Exponential sums and equidistribution

1 02/20 (Vidhu): Convolution

$k = \mathbb{F}_q$ with q a power of p, A is an ℓ-adic coefficient ring and is $K/\mathbb{Q}_\ell$ a finite extension or $\mathcal{O}_K$ or $\mathbb{F}_q$.

Let $C \coloneqq \{\text{lisse sheaves of free } A\text{-modules of finite rank on } \mathbb{G}_m \otimes k \text{ s.t. } F \text{ is tame at } 0, \text{ totally wild at } \infty\}$. We have a multiplication map

$$\pi\colon (\mathbb{G}_m \otimes k) \times (\mathbb{G}_m \otimes k) \to \mathbb{G}_m \otimes k.$$

Take $F, G \in C$ and consider the box product $F \boxtimes G = p_1^* F \otimes p_2^* G$ where p_1, p_2 are the two projections.

We consider $\pi_! (F \boxtimes G)$, and we will write $F * G = R^1 \pi_! (F \boxtimes G) \in C$.

Theorem 1.

(i) $F * G$ is bi-exact compatible with base extension of coefficient rings.

(ii) $R\pi_* (F \boxtimes G)$ commutes with passage to fibers and we have the natural map $R\pi_! (F \boxtimes G) \to R\pi_* (F \boxtimes G)$ is an isomorphism.

(iii) $F^\vee = \operatorname{Hom}_A (F, A) \in C$ and $(F * G) * (F^\vee * G^\vee) \cong A(-1)$

(iv) $\operatorname{rk}(F * G) = \operatorname{rk} F \cdot \operatorname{Sw}_\infty(G) + \operatorname{rk} G \cdot \operatorname{Sw}_\infty(F)$.

(v) $\operatorname{Sw}_\infty (F * G) = \operatorname{Sw}_\infty(F) \operatorname{Sw}_\infty(G)$.

(vi) $H_c^1 (F * G) \cong H_c^1(F) \otimes_A H_c^1 (G), H^1 (F * G) \cong H^1(F) \otimes_A H^1 (G)$ with a corresponding commutative diagram given by forgetting supports.

(vii) If F, G are pure of weights $w(F)$ and $w(G)$, then $F * G$ is also pure of weight $1 + w(F) + w(G)$.

(viii) $a \in \mathbb{F}_q^\times = \mathbb{G}_m (\mathbb{F}_q)$, write $\operatorname{Tr}_F(a) = \operatorname{Tr} (\operatorname{Fr}_a | F_a)$, and $\operatorname{Tr}_{F*G} = -\operatorname{Tr}_F * \operatorname{Tr}_G$, where the latter is convolution as functions.

(ix) Let $\chi\colon \mathbb{F}_q^\times \to A^\times$ be a multiplicative character and let FT be the multiplicative Fourier transform, so that $\operatorname{FT}(\operatorname{Tr}_F(\chi)) = -\operatorname{Tr}\left(\operatorname{Fr} | H_c^1 \left(\mathbb{G}_m \otimes \overline{\mathbb{F}_q}, F \otimes L_\chi\right)\right)$.

(x) The map $[N]\colon \mathbb{G}_m \to \mathbb{G}_m$ is compatible with convolution, and similarly for translation by a.

(xi) $N \le 1$ and an integer, then $[N]^* (F * G) \hookrightarrow [N]^* F * [N]^* G$ and has an A-flat cokernel.

Let's give some idea of how this is proved.

Proposition 2. *$R^i \pi_! (F \boxtimes G)$ is lisse on $\mathbb{G}_m \otimes k$ and commutes with passage to fibers.*

Proof. Take $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ with $(x, y) \mapsto (x, xy)$ and let's try to understand the second projection. In particular, $F \boxtimes G$ becomes $F_x \boxtimes G_{t/x}$. Take the injection $j: \mathbb{G}_m \times \mathbb{G}_m \hookrightarrow \mathbb{P}^1 \times \mathbb{G}_m$ and take the second projection down to $\mathbb{G}_m$.

We have $R^i \pi_! (F \boxtimes G) = R^i p_2 j_! \left(F_x \boxtimes G_{t/x}\right)$. Deligne's semicontinuity theorem tells us that it suffices to show that the $\mathrm{Sw}_0 \left(F_x \boxtimes G_{t/x}\right) + \mathrm{Sw}_\infty \left(F_x \boxtimes G_{t/x}\right)$ is constant as a function of t. This is not too bad to check.

The commuting with passage to fibers claim follows from proper base change. ☺

Proposition 3. *$R^1 \pi_! (F \boxtimes G)$ is tame at 0.*

Let's check that $\mathrm{Tr}_{F*G} = -\mathrm{Tr}_F * \mathrm{Tr}_G$. The former is

$$\mathrm{Tr}\left(\mathrm{Fr}_a \,|H^1_c \left(X = \{xy = a\} \subset \mathbb{G}_m \times \mathbb{G}_m, F \boxtimes G\right)\right) = -\sum_{z \in X(\mathbb{F}_q)} \mathrm{Tr}\left(\mathrm{Fr}_{(x,y)} \,| F \otimes G\right)$$

by Grothendieck-Lefschetz, the key point being that most of the higher pushforwards vanish. The latter is evidently the convolution.

Finally, we'll be able to define the Kloosterman sheaf. Let $\psi: \mathbb{F}_q \to A^\times$ additive and $\chi: \mathbb{F}_q^\times \to A^\times$ multiplicative.

Let $b \in \mathbb{Z}$. We define $[b]$ as the map $\mathbb{G}_m \to \mathbb{G}_m$ as the bth power map. Then, define $\mathrm{Kl}(\psi, \chi, b) = [b]_* (L_\psi \otimes L_\chi)$, and the Kloosterman sheaf in general as the n-wise convolution of these. Let's denote this by $\mathrm{Kl}(\psi, \chi_1, \ldots, \chi_n, b_1, \ldots, b_n)$.

Note that $[N]_* (F * G) = [N]_* F * [N]_* G$, which implies that $[N]_* \mathrm{Kl} = \mathrm{Kl}(\psi, \chi_1, \ldots, \chi_n, Nb_1, \ldots, Nb_n)$.

Proposition 4. *Suppose $N | q-1$ and n a natural number. Let $\chi_1, \ldots, \chi_n$ be multiplicative characters of $\mathbb{F}_q^\times$. Then, there is an isomorphism of lisse sheaves on $\mathbb{G}_m$*

$$[N]_* \mathrm{Kl}\left(\psi_N = \psi(N-), \chi_1^N, \ldots, \chi_n^N, 1, \ldots, 1\right) \otimes \left[\prod_{i=1}^{N} \left(-g(\psi, \rho_i)^n\right)\right]^{\deg} \cong \mathrm{Kl}(\psi, \chi\rho_1, \ldots, \chi\rho_N; 1, \ldots, 1),$$

where $[\varepsilon]^{\deg}$ means the constant sheaf with Frobenius acts by $\varepsilon^{\deg x}$.