The Tychonoff theorem

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2020/12/16

This paper has been submitted as a final project for MATH GU4051: Topology under the instruction of Dr. S. Michael Miller in the Fall 2020 term at Columbia University. Most of the proofs and the explanations of concepts provided in this expository paper are not entirely original, but based on the sources referenced in and after the paper.
Audience: students who have learned the first half semester of MATH GU4051: Topology or students who have read Hatcher's Notes on Introductory Point-Set Topology.

Contents

1 Preparation

We have seen in class that the product of finitely many compact spaces with product topology is compact. One might ask, why finitely many? Is it true for infinitely many factor case? It turns out that this statement not only works for finitely many product case, but also works for the arbitrary product case, which gives us Tychonoff's theorem.

Theorem 1. *Tychonoff's theorem: An arbitrary product of compact space is compact in the product topology.*

With Tychonoff's theorem, we can get some interesting results. For example, the topological space $[0,1]^{[0,1]}$ with product topology is compact as a product of compact space $[0,1]$. In this paper, we will give two proofs of Tychonoff's theorem and show that it is actually equivalent to the Axiom of Choice. The first method of showing Tychonoff's theorem I will give follows [1][1] written by David Wright and [2][2] written by Ken Brown; the second method of showing Tychonoff's theorem I will give follows Chapter 5 of [3] which is the the second edition of Munkres' Topology.

1.1 Basic concepts

To make sense of the statement, we must first recall the definition of two topologies on product space and compactness.

1.1.1 Product topology vs. box topology

Definition 1. *Let $X_i, i \in I$ be a family of topological spaces. The product topology on the Cartesian product $\prod_{i\in I} X_i$ is generated by the basis $\{\prod_{i\in I} U_i | U_i$ is open in X_i and $U_i = X_i$ for all but finitely many $i \in I\}$.*

Definition 2. *Let $X_i, i \in I$ be a family of topological spaces. The box topology on the Cartesian product $\prod_{i\in I} X_i$ is generated by the basis $\{\prod_{i\in I} U_i | U_i$ is open in $X_i\}$.*

1.1.2 Compactness

Definition 3. *A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if $\bigcup_{S\in\mathcal{A}} S = X$. It is called an open cover of X if its elements are open subsets of X.*

Definition 4. *Let $\mathcal{W}$ be a collection of subset of some space X, X is said to be finitely covered by $\mathcal{W}$ if there's a finite subcollection of $\mathcal{W}$ covers X.*

Definition 5. *A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*

We can see that Tychonoff's theorem is not true in the box topology. Intuitively, that is because box topology is finer than product topology, it is more likely that we can find an open cover that doesn't have finite subcover. To see box topology is finer than product topology, here's an example: infinite product of discrete space is not discrete in the product topology, while it is discrete equipped with the box topology.

Example 1. $\{0,1\}^{\mathbb{N}}_{disc}$ is infinite discrete in the box topology, each individual singleton is an open set. Pick the collection contains all the singletons in this space, we can see this open cover doesn't have a finite subcover, thus, $\{0,1\}^{\mathbb{N}}_{disc}$ with the box topology is not compact.

Now, let's recall a theorem from the lecture.

Theorem 2. *If X and Y are compact topological space, then $X \times Y$ is compact with product topology.*

We proved this in class using the tube lemma, but an important step in the tube lemma approach is that there's finitely many factors in the product space. In order to generalize the proof of this "baby Tychonoff's theorem" to the real Tychonoff's theorem, we will give an alternate proof in the next section. We saw that the product of two compact spaces equipped with product topology is compact, following the similar argument, we could show that this result is true for finitely many products of compact space. Our goal was to prove Tychonoff's theorem, which can be seen as a more complicated case. So now, we wonder if we could generalize our theorem to a more complicated case.

The first thing came to our mind is induction. However, the induction we are familiar with is only valid for inferring statement involving natural numbers or statement indexed by countable index sets. To prove Tychonoff's theorem, clearly we need some more powerful techniques. But we could still try and see if the induction we are familiar with can be used to prove the countable case of Tychonoff's theorem, and it turns out that the simple induction works for proving the countable case of Tychonoff's theorem.

[1] https://www.ams.org/journals/proc/1994-120-03/S0002-9939-1994-1170549-2/S0002-9939-1994-1170549-2.pdf
[2] http://pi.math.cornell.edu/ kbrown/4530/tychonoff.pdf

1.2 First Step Generalization

In this section, we will prove the countable case of Tychonoff's theorem by induction.

1.2.1 Method for showing compactness

Usually we prove compactness by the definition. Here, instead of using the definition of compactness, we use the following lemma.

Lemma 3. *X is compact iff for every collection $\mathcal{W}$ of open subsets of X which does not finitely cover X, $\mathcal{W}$ does not cover X either.*

This lemma is clearly true since from the definition of compactness, we can see that "X is compact" and "let $\mathcal{W}$ be a collection of open subsets of X such that $\mathcal{W}$ does not finitely cover X, then $\mathcal{W}$ does not cover X" are contrapositive of each other.
So here our method is to use lemma 3, that is, let $\mathcal{W}$ be a collection of open subsets of X such that there's no finite subcover, we will show $\mathcal{W}$ does not cover X. To help understand how we use it, we first give the proof of theorem 2 via this method. For the finitely many product case, this proof is just a rephrasing of tube lemma approach, but this method allows us to generate the proof to the more complicated cases, as we desired.

Proof. Given any collection $\mathcal{W}$ of open sets in $X \times Y$ that doesn't have a finite subcover, we want to show $\mathcal{W}$ doesn't cover X either.

- Step 1 (Claim 1 of [2]): As an intermiediate step, we want to show there must be a $x_0 \in X$ such that for its any open neighborhood U, $U \times Y$ is not finitely covered by $\mathcal{W}$.

