The KPZ scaling limit of the colored asymmetric simple exclusion process

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(based on joint work with Amol Aggarwal and Ivan Corwin)

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Models and main results
Fix \( q \in [0, 1) \) and place a particle of “color” \(-k\) at location \( k\) for every \( k \in \mathbb{Z}\).

Particles attempt to swap positions to the left and right with rates \( q \) and 1, respectively. Swaps succeed if the initiating particle is of higher color.
Individual particle behavior

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The particles lie in a *rarefaction fan* parametrized by speed $\alpha \in (-1, 1)$. 
The colored ASEP height function $h_{\text{ASEP}}$ is

$$h_{\text{ASEP}}(x, 0; y, t) := \# \text{ particles of initial position } \leq x \text{ to right of } y \text{ at time } t.$$  

A lot is known about $y \mapsto h_{\text{ASEP}}(0, 0; y, t)$ in the $t \to \infty$ limit: e.g., after rescaling,

• $h_{\text{ASEP}}(0, 0; 0, t)$ converges to the GUE Tracy-Widom distribution of RMT [Tracy-Widom '09]

• $y \mapsto h_{\text{ASEP}}(0, 0; y, t)$ converges to the parabolic Airy$_2$ process [Quastel-Sarkar ’22]

We are interested in the joint limit $(x, y) \mapsto h_{\text{ASEP}}(x, 0; y, t)$. 
The limiting object: The Airy sheet

Airy sheet $\mathcal{S}$ arises as the limit of a model of a random directed metric: Brownian last passage percolation (LPP) [Dauvergne-Ortmann-Virág].

With $\mathbf{B} = (B_1, \ldots, B_n)$ i.i.d. Brownian motions,

$$B[(x, n) \rightarrow (y, 1)] = \sup_{\gamma} B[\gamma],$$

where the weight $B[\gamma]$ of a directed path $\gamma$ is the integral of $\mathbf{B}$ over $\gamma$, i.e., sum of increments along the $B_i$. 
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**Theorem (Dauvergne-Ortmann-Virág)**

As $\varepsilon \to 0$,

$$
\varepsilon^{1/3} \left( B[(2\varepsilon^{-2/3} x, \varepsilon^{-1}) \to (\varepsilon^{-1} + 2\varepsilon^{-2/3} y, 1)] - 2\varepsilon^{-1} + 2(x - y)\varepsilon^{-2/3} \right)
$$

converges in distribution to the Airy sheet $\mathcal{S}(x, y)$ as a continuous function on $\mathbb{R}^2$ uniformly on compact sets.

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The scaling exponents $\frac{1}{3}$ and $\frac{2}{3}$ are characteristic of the Kardar-Parisi-Zhang universality class.
Main result: Colored ASEP to Airy sheet

Recall

\[ h^{\text{ASEP}}(x, 0; y, t) = \# \text{ particles of initial position } \leq x \text{ to right of } y \text{ at time } t. \]

**Theorem (Aggarwal-Corwin-H.)**

Fix \( q \in [0, 1) \) and \( \alpha = 0 \). The rescaled colored ASEP height function

\[
\varepsilon^{1/3} \left( \varepsilon^{-1} + 2(x - y)\varepsilon^{-2/3} - 2h^{\text{ASEP}}(2\varepsilon^{-2/3}x, 0; 2\varepsilon^{-2/3}y, 2\varepsilon^{-1}(1 - q)^{-1}) \right)
\]

converges in distribution, as \( \varepsilon \to 0 \), to the Airy sheet \( S(x, y) \) as continuous functions on \( \mathbb{R}^2 \) uniformly on compact sets.

The case of general \( \alpha \in (-1, 1) \) holds too, with explicit \( \alpha \)-dependent scaling coefficients.
Colored S6V model

Introduced by [Jimbo ’86], [Bazhanov ’85], [Gwa-Spohn ’93], [Kuniba-Mangazeev-Maruyama-Okado ’16], [Borodin-Wheeler ’22].

Quantum parameter $q \in [0, 1)$, spectral parameter $z \in (0, 1)$. At most one arrow per edge.

\[
\begin{array}{|c|c|c|c|c|}
\hline
i & j & i & j & i \\
\hline
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & q(1-z) & 1-z & 1-q & z(1-q) \\
\frac{1}{1-qz} & \frac{1}{1-qz} & \frac{1}{1-qz} & \frac{1}{1-qz} \\
\hline
\end{array}
\]

Simulation by Leo Petrov
The colored S6V height function $h^{S6V}(x, 0; y, t)$ is the number of arrows of color $\geq x$ exiting horizontally from vertical line $t$ at height $y$ or higher.

