F}_q$ that is pure of weight 0. Assume that the associated arithmetic and geometric monodromy groups are equal: $G = G_{\text{arith}} = G_{\text{geom}}$. Then the conjugacy classes of the Frobenius elements F_u, for $u \in U$, are equidistributed in a maximal compact subgroup K of G.*

[1] We can get by with the condition that the image of π_1^a is in G.

More precisely, define the following sequences.

$$X_n = \frac{1}{|U(\mathbb{F}_{q^n})|} \sum_{\deg(u)=n} \delta(\theta(u)^{n/\deg(u)})$$
$$Y_n = \frac{1}{|\deg u = n|} \sum_{\deg(u)=n} \delta(\theta(u))$$
$$Z_n = \frac{1}{|\deg u \le n|} \sum_{\deg(u)\le n} \delta(\theta(u))$$

All these sequences of measures tend to $\mu^\natural$ on $K^\natural$. In other words, for any continuous $\mathbb{C}$-valued function f on $K^\natural$, we have

$$\int_{K^\natural} f d\mu^\natural = \lim_n \int_{K^\natural} f dX_n = \lim_n \int_{K^\natural} f dY_n = \lim_n \int_{K^\natural} f dZ_n.$$

Remark. This result holds in more generality. Specifically, we can replace the curve U with a smooth, geometrically connected scheme of arbitrary dimension over a finite field. See [KS99, Theorem 9.2.6]

3 Proof of the theorem

We will first briefly review some ingredients to the proof.

3.1 Swan conductors

Let us briefly review the notion of a Swan conductor, which will be used in the proof. Given a Henselian DVR R with perfect residue field k of cahracteristic p and fraction field K, we have the short exact sequences

$$1 \to I \to \mathrm{Gal}(K^{\mathsf{sep}}/K) \to \mathrm{Gal}(k^{\mathsf{sep}}/k) \to 1, \quad 1 \to P \to I \to \prod_{l \neq p} \mathbb{Z}_l(1) \to 1.$$

There is an upper numbering filtration and a lower numbering filtraion on I (both decreasing), whose precise definition we do not need. Some useful properties are: $I = I^{(0)}$, P is the closure of all $I^{(r)}$ for all $r > 0$.

P is the ramification group. If P acts through a finite discrete quotient on a $\mathbb{Z}[1/p]$-module M, then $M = \oplus M(x)$ for some $x \ge 0$, where $M(0) = M^P$, $(M(x))^{I(x)} = 0$ for $x > 0$ and $(M(x))^{I(y)} = M(x)$ for $y > x$. The finitely many x for which $M(x)$ is non-zero are known as the *breaks* of M. These can be cahracterized as the x for which

$$\bigcup_{y > x} \rho(I^{(y)}) \subsetneq \rho(I^{(x)}).$$

Changing settings slightly with M an A-module where A is a noetherian local ring of residue characteristic $\neq p$, we have the following definition.

Definition 3.1. *The Swan conductor of M is defined as*

$$\mathrm{Swan}(M) = \sum_{x \ge 0} x \,\mathrm{rank}_A(M(x)).$$

In particular, $\mathrm{Swan}(M) = 0$ iff M is trivial as a representation of P.

3.2 Theorems of étale cohomology

We state without proof several important results from étale cohomology.

Proposition 3.2. *Let $\mathcal{F}$ be lisse on $U \subset C$, a smooth proper geometrically connected curve over a perfect field k. Then*

$$H^0(U_{k^{sep}}, \mathcal{F}) = (\mathcal{F}_{\bar{\eta}})^{\pi_1^{\text{geom}}}, \quad H^2_c(U_{k^{sep}}, \mathcal{F}) = (\mathcal{F}_{\bar{\eta}})_{\pi_1^{\text{geom}}}(-1).$$

Theorem 3.3 (Grothendieck-Ogg-Shafarevich)**.** *With the same conditions, we have*

$$\chi(U_{k^{sep}}, \mathcal{F}) = \chi_c(U_{k^{sep}}, \mathcal{F}) = \operatorname{rank}(\mathcal{F})\chi_c(U_{k^{sep}}) - \sum_{x \in C-U} \deg_k(x) \operatorname{Swan}_x(\mathcal{F}).$$

Theorem 3.4 (Grothendieck trace formula)**.** *Same conditions, but let k be a finite field and let F be the geometric Frobenius with F_x its image in $\pi_1(U)$ through D_x/I_x. Then*

$$\sum_{x \in U(k)} \operatorname{Tr}(F_x, \mathcal{F}_{\bar{\eta}}) = \sum_i (-1)^i \operatorname{Tr}(F | H^i_c(U_{k^{sep}}, \mathcal{F})).$$

Theorem 3.5 (Weil II)**.** *Let $X/\mathbb{F}_q$ be a scheme of finite type and let $\mathcal{F}$ be a mixed sheaf of weights $\leq w$. Then $H^i(X_{\overline{\mathbb{F}_q}}, \mathcal{F})$ is mixed of weights $\leq w + i$.*

3.3 The proof

Recall the statement:

$$\int_{K^\natural} f d\mu^\natural = \lim_n \int_{K^\natural} f dX_n = \lim_n \int_{K^\natural} f dY_n = \lim_n \int_{K^\natural} f dZ_n.$$

for functions f on $K^\natural$. Here we began with a lisse sheaf $\mathcal{F}$ on U, pure of weight 0, which we will identify with a representation $\rho\colon \pi_1^a \to G$, with $G = G_{\text{arith}} = G_{\text{geom}}$.