  Proof. Assume, towards a contradiction, there's no such $x_0 \in X$. This means that for every $x \in X$, $\mathcal{W}$ finitely covers $U_x \times Y$, where U_x is some open set contains x.
  Then we can find an open neighborhood U_x around any $x \in X$, such that $U_x \times Y$ is finitely covered by $\mathcal{W}$. Such open neighborhood around every $x \in X$ form a open cover of X, call it $\mathcal{U}$. By the compactness of X, we know $\mathcal{U}$ has a finite subcover $\{U_1, U_2, \cdots, U_n\}$.
  Each $U_i \times Y$ is finitely covered by $\mathcal{W}$, and $\{U_1 \times Y, U_2 \times Y, \cdots, U_n \times Y\}$ covers $X \times Y$. This implies that $X \times Y$ is finitely covered by $\mathcal{W}$, which contradicts with our assumption that $\mathcal{W}$ doesn't have a finite subcover.
  Therefore, there must be a $x_0 \in X$ such that for its any open neighborhood U, $U \times Y$ is not finitely covered by $\mathcal{W}$. ☐

- Step 2 (Claim 2 of [2]): Recall the definition of x_0 in Step 1, we want to show there must be a $y_0 \in Y$ such that any open set $U \times V$ contains (x_0, y_0) is not finitely covered by $\mathcal{W}$

  Proof. Assume towards contradiction, there's no such $y_0 \in Y$.
  This means that for every $y \in Y$, $\mathcal{W}$ finitely covers $U_y \times V_y$, where $U_x \times V_y$ is some open set contains (x_0, y). Such V_y around every $y \in Y$ form an open cover of Y, call it $\mathcal{V}$. By the compactness of Y, $\mathcal{V}$ has a finite subcover, so there's a finite subset $K \subset Y$ such that $Y = \bigcup_{y \in K} V_y$. Now let $U = \bigcap_{y \in K} U_y$.
  We have $U \times Y = \bigcap_{y \in K} U_y \times \bigcup_{y \in K} V_y \subset \bigcup_{y \in K} U_y \times V_y$.
  Since K is a finite subset of Y and each $U_y \times V_y$ is finitely covered by $\mathcal{W}$, so $\bigcup_{y \in K} U_y \times V_y$ is finitely covered by $\mathcal{W}$, thus $U \times Y$ is finitely covered by $\mathcal{W}$. And we know $x_0 \in U$ since $x_0 \in U_y$ for each $y \in K$.
  Notice that this contradicts with our Step 1.
  Therefore, there must be a $y_0 \in Y$ such that any open set $U \times V$ contains (x_0, y_0) is not finitely covered by $\mathcal{W}$. ☐

By Step 2, we know there's a $(x_0, y_0) \in X \times Y$ such that none of its open neighborhood is finitely covered by $\mathcal{W}$. In particular, there's no open set in $\mathcal{W}$ covers (x_0, y_0). This implies that $\mathcal{W}$ doesn't cover (x_0, y_0). Because if (x_0, y_0) is in some open set $W \in \mathcal{W}$, there will be a basic open set $U \times V$ with $(x_0, y_0) \in U \times V \subset W$, which contradicts with the result of Step 2.
Therefore,(x_0, y_0) is not covered by $\mathcal{W}$, and so $X \times Y$ is not covered by $\mathcal{W}$ as desired. ☐

Now, we have a sense of how to use this method, let's apply it to the simple version of Tychonoff's theorem, which is the case of countably many product of compact spaces with product topology.

1.2.2 The proof for countable case of Tychonoff's theorem

It might be weird that for the "baby Tychnnoff's theorem", we rephrase the tube lemma approach and results in a contradiction at the end. But the contradiction at the end is actually the inspiration of proving more complicated case.
Here I will explain how we use this strategy to prove the countable case of Tychonoff's theorem.
Let $\mathcal{W}$ be a collection of open subset of the product space $X = \prod_{i\in\mathbb{N}} X_i$. Our idea is to assume towards contradiction that $\mathcal{W}$ covers X, so for every point in the product space $X = \prod_{i\in\mathbb{N}} X_i$, it has a basic open neighborhood being finitely covered by $\mathcal{W}$, and we know for this basic open neighborhood, there must be a largest $m \in \mathbb{N}$ such that the $m-th$ term factor $U_m \neq X_m$. And we will show this is impossible by constructing a point $(x_i)_{i\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, there's no basic open set of the form $U_1 \times \cdots U_n \times \prod_{i\in\mathbb{N}\setminus\{1,\cdots,n\}} X_i$ contains $(x_i)_{i\in J}$ is finitely covered by $\mathcal{W}$.
To simplify our proof, we first make a lemma:

Lemma 4. (Lemma in [2]) *Let $\mathcal{W}$ be a family of open sets in a product $X \times Y \times Z$. Assume there is a point $x_0 \in X$ such that no open set $U \times Y \times Z$ with $x_0 \in U$ is finitely covered by $\mathcal{W}$. If Y is compact, then there is a point $y_0 \in Y$ such that no open set $U \times V \times Z$ with $(x_0, y_0) \in U \times V$ is finitely covered by $\mathcal{W}$.*

The proof of this lemma is pretty much using the same idea as Step 2 above, because we can see that basically the space Z doesn't really affect anything. So that we can assume towards contradiction that there's no such $y_0 \in Y$, then by the compactness of Y, we get to the contradiction with the fact that there is a point $x_0 \in X$ such that no open set $U \times Y \times Z$ with $x_0 \in U$ is finitely covered by $\mathcal{W}$.

Now let's use what we have so far and induction to show the countable case of Tychonoff's theorem.

Theorem 5. (The 2nd Theorem in [2]) *Let $\{X_i | i \in \mathbb{N}\}$ be a countable collection of compact topological spaces, then the product $\prod_{i\in\mathbb{N}} X_i$ with product topology is also compact.*

Proof. Given any collection of open set $\mathcal{W}$. We want to show there's an $(x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i$ such that there's no basic open set in the form of $U_1 \times U_2 \times \cdots \times U_n \times \prod_{i\in\mathbb{N}, i>n} X_i$ with $(x_1, \cdots x_n) \in U_1 \times \cdots \times U_n$ is finitely covered by $\mathcal{W}$ for each $n \in \mathbb{N}$.

