Geometric aspects of general multiple Dirichlet series over function fields

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Abstract

Multiple Dirichlet series were originally defined as multi-variate Dirichlet series satisfying certain functional equations with conjecturally nice analytic properties that would give precise asymptotics for moments of L-functions. Generalizing an observation of Chinta, Sawin recently gave an axiomatic characterization of a general class of multiple Dirichlet series over function fields that is independent of their functional equations. Moreover, he proved their existence as formal power series by exhibiting the coefficients as trace functions of explicit perverse sheaves.

In this talk, I'll explain how to 1. prove analyticity of these series in a suitable non-empty region of convergence, and 2. establish some (but not all) of the functional equations that they satisfy. The methods for both are completely geometric: analyticity is a consequence of bounding the cohomology of local systems on a compactification of a configuration space, and the functional equations follow from a density trick for irreducible perverse sheaves.

1 Introduction

Recall that a Dirichlet series is a series of the form

$$L(s,a) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $a(n) \in \mathbb{C}$ satisfy a multiplicativity condition: $a(mn) = a(m)a(n)$ for $\gcd(m,n) = 1$.

For interesting examples such as $a(n) = \left(\frac{n}{m}\right)$, the Legendre symbol, these Dirichlet series satisfy functional equations that relate $L(s,a)$ and $L(1-s,a)$.

Consider a natural multi-variate generalization of this:

$$L(s_1,\ldots,s_k,a) = \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{a(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}},$$

where $a(n_1,\ldots,n_k) \in \mathbb{C}$ satisfy a *twisted* multiplicativity condition:

$$a(n_1m_1,\ldots,n_km_k) = a(n_1,\ldots,n_k)\, a(m_1,\ldots,m_k) \prod_{1\le i,j\le k} \left(\frac{n_i}{m_j}\right)^{c_{ij}}$$

for $\gcd(n_1\cdots n_k, m_1\cdots m_k) = 1$.

Example 1. Consider

$$\sum_{n_1,n_2,n_3,n_4} \left(\frac{n_1 n_2}{n_3 n_4}\right) n_1^{-s_1} n_2^{-s_2} n_3^{-s_3} n_4^{-s_4} = \sum_{n_1,n_2} L\left(s_3, \chi_{n_1 n_2}\right) L\left(s_4, \chi_{n_1 n_2}\right) n_1^{-s_1} n_2^{-s_2}.$$

Why care about this?

(i) Applications to moments of L-functions: In the example, set $s_3 = s_4 = 1/2$ and $n_1 = n_2$, which says something about the second moment of quadratic L-functions weighted by the number of divisors of the conductor.

(ii) Intrinsically interesting: For what kinds of interesting examples do we get (many) functional equations?

It turns out that the coefficients $\left(\frac{n_1 n_2}{n_3 n_4}\right)$ should be replaced with coefficients $a_{n_1,\ldots,n_4}$ that typically agree (i.e. when the n_i are square-free and co-prime to each other) so that the modified series has better analytic properties, which are obtained from satisfying functional equations.

Using this better-behaved modified series, one can then go back and say something about moments of L-functions.

These multi-variate series with good analytic properties are roughly called **multiple Dirichlet series (MDS)**. Some prior work:

- Diaconu, Goldfeld, and Hoffstein '03: applied to conjectures on asymptotics for moments of L-functions, defined MDS in terms of functional equations and twisted multiplicativity.
- Chinta, Friedberg, and Hoffstein '06: discovered a certain local-to-global property in the function field setting.
- Diaconu and Pasol '18: gave a more restrictive definition of function field MDS in terms of local-to-global properties, proved uniqueness and existence for a specific family of cases corresponding to moments of quadratic Dirichlet L-series.
- Whitehead '14: obtained expected functional equations in the Diaconu-Pasol setting and established meromorphic continuation to a certain region.
- Sawin '22: generalized and simplified Diaconu and Pasol's work by using the language of perverse sheaves.

There is much much more out there, and I'm definitely omitting many key developments and related advances!

What did I do?

Sawin's "general multiple Dirichlet series" are purely formal. I proved the following:

(i) They are genuine analytic functions (have some radius of convergence).

(ii) They satisfy some functional equations.

Both are proved geometrically:

(i) Analyticity follows from bounding the coefficients, which is done by bounding cohomology of sheaves on the Kontsevich space of stable maps.