**Theorem (Aggarwal-Corwin-H.)**

Fix $q \in [0, 1)$, $z \in (0, 1)$, $\alpha \in (z, z^{-1})$. For explicit scaling coefficients $\mu$, $\sigma$ and $\beta$ (depending on $\alpha$), the rescaled colored S6V height function

$$
\sigma^{-1} \varepsilon^{1/3} \left( h^{S6V}(\beta \varepsilon^{-2/3} x, 0; \alpha \varepsilon^{-1} + \beta \varepsilon^{-2/3} y, \varepsilon^{-1}) - \mu \varepsilon^{-1} - \mu' \beta(y-x) \varepsilon^{-2/3} + \beta x \varepsilon^{-2/3} \right)
$$

converges in distribution, as $\varepsilon \to 0$, to the Airy sheet $S(x, y)$ as continuous functions on $\mathbb{R}^2$ uniformly on compact sets.
Proof ingredients
The Airy line ensemble $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots)$ is an $\mathbb{N}$-indexed collection of random non-intersecting curves on $\mathbb{R}$ [Prähofer-Spohn ’02, Corwin-Hammond ’14]:

It arises as the edge scaling limit of Dyson Brownian motion.

$\mathcal{S}$ was defined by Dauvergne-Ortmann-Virág via an infinite LPP problem in $\mathcal{P}$. 
The path from Brownian LPP to the Airy sheet

Br LPP → RSK → Dyson BM

Airy sheet

Parabolic Airy line ensemble

infinite LPP

edge scaling

RSK isn’t applicable to S6V or ASEP.
The path from Brownian LPP to the Airy sheet

Br LPP \rightarrow RSK \rightarrow \text{Dyson BM}

\text{Airy sheet} \rightarrow \text{edge scaling} \rightarrow \text{Parabolic Airy line ensemble}

\text{infinite LPP} \\

\text{RSK isn’t applicable to S6V or ASEP.}
The colored $q$-Boson model is a colored vertex model on a semi-infinite strip of fixed height. It allows \textit{arbitrarily} many arrows on \textit{vertical} edges.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
& $j$ & $i$ \\
\hline
1 & $(1 - q^A_i)q^{A_{i+1,N}}$ & 1 \\
\hline
$i$ & $j$ & $i$ \\
\hline
$(1 - q^A_j)q^{A_{j+1,N}}$ & 0 & $q^{A_{i+1,N}}$ \\
\hline
\end{tabular}
\end{table}

Arrows enter at $-\infty$ and travel \textit{straight} except for finitely many columns.
The Yang-Baxter equation

The colored $q$-Boson model and the colored S6V model are related via the Yang-Baxter equation.

$$\sum_{\substack{b_1,j_1 \in [0,N], \\ K \in \mathbb{Z}^N_{\geq 0}}} a_1 \ b_1 \ j_1 \ b_2 = \sum_{\substack{b_1,j_1 \in [0,N], \\ K \in \mathbb{Z}^N_{\geq 0}}} \ i_1 \ j_1 \ b_2$$

Gives a way to manipulate vertex models graphically while preserving partition functions/distributions, and is the source of “integrability.”
A matching via Yang-Baxter

\[ \mathcal{R}_1 = \mathcal{R}_2 \]

So the colored $S^6V$ height function is distributed as colored arrow counts in the last column of $q$-Boson. The uncolored case was shown in [Borodin-Bufetov-Wheeler '16], and the colored case in [Aggarwal-Borodin '24].
A matching via Yang-Baxter

\[ R_1 R_2 = \begin{array}{c} \text{frozen, trivial weight} \\
\vdots \\
R_2 \\
\end{array} \]

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The colored Hall-Littlewood line ensemble from the colored $q$-Boson model

Colored line ensemble $L^{\text{col}} = (L^{(1)}, \ldots, L^{(N)})$, with $L^{(k)} = (L^{(k)}_1, L^{(k)}_2, \ldots)$ a line ensemble defined by

$$L^{(k)}_i(y) = \# \{ y' > y : \text{color exiting horizontally from } (-i, y') \text{ is } \geq k \}.$$
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$$L^{(k)}_i(y) = \# \left\{ y' > y : \text{color exiting horizontally from } (-i, y') \text{ is } \geq k \right\}.$$

Yang-Baxter: $L^{(k)}_1$ is distributed as the color $k$ height function $h^{S6V}(k, 0; \cdot, t)$. 
Recall that $\mathcal{S}$ is represented as a last passage percolation (LPP) problem in the parabolic Airy line ensemble $\mathcal{P}$.