- Base Case: Let $X = \prod_{i\in\mathbb{N}} X_i$, $Y = \prod_{i\in\mathbb{N}\setminus\{1\}} X_i$, then we can see that $X = X_1 \times Y$. As we showed in Step 1 in the proof of theorem 2, there's an $x_1 \in X_1$ such that there's no open set in the form of $U_1 \times Y = U_1 \times \prod_{i\in\mathbb{N}\setminus\{1\}} X_i$ with $x_1 \in U_1$ is finitely covered by W.

- Induction Hypothesis: Assume there's an $(x_1, \cdots x_n) \in X_1 \times \cdots \times X_n$ such that there's no open set in the form of $U_1 \times U_2 \times \cdots \times U_n \times X_{n+1} \times \cdots$ with $(x_1, \cdots x_n) \in U_1 \times \cdots \times U_n$ is finitely covered by $\mathcal{W}$ for some $n \in \mathbb{N}$.

- Induction Step: Now, let $A = X_1 \times \cdots \times X_n$, $C = \prod_{i\in\mathbb{N}\setminus\{1,\cdots n+1\}} X_i$, then $\prod_{i\in\mathbb{N}} X_i = A \times X_{n+1} \times C$. Since X_{n+1} is compact, by lemma 4 and our induction hypothesis, we know there's an $(x_1, \cdots x_{n+1}) \in X_1 \times \cdots \times X_{n+1}$ such that there's no open set of the form $U_1 \times U_2 \times \cdots \times U_i \times \prod_{i\in\mathbb{N}\setminus\{1,\cdots n+1\}} X_i$ with $(x_1, \cdots x_{n+1}) \in U_1 \times \cdots \times U_{n+1}$ is finitely covered by $\mathcal{W}$.

So now, we have a way to construct an element $(x_1, x_2, \cdots) \in \prod_{i\in J} X_i$ such that there's no basic open set contains $(x_1, x_2, \cdots)$ is finitely covered by $\mathcal{W}$, which shows this element $(x_1, x_2, \cdots)$ is not covered by $\mathcal{W}$ as desired. So we conclude that $\prod_{i\in\mathbb{N}} X_i$ is compact. □

Here we used induction and the contrapositive to show the countable case of Tychonoff's theorem is true. But we can see that because of the "countability" of mathematical induction we are familiar with, in order to prove Tychonoff's theorem, we need to either improve our technique, or seek for other ideas.

1.3 Theorems needed to prove Tychonoff's theorem

1.3.1 Set theory

Axiom 1. *Axiom of choice: Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, a set C such that C is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.*

Here we are dealing with the arbitrary product of compact spaces, the Axiom of Choice is important for proving Tychonoff's theorem. Notice that in the countable case, we are using the fact that there's a bijection between our collection of compact spaces and $\mathbb{N}$, and the properties of $\mathbb{N}$ (in particular, well-ordering principle). But now, for the more general case, if we want to use techniques like induction, we have to make sure we know there is a next element and what that element is, here's where we need the Axiom of Choice and its beautiful consequences. In fact, we need to prove Tychonoff's theorem under the assumption that the axiom of choice is true. Furthermore, Tychonoff's theorem is equivalent to the axiom of choice, and we will elaborate on this in the last section.

Definition 6. *A relation C on a set A is called order relation(simple order) if it has the following properties:*

- *Comparability: For every x and y in A for which $x \neq y$, either xCy or yCx.*
- *Nonreflexivity: For no x in A does the relation xCx hold.*
- *Transitivity: If xCy and yCz, then xCz.*

And a set is said to be simply ordered if there's an order relation on it.

We can see that $<$ and $>$ on $\mathbb{R}$ are order relations. But there's a lot of other order relations:

Example 2. The relation $\ll$ on $\mathbb{R}$: $x \ll y$ if $x^2 < y^2$ or if $x^2 = y^2$ and $x > y$ is an order relation, $\mathbb{R}$ is simply ordered with $\ll$.

Example 3. The relation C on $\mathbb{Z} \times \mathbb{Z}$: $(m,n)C(s,t)$ if $m < s$ or $m = s, n < t$. For instance, $(0,-1)C(0,0)$, $(0,10)C(1,0)$. This is also an order relation and it is called the dictionary order on $\mathbb{Z} \times \mathbb{Z}$.

Definition 7. *A well-ordering(or well-ordered relation) on a set S is an order relation on S with the property that every non-empty subset of S has a least element in this ordering. A set A with an well-ordering is called a well-ordered set.*

Example 4. $\mathbb{N}$ with order relation $<$ is well ordered; $\mathbb{Z}$ with $<$ is not because there's no least element of $\mathbb{Z}$.

Example 5. Let the relation $\ll$ on $\mathbb{Z}$ be $m \ll n$ if $m^2 < n^2$ or $m^2 = n^2, m < n$. That is saying $\mathbb{Z}$ is in the order of $0 \ll -1 \ll 1 \ll -2 \ll 2 \cdots$. $\mathbb{Z}$ with $\ll$ is well ordered.

Theorem 6. *Well-ordering theorem: If A is a set, there exists an order relation on A that is well-ordering.*

That says, for every set, there's an order relation on it makes the set well-ordered. This theorem is a consequence of the Axiom of Choice (in fact, is equivalent to the Axiom of Choice), which was proved by Zermelo in 1904. Here we wouldn't give the rigorous proof, since it is long and include some more set theoretic concepts. Most people think the Axiom of Choice is trivially true and the well-ordering theorem is obviously false, but it turns out that these two are equivalent to each other. Sometimes intuition helps us understand concepts but we couldn't fully rely on it, we need rigorous logic too.

Definition 8. *Given a set A. A relation $\prec$ is called a strict partial order on A if it has the following properties:*

- *Nonereflexivity: The relation $x \prec x$ never holds.*
- *Transitivity: if $x \prec y$, $y \prec z$, then $x \prec z$.*

Notice that these are the second and third property of order relation. So if A is a set with a strict partial order relation, it is possible that some subset B of A is simply ordered with this relation, it just requires that every pair of elements in B are comparable under this relation.