(ii) The functional equations follow from establishing the result first for a dense subset of tuples, and then using properties of perverse sheaves to extend it to all tuples. More specifically, the key point is to generalize to the multi-variate situation the identity relating the Fourier transform of a Dirichlet character to its conjugate.

2 Sawin's general multiple Dirichlet series

Let $\mathbb{F}_q[t]^+$ be the set of monic polynomials and $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ a character of order n.

The resultant $\operatorname{Res}(f,g)$ of $f,g \in \mathbb{F}_q[T]$ is defined as the product of values of f at the roots of g. In particular, $\operatorname{Res}(f,g) = 0$ iff f and g share a common root.

Sawin defined the following:

Definition 2. Let M be a symmetric $m \times m$ matrix with integer entries. A **general multiple Dirichlet series** is a multi-variate series

$$\begin{aligned} L(s_1,\dots,s_m;M) &= \sum_{f_1,\dots,f_m \in \mathbb{F}_q[t]^+} a(f_1,\dots,f_m) \prod_{i=1}^m q^{-s_i \deg f_i} \\ &= \sum_{d_1,\dots,d_m \geq 0} \left(\sum_{\deg(f_i)=d_i} a(f_1,\dots,f_m;M) \right) \prod_{i=1}^m q^{-s_i d_i} \end{aligned}$$

satisfying twisted multiplicativity and a local-to-global property relating the coefficients of the two series representations:

(i) (twisted multiplicativity)

$$a(f_1 g_1,\cdots,f_m g_m;M) = a(f_1,\cdots,f_m;M)\, a(g_1,\cdots,g_m;M) \prod_{i,j} \left(\frac{f_i}{g_j}\right)_\chi^{M_{ij}}$$

for $\gcd(f_1\cdots f_m, g_1\cdots g_m) = 1$, with $\left(\frac{f}{g}\right)_\chi = \chi(\operatorname{Res}(f,g))$.

(ii) (local-to-global) For a prime π, the value of $a(\pi^{d_1},\dots,\pi^{d_m};M)$ is related to the value of the sum $\sum_{\deg(f_i)=d_i} a(f_1,\dots,f_m;M)$.

Note that $\mathbb{A}^d_{\mathbb{F}_q}$ can be viewed as a moduli space for monic polynomials of degree d. Indeed, for an $\mathbb{F}_q$-algebra R, $\mathbb{A}^d_{\mathbb{F}_q}(R) = R^d = \{(r_{d-1},\dots,r_0) : r_i \in R\}$, which we identify with the set of monic single-variable polynomials over R of degree d: $\{t^d + r_{d-1}t^{d-1} + \cdots + r_0 : r_i \in R\}$.

Consequently, for non-negative integers $d_1,\dots,d_m$, we can view $\prod_{i=1}^m \mathbb{A}^{d_i}_{\mathbb{F}_q}$ as a moduli space for tuples of monic polynomials of fixed degrees $d_1,\dots,d_m$.

Sawin's idea is to construct a sheaf on $\prod_{i=1}^m \mathbb{A}^{d_i}_{\mathbb{F}_q}$ and realize the coefficients $a(f_1,\dots,f_m;M)$ as the trace of Frobenius acting on the stalk of this sheaf at the point $(f_1,\dots,f_m) \in \prod_{i=1}^m \mathbb{A}^{d_i}_{\mathbb{F}_q}$.

Let me quickly describe how this sheaf is defined:

(i) Define the polynomial function

$$F_{d_1,\dots,d_m} = \prod_{i=1}^{m} \operatorname{Res}\left(f_i', f_i\right)^{M_{i,i}} \prod_{1\le i<j\le r} \operatorname{Res}\left(f_i, f_j\right)^{M_{i,j}}$$

on $\prod_{i=1}^{m} \mathbb{A}^{d_i}$.

(ii) Let U be the open subset for which $F_{d_1,\dots,d_m}$ is invertible, i.e. $F_{d_1,\dots,d_m}$ defines a morphism $\prod_{i=1}^{m} \mathbb{A}^{d_i} \to \mathbb{A}^1$ and U is the preimage of $\mathbb{G}_m \subset \mathbb{A}^1$.