Proving convergence to the Airy sheet comes down to two main components:

1. Show colored height function (i.e., $L^{(k)}_1$) is approximately LPP in $L^{(1)}$.
2. Show convergence of $L^{(1)}$ to $\mathcal{P}$.

The colored and uncolored line ensembles each have Gibbs properties. Colored Gibbs is the tool for (1) and uncolored Gibbs the tool for (2).
An approximate LPP representation

Given $L^{(1)}$, $L^{(k)}$ satisfies a (colored Hall-Littlewood) Gibbs property. Can be represented in terms of a variational problem: when $q = 0$, it holds that

$$L_i^{(k)} = \text{PT} \left( L_i^{(1)}, L_{i+1}^{(k)} \right),$$

$$\text{PT} \left( f, g \right) (x) = f(x) + \max_{0 \leq y \leq x} \left( g(y) - f(y) \right),$$

and for $q > 0$,

$$\mathbb{P} \left( \max_y \left| L_i^{(k)}(y) - \text{PT} \left( L_i^{(1)}, L_{i+1}^{(k)} \right) (y) \right| \geq m \right) \leq q^{cm^2}.$$
The (uncolored) Hall-Littlewood Gibbs property of $L^{(1)}$
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$\mathcal{P}$ has **Brownian Gibbs** property: given by **non-intersecting** Brownian bridges.
The (uncolored) Hall-Littlewood Gibbs property of $L^{(1)}$

$L^{(1)}$’s Gibbs property is more complicated.

\[ \Delta_0(x + 1) = 2 \]

\[ \Delta_1(x) = 1 \]

Law of top $k$ curves of $L^{(1)}$ on $[a, b]$ is a collection of non-crossing Bernoulli random walk bridges, reweighted by a RN derivative

\[
\prod_{i=0}^{k} \prod_{x=a+1}^{b} \left(1 - q^{\Delta_i(x-1)} \mathbb{1}_{\Delta_i(x) = \Delta_i(x-1)-1}\right),
\]

where $\Delta_i(x)$ is separation of $(i - 1)^{\text{st}}$ and $i^{\text{th}}$ curve at $x$ [Corwin-Dimitrov ’18].
Convergence of $L^{(1)}$ to $\mathcal{P}$ and a lack of monotonicity

Showing $L^{(1)} \rightarrow \mathcal{P}$ comes down to establishing

1. tightness of $L^{(1)}$ at the edge, and
2. showing all subsequential limits have Brownian Gibbs.

Then can use [Aggarwal-Huang ’23] which characterizes $\mathcal{P}$ as the unique law among Brownian Gibbsian line ensembles with parabolic decay of $-x^2$. 
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Many works have proved tightness of line ensembles, but all rely heavily on monotone coupling properties of the line ensembles.

These do not exist for the Hall-Littlewood line ensemble!

We give a new proof framework for tightness using only “weak monotonicity” of partition functions [Corwin-Dimitrov ’18].
Next directions

- Including time in the scaling limit (✓ for ASEP and S6V)
- Scaling limit under general initial conditions (✓ for ASEP)
- Extend to other models
- Use to investigate other phenomena, e.g. mixing times, stationary measures, scaling limits of particle trajectories...
Summary

- Colored ASEP and colored S6V height functions converge to the Airy sheet, directed landscape, KPZ fixed point.
- Use Yang-Baxter to relate colored height functions with colored line ensembles defined via the colored $q$-Boson model.
- Colored Gibbs property $\Rightarrow$ approximate LPP representation:
  \[
P \left( \max_y \left| L_i^{(k)}(y) - \text{PT} \left( L_i^{(1)}, L_{i+1}^{(k)} \right)(y) \right| \geq m \right) \leq q^{cm^2}.
  \]
- Line ensemble tightness via only uncolored Gibbs & weak monotonicity.
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Thank you!