Example 6. If (x_1, y_1) and (x_2, y_2) are two points in $\mathbb{R}^2$, define $(x_1, y_1) \prec (x_2, y_2)$ if $y_1 = y_2, x_1 < x_2$. This is a partially strict order on $\mathbb{R}^2$. Different points are comparable if and only if they are on the same horizontal line. So the maximal simply ordered subsets are horizontal lines in $\mathbb{R}^2$.

This discussion leads to the following principle:

Theorem 7. *The Maximum Principle: Let A be a set, let $\prec$ be a strict partial order on A. Then there exists a maximal simply ordered subset B of A.*

The maximal here doesn't necessarily mean the subset B is the simply ordered subset with the largest cardinality, instead, it means that there's no simply ordered subset of A properly contains B. To see this, consider the following example:

Example 7. Consider the set $\{1, 2, 3, 4\}$ with the order $\prec$ such that $a \prec b$ if a is a divisor of b and $a \neq b$. One can check that $\prec$ is a strict partial order. We can see that $\{1, 3\}$ and $\{1, 2, 4\}$ are both maximal simply ordered subset of $\{1, 2, 3, 4\}$ since both the pair 2, 3 and 3, 4 are not comparable under $\prec$. But apparently $\{1, 3\}$ is not the simply ordered subset with the largest cardinality.

This theorem was first formulated by Hausdorff in 1914. Here we give an intuitive explanation of the maximum principle, this explanation follows Chapter 1 Section 11 of [3], which is the second edition of Munkres' Topology.

Imagine one has a set A with a partial strict order and to put elements of A a box. First we could pick an element and put it in the box. Then we pick another element from the set, if it is comparable with the first element, then we put it in the box; otherwise, we throw it away.

We proceed this process, in each step, we compare the element we pick with every element in the box, if it is comparable with every one of them, then we put it in the box; otherwise, we throw it away.

Keep doing this till we have checked all the elements in the set. The elements in the box form the maximal simply ordered subset as we desired.

This explanation is rigorous if our set is finite, but is not always rigorous because we couldn't always say "keep doing this till we have checked all the elements". Here's where we need induction and even more powerful technique–transfinite induction–we will talk about this technique in the latter section.

Definition 9. *Let A be a set and let $\prec$ be a strict partial order on A. If B is a subset of A, an upper bound of B is an element c of A such that for every element b in B, either $b = c$ or $b \prec c$. A maximal element of A is an element m of A such that for no element a of A does the relation $m \prec a$ hold.*

Here, notice that based on the definition of upper bound, c must be comparable to every element in B to be the upper bound of B.

Example 8. Again, consider the set $\{1, 2, 3, 4\}$ with the order $\prec$ such that $a \prec b$ if a is a divisor of b and $a \neq b$. 4 is the upper bound of $\{1, 2, 4\}$, but not the upper bound of $\{1, 2, 3, 4\}$, because 4 is not comparable to 3. And we can see that 3 and 4 are both maximal elements.

Lemma 8. *Zorn's lemma: Let A be a set that's strict partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.*

Zorn's lemma is an consequence of the maximum principle.

Proof. The maximum principle says we can find a maximal subset B of A that is simply ordered, then by the hypothesis of Zorn's lemma, B has an upper bound, call it c. Then this c is a maximal element of A. To see why this is the case, think about it by contradiction:

If there's an element $d \in A$ such that $c \prec d$. Then since c is the upper bound of B, $d \notin B$. By the transitivity of $\prec$, from $c \prec d$, $b \prec c$ for every $b \in B$, we know that $b \prec d$ for every $d \in B$. So then this implies that $B \cup \{d\}$ is a simply ordered subset of A.

$B \subset B \cup \{d\}$, which contradicts the fact that B is a maximal simply ordered subset of A. □

1.3.2 Finite intersection property

Definition 10. *A collection $\mathcal{C}$ of subsets of X is said to have finite intersection property (or FIP for short) if for every finite subcollection $\{C_1, \cdots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.*

Theorem 9. (Theorem 26.9 of [3]) *A topological space X is compact iff every collection $\mathcal{K}$ of closed subset of X with FIP has the property that the intersection of all the sets in $\mathcal{K}$ is non-empty.*

Proof. Here we recall lemma 3. Note that if X is compact, we have if a collection $\mathcal{W}$ of open subsets that doesn't cover X, then $\mathcal{W}$ does not cover X.

- compact $\Rightarrow$ closed collection with FIP has the property of all sets in $\mathcal{O}$ is nonempty

  Proof. Given any collection $\mathcal{K}$ of closed subsets of X with FIP. Take the complement of each closed subset in $\mathcal{K}$; we get a collection of open subsets of X, call it $\mathcal{W}$. Since $\mathcal{K}$ has the finite intersection property, we can see that any finite subcollection of $\mathcal{W}$ doesn't cover X since the intersection of their complements is not empty. That says $\mathcal{W}$ has no finite subcover, then by the compactness of X and lemma 3, we know $\mathcal{W}$ doesn't cover X, which is equivalent of the intersection of all sets in $\mathcal{K}$ is nonempty. □

- closed collection with FIP has the property of all sets in $\mathcal{O}$ is nonempty $\Rightarrow$ compact

  Proof. Given any collection $\mathcal{O}$ of open subsets of X. Find the complement of each open subset in $\mathcal{O}$, they form a collection of closed subset of X, call it $\mathcal{K}$.
  Here we use lemma 3 to show compactness again. If X is not finitely covered by $\mathcal{O}$, then we know any finite union of sets in $\mathcal{O}$ does not cover X, which means any finite intersection of sets in $\mathcal{K}$ is not empty, thus $\mathcal{K}$ has finite intersection property.
  Then by our hypothesis, the intersection of all sets in $\mathcal{K}$ is not empty, which is equivalent to say the union of all sets in $\mathcal{O}$ is not X, thus $\mathcal{O}$ doesn't cover X.
  Therefore, by lemma 3, X is compact. □

□

2 Proofs

2.1 The first method of proving Tychonoff's theorem

As we discussed above, we need to either improve the technique or seek for new ideas, our first method of proving Tychonoff's theorem is came up with by improving our technique–we generalized the mathematical induction from countable index sets to arbitrary index sets.