(iii) $\mathbb{G}_m$ has a locally constant sheaf called the Kummer sheaf, which can be understood using the correspondence between locally constant sheaves on $\mathbb{G}_m$ and representations of the fundamental group of $\mathbb{G}_m$: $\mathcal{L}_\chi$ is defined as the composition $\pi_1\left(\mathbb{G}_m\right) \twoheadrightarrow \mathbb{F}_q^\times \xrightarrow{\overline{\chi}} \mathbb{C}^\times$.

(iv) Pulling back $\mathcal{L}_\chi$ along $F_{d_1,\dots,d_m}: U \to \mathbb{G}_m$ gives a sheaf called $\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)$ on U whose trace function is $\chi\left(F_{d_1,\dots,d_m}\right)$.

(v) Let $j: U \hookrightarrow \prod_{i=1}^{m} \mathbb{A}^{d_i}$ be the inclusion. There is a "best extension" of $\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)$ on U to $\prod_{i=1}^{m} \mathbb{A}^{d_i}$, denoted by $K_{d_1,\dots,d_m} \coloneqq j_{!*}\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)$, i.e. $j^* j_{!*}\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right) = \mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)$.

Remark 3. If you're familiar with etale cohomology, there is an abelian category of "perverse sheaves" inside the derived category of ℓ-adic sheaves, which is given by the heart of a certain t-structure. There are two important relevant examples of perverse sheaves:

(i) If X is a smooth variety and L is a lisse sheaf on X, then $L\left[\dim X\right]$ is perverse.

(ii) If X is a variety, $j: U \subset X$ is the inclusion of an open subset, and A is a perverse sheaf on U, then there is a perverse sheaf $j_{!*}A$ on X, called the intermediate extension of A, defined as the unique extension of A that has no non-trivial sub-objects or quotients supported on $X \backslash U$.

Combining these two examples, $j_{!*}\left(\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)\left[d_1 + \cdots + d_m\right]\right)$ is a perverse sheaf on $\prod_{i=1}^{m} \mathbb{A}^{d_i}$, and Sawin defines

$$K_{d_1,\dots,d_m} = j_{!*}\left(\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)\left[d_1 + \cdots + d_m\right]\right)\left[-d_1 - \cdots - d_m\right],$$

shifted up so that generically $K_{d_1,\dots,d_m}$ agrees with $\mathcal{L}_\chi\left(F_{d_1,\dots,d_m}\right)$.

From this sheaf, Sawin constructs the a-coefficient as follows:

Given a tuple $\left(f_1, \dots, f_m\right) \in \prod_{i=1}^{m} \mathbb{A}^{d_i}$, let

$$a\left(f_1, \dots, f_m; M\right) = \operatorname{Tr}\left(\operatorname{Fr}_q, \left(K_{d_1,\dots,d_m}\right)_{\left(f_1,\dots,f_m\right)}\right),$$

where Fr_q is the geometric Frobenius.

The main result of Sawin's paper is the following:

Theorem 4. The axioms—local-to-global and twisted multiplicativity—of a MDS uniquely characterize $a\left(f_1, \dots, f_m; M\right)$, whose existence can be realized as the trace of Frobenius on $\left(K_{d_1\dots,d_m}\right)_{\left(f_1,\dots,f_m\right)}$.

3 Analyticity

Theorem 5. Sawin's MDS $L(s_1, \ldots, s_m; M)$ is an analytic function with a non-empty region of convergence.

Let $\mathcal{M}_d$ denote the set of monic polynomials of degree d. Let

$$\lambda(d_1, \ldots, d_m; M) = \sum_{\deg f_1 = d_1, \ldots, \deg f_m = d_m} a(f_1, \ldots, f_m; M).$$

The general strategy is as follows:

(i) The local-to-global axiom relates $\lambda(d_1, \ldots, d_m; M)$ to $a(f_1, \ldots, f_m; M)$, allowing you to bound the latter in terms of the former.