We will just do the proof for X_n, with the other cases being analogous. By the Peter-Weyl theorem, it suffices to prove the result for characters of irreducible representations. The trivial representation reduces to $1 = 1$. For a nontrivial irreducible representation ψ (of G, as those of K are restrictions from those of G), orthogonality of characters gives

$$\int_{K^\natural} f d\mu^\natural = 0.$$

Thus it will certainly suffice to prove that

$$\left| \int_{K^\natural} \operatorname{Tr}(\psi) dX_n \right| \leq O\left(\frac{\dim(\psi)}{q^{n/2}}\right).$$

We will do this by studying the composite representation

$$\pi_1(U) \xrightarrow{\rho} G \xrightarrow{\psi} \operatorname{GL}(m, \overline{\mathbb{Q}_l}),$$

whose corresponding sheaf we will denote with $\mathcal{F}(\psi)$. The first thing to note is that $\mathcal{F}(\psi)$ is still pure of weight 0, because ψ ends up making $\mathcal{F}(\psi)$ a subquotient of some $\mathcal{F}^{\otimes m} \otimes (\mathcal{F}^\vee)^{\otimes m}$.

Now note that the LHS of the Grothendieck-Lefschetz trace formula satisfies

$$\sum_{x \in U(\mathbb{F}_{q^n})} \operatorname{Tr}(F_x | \mathcal{F}(\psi)) = |U(\mathbb{F}_{q^n})| \int_{K^\natural} \operatorname{Tr}(\psi) dX_n.$$

Indeed, the contribution of some x with residue field of size q^a in the measure X_n is indeed given by the the n/ath power of the image of the Frobenius $x \mapsto x^{q^a}$ under ψ, which is indeed the image of Frobenius on the left which maps $x \mapsto x^{q^n}$. Thus the trace formula applied to $\mathcal{F}(\psi)$ gives

$$|U(\mathbb{F}_{q^n})| \int_{K^\natural} \mathrm{Tr}(\psi) dX_n = \sum_{i=0}^{2} (-1)^i \mathrm{Tr}(F^n | H_c^i(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi))).$$

Since ψ is irreducible and non-trivial, its invariants and co-invariants under $\pi_1(U)$ are both 0. Since $\pi_1(U)$ is dense in G, this implies that

$$H_c^0(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi)) = H_c^2(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi)) = 0.$$

Next, since $\mathcal{F}(\psi)$ is pure of weight 0 on $U_{\overline{\mathbb{F}_q}}$, we have that $H_c^1(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi))$ is mixed of weight ≤ 1. Thus we have

$$\left| \int_{K^\natural} \mathrm{Tr}(\psi) dX_n \right| \leq \frac{|\chi_c(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi))| q^{n/2}}{|U(\mathbb{F}_{q^n})|}. \tag{1}$$

The denominator is bounded below by

$$|U(\mathbb{F}_{q^n})| \geq q^n - h_c^1(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi)) q^{n/2},$$

which is approximately q^n. The numerator can be bounded using the Grothendieck-Ogg-Shafarevich formula:

$$\chi_c(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi)) = \chi_c(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi)) - \sum_{y \in (C-U)(\overline{\mathbb{F}_q})} \mathrm{Swan}_y(\mathcal{F}(\psi)).$$

Note that at each $y \in (C-U)(\overline{\mathbb{F}_q})$, the biggest break of $\mathcal{F}(\psi)$ is less than that of $\mathcal{F}$, because I_y acts on $\mathcal{F}$ through ρ but on $\mathcal{F}(\psi)$ through $\psi \circ \rho$. Therefore, if $r_1, \ldots, r_N$ are these biggest breaks of $\mathcal{F}$, then we have

$$|\chi_c(U_{\overline{\mathbb{F}_q}}, \mathcal{F}(\psi))| \leq \left(2g - 2 + N + \sum_{i=11}^{N} r_i \right) \dim(\psi).$$

Combining this with (1), we have

$$\left| \int_{K^\natural} \mathrm{Tr}(\psi) dX_n \right| \leq O\left(\frac{\dim(\psi)}{q^{n/2}} \right),$$

as desired.

References

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