2.1.1 Transfinite Induction

Transfinite induction is valid for well-ordered sets. Similar to the simple induction that we are familiar with, we want to use the fact that the statement is true for the "smaller" elements, to prove the statement is true for the "next" or the "larger" elements. To introduce transfinite induction, we shall first introduce the ordinal number.
Ordinal numbers can be thought of the "position" of elements in some well-ordered set. There are several equivalent definitions of ordinal numbers, here we will follow the definition of von Neumann ordinals, whose main idea is "each ordinal is the well-ordered set of all smaller ordinals."

Definition 11. *Ordinal number: A set S is an ordinal number if and only if S is well-ordered and every element of S is also a subset of S.*

This might seem confusing, in order to make it more intuitive, we construct natural numbers in the following way so that natural numbers are ordinal numbers, this construction is known as von Neumann construction.
$\varnothing$ satisfies the definition of being an ordinal number: there's no element in it, so it is vacuously true that it is simply ordered; there's no non empty subset of it, so it is well-ordered; Also, there's no elements in it, so it is vacuously true that every element of it is a subset of it. We conclude that $\varnothing$ is an ordinal number, so we define:

- $0 := \varnothing$
- $1 := 0 \cup \{0\} = \varnothing \cup \{0\} = \{0\} = \{\varnothing\}$
- $2 := 1 \cup \{1\} = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$
- $n + 1 := n \cup \{n\}$ for all $n \in \mathbb{N}$

The order relation on these ordinals is membership inclusion $\in$.
After constructing $\mathbb{N}$, we extend this construction (still with order relation $\in$) and continue with:

- $\omega := \mathbb{N} = \{0, 1, 2, 3, \cdots\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\} \cdots\}$, which we can see is the union of every element in ω.
- $\omega + 1 := \omega \cup \{\omega\} = \mathbb{N} \cup \{\omega\} = \{0, 1, 2, 3, \cdots\} \cup \{\omega\} = \{0, 1, 2, 3, \cdots\} \cup \{\{0, 1, 2, 3, \cdots\}\}$
- $\omega + n + 1 := \omega + n \cup \{\omega + n\}$

And eventually, we have $\omega 2 := \{1, 2, 3, \cdots\} \cup \{\omega + 1, \omega + 2, \omega + 3, \cdots\}$[3]. Then followed by this, there's $\omega 2 + 1$, $\omega 2 + 2 \cdots$, $\omega 3$, $\omega 4$, $\cdots$
Furthermore, we have ω^2, then later we even get to ω^ω. In fact, all the ordinal numbers we mentioned till now are countable sets. So we define ω_1 to be the first uncountable ordinal number we meet following this construction, ω_2 to be the first ordinal number whose cardinality exceeds the cardinality of ω_1, and so on.
One might have already observed that the ordinal numbers like 0 or ω are slightly different than others. We know that except for 0, which is the smallest ordinal number, others are all defined by smaller ordinal numbers. For example, we defined $2 = 1 \cup \{1\}$; however, we defined w in the way of $w = \mathbb{N}$, which is the set containing all the smaller ordinal numbers, and that is because we couldn't find the immediate predecessor of w. So we give the definition of these two kinds of ordinal numbers:

Definition 12. *An ordinal number β is called a successor ordinal if there is an ordinal number α such that $\beta = \alpha \cup \{\alpha\}$.*
An ordinal number γ is called a limit ordinal if $\gamma > 0$ and γ is not a successor ordinal.

Example 9. We can see that ω, ω^2, and ω^ω are all limit ordinals; all the positive natural numbers are successor ordinals; 0 is neither successor ordinal nor limit ordinal.

Using ordinals we could index well-ordered sets. And in fact, for every well-ordered set, we can index it with exactly one ordinal number, the proof of this fact can be found at theorem 1 of [4][4] written by Terence Tao. One might ask: why do we define so many different ordinals that are all countable, why don't we just use $\mathbb{N}$? Because well-ordered sets with the same cardinality are not necessarily "equivalent" in the sense of ordering, to see what the difference is, we shall introduce the definition of order type.

Definition 13. *Suppose A and B are two sets with order relation $<_A$ and $<_B$ respectively. We say that A and B have the same order type if there is a bijective correspondence between them that preserves order; that is, if there exists a bijection $f : A \to B$ such that $a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2)$.*

Example 10. The set of real numbers $\mathbb{R}$ with the order relation $<$ has the same order type as $(0, \infty)$ with the order relation $<$, the order-preserving bijection is $f : \mathbb{R} \to (0, \infty), f(x) = e^x$.

By the definition of having the same order type, we know if two sets have the same order type, they must have the same cardinality. But does the converse hold? For well-ordered finite sets with the same cardinality, we can always find a bijection which preserves order; however, for infinite sets, well-ordered sets with the same cardinality do not necessarily have the same order type.

Example 11. Let A be the set $\mathbb{N} \cup \{0'\}$. Define $<_1$ as $a <_1 b$ if $a, b \in \mathbb{N}, a < b$ or $a \in \mathbb{N}, b = 0'$ and $<_2$ by $a <_2 b$ if $a, b \in \mathbb{N}, a < b$ or $a = 0', b \in \mathbb{N}$. We can argue A with $<_1$ and A with $<_2$ have different order type: there's a maximal element in $(A, <_1)$, which is $0'$, because there's no element a in A such that $a <_1 0'$; but there's no maximal element in $(A, <_2)$. In fact, $(A, <_1)$ has the same order type as $\omega + 1$ and $(A, <_2)$ has the same order type of ω.