(ii) Apply the Grothendieck-Lefschetz trace formula to see that

$$\lambda(d_1, \ldots, d_m; M) = \sum_i (-1)^i \operatorname{Tr}\left(\mathrm{Fr}_q, H_c^i\left(\prod_{j=1}^m \mathbb{A}^{d_j}, K_{d_1,\ldots,d_m}\right)\right).$$

(iii) Use Deligne's theory of weights to see that

$$|\lambda(d_1, \ldots, d_m)| \le \sum_{i=0}^{2r}\left(\dim H_c^i\left(\prod_{j=1}^m \mathbb{A}^{d_j}, K_{d_1,\ldots,d_m}\right)\right) q^{i/2}.$$

(iv) Compactify $\prod_{j=1}^m \mathbb{A}^{d_j}$ using a quotient by $S_{d_1} \times \cdots \times S_{d_m}$ of the Konstevich moduli space $\overline{\mathcal{M}}_{0,d_1+\cdots+d_m}(\mathbb{P}^1, 1)$ (call this $\mathcal{M}$), and use the decomposition theorem to express $H_c^i\left(\prod_{j=1}^m \mathbb{A}^{d_j}, K_{d_1,\ldots,d_m}\right)$ as a direct summand of $H_c^i(\mathcal{M}, \mathcal{L})$ for some sheaf $\mathcal{L}$ on $\mathcal{M}$.

(v) Use a "comparison trick" to relate $\dim H_c^i(\mathcal{M}, \mathcal{L})$ to the analogous setting in characteristic zero.

(vi) Bound $\dim H_c^i(\mathcal{M}, \mathcal{L})$ by finding a cell structure on $\mathcal{M}$, and bounding the number of cells (something exponential in m)!

4 Functional equations

Write $L(s_1, \ldots, s_m; M)$ as

$$\sum_{d_1,\ldots,d_m \ge 0} \sum_{\deg f_1 = d_1, \ldots, \deg f_m = d_m} a(f_1, \ldots, f_m, M)\, q^{-s_1 d_1} \cdots q^{-s_m d_m}$$

and write $L_{\text{mod}}(s_1, \ldots, s_m; M)$ as a slight variant of this.

Theorem 6. If $M_{1,1} = 0, M_{1,2}, \ldots, M_{1,m} \neq 0$, and n even, there is a functional equation relating

$$L(s_1, \ldots, s_m, M)$$

to sums of

$$L_{\text{mod}}(1 - s_1, \omega_2(s_1 + s_2) \ldots, \omega_m(s_1 + s_m), M')$$

ranging over certain roots of unity ω_i, where M' is a slight modification of M.

The general strategy is as follows:

(i) By definition, we have

$$a\left(f_1,\dots,f_m;M\right) = \prod_{j>i\geq 1}\left(\frac{f_i}{f_j}\right)_\chi^{M_{ij}}\prod_{i\geq 1}\left(\frac{f_i'}{f_i}\right)_\chi^{M_{ii}}$$

for the open subset U where $F_{d_1,\dots,d_m}$ is invertible. We first show that this identity is still true for the slightly larger open subset X where we do not assume $(f_1,f_i)=1$ for any i by describing the intermediate extension $j_{!*}$ as a sheaf! The idea is that the complement of U in X is a divisor with normal crossings, i.e. locally looks like a union of coordinate hyperplanes.

(ii) Recall the relationship between the conjugate Dirichlet character $\overline{\chi}\colon (\mathbb{F}_q[t]/g)^\times \to \mathbb{C}^\times$ and the Fourier transform:

$$\overline{\chi}(f) = \text{constant}(\chi)\cdot \underbrace{\sum_{h\in\mathbb{F}_q[t]/g}\chi(h)e\left(\frac{fh}{g}\right)}_{\text{Fourier transform}},$$

where $e(a) = e^{\frac{2\pi i\,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)}{p}}$ and a_1 is the residue of a, i.e. a non-trivial additive character.

(iii) Mimic the above:

$$a\left(f_1,\dots,f_m;M'\right) = \text{constant}\left(d_i,M\right)\cdot \underbrace{\sum_{\deg h = d_2+\cdots+d_m} a\left(h,f_2,\dots,f_m;M\right)e\left(\frac{hf_1}{f_2\cdots f_m}\right)}_{\text{Fourier transform}}.$$

This is an easy computation for the open subset X above. To extend:

(a) Express both sides as trace functions of K_1 and K_2.

(b) Check that K_1 and K_2 are irreducible perverse sheaves.

(c) On an open subset, K_1, K_2 are irreducible and lisse with the same trace function, so K_1, K_2 are intermediate extensions of the same sheaf.