[3]Some people write it as $\omega \cdot 2$. And actually $2\omega \neq \omega 2$.
[4]https://terrytao.wordpress.com/2009/01/28/245b-notes-7-well-ordered-sets-ordinals-and-zorns-lemma-optional/

The above example explained why we need so many different ordinal numbers that are all countable: infinite well-ordered sets with the same cardinality could have different order type. And in fact, recall that for every well-ordered set we can index it with exactly one ordinal number, that actually means: every well-ordered set has the same order type with some ordinal number, and that ordinal number is unique for every well-ordered set.

Ordinal numbers are usually thought as the representation of order type of well-ordered sets or the representation of the positions of elements in well-ordered sets. As a set, ordinal numbers have order-preserving bijective correspondences with some well-ordered sets, so ordinal numbers can be used to describe the order type of well-ordered sets; however, they also represents the "position" of elements in well-ordered sets—we consider the order-preserving bijective correspondences between ordinals and well-ordered sets again, the elements of ordinals are actually smaller ordinals, and the position of those smaller ordinals represents the position of their corresponding elements.

Here are some examples about how to use ordinal numbers to describe order type of well-ordered sets and the position of elements:

Example 12. Let $A = \{2, 3, 4\}$, it's clear that A has the same order type as the ordinal number $3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\}$. We can see that the ordinal number 3 represents the order type of A; and the elements in 3 describes the position of their corresponding elements in A, for example, 0 is the 1st element in 3, which means 2 is the 1st one in A.

Example 13. Consider the set $\mathbb{N} \cup \{0^{'}\}$ with the order relation $<_1$ as defined before, this set has the same order type as $w + 1$; the corresponding element of $0^{'}$ in $\omega + 1$ is ω, which is the $(\omega + 1)st$ element in ordinal $\omega + 1$.

Now that after defining ordinal numbers, and having seen the fact that every well ordered set can be indexed by an ordinal number, we finally can generalize simple induction to transfinite induction.

Let A be a well-ordered set with order relation $<$ and $P(x)$ be a proposition with domain of the index set of A, which is the corresponding ordinal number, call it λ. We will use the following step to show P is true for all elements in λ:

- Base Case: Demonstrate $P(0)$ is true.
- Successor Step: Assume $P(\alpha)$ is true for arbitrary α, prove $P(\alpha \cup \{\alpha\})$ is true.
- Limit Step: For all limit ordinals γ, assuming $P(\alpha)$ is true for all ordinals $\alpha < \gamma$, prove $P(\gamma)$ is true.

To see why transfinite induction proves P is true for all elements in λ, assume towards contradiction, there's some ordinal $\alpha \in \lambda$ such that $P(\alpha)$ fails, we can find a subset C of λ whose elements are such ordinals, and C is not empty. Since we know λ is well-ordered, then C has a least element β. The fact that the proposition for ordinals smaller than β are true but $P(\beta)$ fails contradicts with either the successor step or the limit step.

Recall that we needed a more powerful technique for proving the maximum principle. Now after we introduced transfinite induction, we can now use the transfinite induction to give a rigorous proof of the maximum principle following the idea we gave in the non-rigorous explanation.

2.1.2 The proof of Tychonoff's theorem

By the well-ordering theorem, we know the set of arbitrarily many compact spaces is well-ordered with some order relation, so we can index it by some ordinal J, and we denote the order on J by $<$.

Proof. Given any collection $\mathcal{W}$ of open sets that doesn't finitely cover $X = \prod_{\alpha \in J} X_\alpha$. Our main idea here is basically the same as that for the countable case, the only difference is that our index is now arbitrary, so we have to use transfinite induction to find the element $(x_\lambda)_{\lambda \in J}$ which satisfies the property that "none of its basic open neighborhood is finitely covered by $\mathcal{W}$".

That means we want to show there's a $(x_\lambda)_{\lambda \in J}$ such that for each $\alpha \in J$, there's no open set of the form $\prod_{\beta < \alpha} U_\beta \times X_\alpha \times \prod_{\alpha < \kappa} X_\kappa$ is finitely covered by $\mathcal{W}$. Note that $\prod_{\beta < \alpha} U_\beta$ here represents any basic open set in $\prod_{\beta < \alpha} X_\beta$, which is saying that $U_\beta = X_\beta$ for all but finitely many $\beta < \alpha$.

- Base Case: We want to show there's no basic open set of the form $\prod_{\beta \in J} X_\beta$ is finitely covered by $\mathcal{W}$. This is trivial since this follows from the construction of $\mathcal{W}$.

- Successor Step: Assume there's an ordinal γ such that there's an element $(x_\beta)_{\beta<\gamma}$ such that there's no basic open sets of the form $\prod_{\beta<\gamma} U_\beta \times X_\gamma \times \prod_{\gamma<\kappa} X_\kappa$ with $(x_\beta)_{\beta<\gamma} \in \prod_{\beta<\gamma} U_\beta$ is finitely covered by $\mathcal{W}$. We want to show that this is also true for the ordinal $\gamma + 1 = \gamma \cup \{\gamma\}$.
  The proof of the successor case is roughly the same as the induction step of the proof of theorem 5.

- Limit Step: Assume there's a $(x_\beta)_{\beta<\gamma}$ such that there's no open set of the form $\prod_{\beta<\gamma} U_\beta \times X_\gamma \times \prod_{\gamma<\kappa} X_\kappa$ with $(x_\beta)_{\beta<\gamma} \in \prod_{\beta<\gamma} U_\beta$ is finitely covered by $\mathcal{W}$ for all $\gamma \in J, \gamma < \alpha$. Now we want to show this is also true for $\gamma = \alpha$.
  Considering $X = \prod_{\beta<\alpha} X_\beta \times X_\alpha \times \prod_{\alpha<\kappa} X_\kappa$, by the hypothesis, we know that there's an element $(x_\beta)_{\beta<\alpha} \in \prod_{\beta<\alpha} X_\alpha$ such that there's no basic open set of the form $\prod_{\beta<\alpha} U_\beta \times X_\alpha \times \prod_{\alpha<\kappa} X_\kappa$, where $(x_\beta)_{\beta<\alpha} \in \prod_{\beta<\alpha} U_\beta$, is finitely covered $\mathcal{W}$. (Again, here we note that $\prod_{\beta<\alpha} U_\beta$ is a basic open set in $\prod_{\beta<\alpha} X_\beta$, this is important because that's why we can get this from the hypothesis—for basic open sets in $\prod_{\beta<\alpha} X_\beta$, there's a maximal index $\eta < \alpha$ such that $U_\eta \neq X_\eta$).

So we constructed the $(x_\lambda)_{\lambda \in J}$ such that there's no basic open set of the form $\prod_{\beta<\alpha} U_\beta \times X_\alpha \times \prod_{\alpha<\kappa} X_\kappa$ containing $(x_\lambda)_{\lambda \in J}$ is finitely covered by $\mathcal{W}$ via the transfinite induction. Therefore, $(x_\lambda)_{\lambda \in J}$ is not covered by $\mathcal{W}$. Thus, $\mathcal{W}$ doesn't cover $X = \prod_{\lambda \in J} X_\lambda$, which by lemma 3 means X is compact. □

2.2 The second method of proving Tychonoff's theorem

The first method used the well-ordering theorem. Here, we give an alternate proof making use of the Zorn's lemma. As mentioned at the beginning of the paper, this method follows Chapter 5 of [3], which is the second edition of Munkres' Topology.

2.2.1 Some useful lemma

Lemma 10. (Lemma 37.1 of [3]) *let X be a set, if there's a collection $\mathcal{A}$ of subsets of X has FIP, then there's a maximal collection $\mathcal{D}$ of subset of X among collections of subset of X with finite intersection property such that $\mathcal{A} \subset \mathcal{D}$.*

As you might expect, we are using Zorn's Lemma to construct this maximal collection. In order to use Zorn's lemma, we shall construct a set whose elements are the collections of subsets of X with FIP, call this kind of set "superset", and we let the strict partial order on our superset to be proper inclusion $\subsetneq$.

Proof. Since we have concepts of sets, collection of sets, and the sets of collections in this proof, we will first clarify our notations.

- a: the element of X
- A: the subset of X
- $\mathcal{A}$: the collection of subsets of X
- $\mathbb{A}$: the superset whose elements are the collection of subsets of X [5]

By the hypothesis, $\mathcal{A}$ is a collection of subsets of X that has FIP, now let $\mathbb{A}$ be the superset whose elements are all collection $\mathcal{B}$ such that $\mathcal{B}$ has FIP and $\mathcal{A} \subset \mathcal{B}$[6]. As we discussed above, the strict partial order on $\mathbb{A}$ is proper inclusion $\subsetneq$. Recall that the hypothesis for Zorn's lemma requires every simply ordered subset of $\mathbb{A}$ has a upper bound in $\mathbb{A}$. So we need to show if $\mathbb{B} \subset \mathbb{A}$ is simply ordered, then $\mathbb{B}$ has a upper bound $\mathcal{C}$.
I claim, $\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}$ is the upper bound of $\mathbb{B}$ in $\mathbb{A}$.
First we notice that $\mathcal{C}$ satisfies $\mathcal{A} \subset \mathcal{C}$, since $\mathcal{A} \subset \mathcal{B}$ for every $\mathcal{B} \in \mathbb{B}$, so to show $\mathcal{C}$ is indeed an element of $\mathbb{A}$, we just need to show $\mathcal{C}$ has finite intersection property.
Given any finitely many element of $\mathcal{C}$, $C_1, \cdots, C_n$, by the construction of $\mathcal{C}$, we know for each C_i, $C_i \in \mathcal{B}_i$

[5]We denote the subset of supersets as subsuperset.
[6]We require $\mathcal{A} \subset \mathcal{B}$ to make sure the maximal element we find contains $\mathcal{A}$.

for some $\mathcal{B}_i \in \mathbb{B}$. As our hypothesis, $\mathbb{B}$ is a simply ordered subset of $\mathbb{A}$ under the relation $\subsetneq$. So we can say that there's a k, $1 \leqslant k \leqslant n$, such that $\mathcal{B}_k = max\{\mathcal{B}_1, \cdots, \mathcal{B}_n\}$, which means $C_i \in \mathcal{B}_k$ for all $1 \leqslant i \leqslant n$. Since $\mathcal{B}_k \in \mathbb{B} \subset \mathbb{A}$, so $\mathcal{B}_k$ has finite intersection property, thus, $C_1 \cap \cdots \cap C_n \neq \varnothing$ as desired.
Therefore, since we just showed every subsuperset of $\mathbb{A}$ with FIP has a upper bound in $\mathbb{A}$, then by Zorn's lemma, $\mathbb{A}$ has a maximal element, which is the maximal collection $\mathcal{D}$ of subset of X has finite intersection property such that $\mathcal{A} \subset \mathcal{D}$. □

Lemma 11. (Lemma 37.2 of [3]) *Let X be a set; Let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:*

- *i. Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.*
- *ii. If A is a subset of X that intersects every element of $\mathcal{D}$ then A is an element of $\mathcal{D}$.*

Proof.

- i: Given any finitely many elements of $\mathcal{D}$, let U be their intersection, then let $\mathcal{U} = \mathcal{D} \cup \{U\}$. We want to show $\mathcal{U}$ has finite intersection property, so then since $\mathcal{D}$ is maximal with respect to the finite intersection property, we can get $\mathcal{U} \subset \mathcal{D}$, which implies $B \in \mathcal{D}$.
  Now given any finitely many element of $\mathcal{U}$. If none of them is U then by the FIP of $\mathcal{D}$ we know their intersection is not empty. If one of them is U then their intersection $D_1 \cap \cdots \cap D_n \cap U$ is also a finite intersection of element of $\mathcal{D}$ since U is also a finite intersection of elements in $\mathcal{D}$, thus nonempty. So $\mathcal{U}$ has FIP as desired
  Therefore i is true.
- ii: Again, we want to show $\mathcal{V} = \mathcal{D} \cup \{A\}$ has finite intersection property, thus, is contained in $\mathcal{D}$, which shows $A \in \mathcal{D}$.
  Given any finitely many elements in $\mathcal{V}$. If none of them is A, then since $\mathcal{D}$ has FIP, we know their intersection is nonempty. If one of them is A, their intersection is $D_1 \cap \cdots \cap D_m \cap A$. By i, we know $D_1 \cap \cdots \cap D_m$ is an element of $\mathcal{D}$, then by our hypothesis, $D_1 \cap \cdots \cap D_m \cap A \neq \varnothing$ So $\mathcal{V}$ has FIP as desired.
  Therefore, ii is true.

□

2.2.2 The proof of Tychonoff's theorem

Now we use these two lemmas to prove Tychonoff's theorem.

Proof. Given a collection of compact space $\{X_i | i \in J\}$, let $X = \prod_{i \in J} X_i$ is compact. To show this, we will use theorem 9, that is, we will show any collection of closed subset of X with FIP has the property that the intersection of all element in it is nonempty.
Let $\mathcal{A}$ be a collection of subsets of X with FIP, we want to show $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \varnothing$.
By lemma 2.2.1, we can find a maximal collection $\mathcal{D}$ with respect to FIP and $\mathcal{A} \subset \mathcal{D}$. It suffices to show $\bigcap_{D \in \mathcal{D}} D \neq \varnothing$.
Given any $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map. Now consider the collection $\{\pi_\alpha(D) | D \in \mathcal{D}\}$. We know $\mathcal{D}$ has FIP so this collection has FIP too. So then $\{\overline{\pi_\alpha(D)} | D \in \mathcal{D}\}$ is a collection of closed subsets of X_α that has finite intersection property. By the compactness of X_α, we know $\bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$ is not empty. So then we can find an $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$ for every $\alpha \in J$.
Let $x = (x_\alpha)_{\alpha \in J}$, we want to show $x \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$, that is $x \in \overline{D}$ for every $D \in \mathcal{D}$. It suffices to show x is a limit point of every $D \in \mathcal{D}$.
Given any basis U containing x, it is in the form of $U = \prod_{\alpha \in J} U_\alpha$ where $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$. Notice that $U = \bigcup \pi_\beta^{-1}(U_\beta)$ where $U_\beta \neq X_\beta$. Now pick any $U_\beta \neq X_\beta$, given any $D \in \mathcal{D}$. We know $x_\beta \in U_\beta$ and $x_\beta \in \pi_\alpha(D)$, so $U_\beta \cap \pi_\alpha(D) \neq \varnothing$, which means there's a $y_\beta \in D$ such that $\pi_\beta(y) \in U_\beta$, thus, $y \in \pi_\beta^{-1}(U_\beta) \cap D$.
Since D is arbitrary, so $\pi_\beta^{-1}(U_\beta) \cap D \neq \varnothing$ for all $D \in \mathcal{D}$. By (b) of lemma 11, and $\mathcal{D}$ is maximal with respect to the finite intersection property, we know $\pi_\beta^{-1}(U_\beta)$ belongs to $\mathcal{D}$. Since U_β is an arbitrary set satisfies $U_\beta \neq X_\beta$, so the arbitrary basis we found containing x, U, belongs to $\mathcal{D}$ by (a) of lemma 11. So we can conclude that every basis containing x is an element of $\mathcal{D}$.

Since $\mathcal{D}$ has finite intersection property, so every basis containing x intersect with every $D \in \mathcal{D}$, which implies that x is a limit point of every $D \in \mathcal{D}$. So $x \in \bigcap_{D \in \mathcal{D}} \overline{D} \neq \varnothing$ as desired. □

3 Tychonoff's theorem implies the Axiom of Choice

An important application of Tychonoff's theorem is that people can actually use it as the Axiom of Choice, that says, Tychonoff's theorem is equivalent to the Axiom of Choice. In the first two Sections, we have seen how to use the Axiom of Choice and its consequences to show Tychonoff's theorem. So now, we will show Tychonoff's theorem implies the Axiom of Choice. The proof follows [5][7] written by John Terilla.

It's not hard to see that the axiom of choice is equivalent to the statement "the cartisian product of nonempty sets is nonempty". So now we will show Tychonoff's theorem implies the cartesian product of non-empty sets is nonempty, that is, if $\{X_i | i \in J\}$ is a collection of nonempty sets, then $\prod_{i \in J} X_i \neq \varnothing$.

Theorem 12. (Theorem 9 of [5]) *Tychonoff's theorem is equivalent to the Axiom of Choice.*

Proof. As discussed above, we have already shown the Axiom of Choice implies Tychonoff's theorem in the first two sections. So now, we want to show Tychonoff's theorem implies the Axiom of Choice.

Let $Y_i = X_i \cup \{i\}$ for each $i \in J$, then we define the only nontrivial open sets in each Y_i to be X_i and $\{i\}$, it's easy to show that this indeed defined a topology on Y_i. Since there's only finite open sets in each Y_i, we know Y_i is compact for all $i \in J$. Then, by Tychonoff's theorem, $\prod_{i \in J} Y_i$ is also compact.

Here, we got $\prod_{i \in J} Y_i$ is compact from Tychonoff's theorem, but there's another important thing to check, which is $\prod_{i \in J} Y_i \neq \varnothing$, otherwise our proof doesn't work. To show that, we couldn't use the Axiom of Choice since that's what we are trying to show. However, we do not need the Axiom of Choice to make sure we have a choice function to pick an element from $\prod_{i \in J} Y_i$. Because we already have a nice choice function, which is choosing $i \in Y_i$ for every i, and that gives us $(x_i)_{i \in J} \in \prod_{i \in J} Y_i$ where $x_i = i$ for every $i \in J$.

Now, define sets $C_i = X_i \times \prod_{j \in J, j \neq i} Y_j$ for every $i \in J$. We know $(\prod_{i \in J} Y_i) \backslash C_i$ is the product of $\{i\}$ and all the $Y_j, j \neq i$, which is open in the product space $\prod_{i \in J} Y_i$ with product topology. So C_i is closed for each $i \in J$.

Let $\mathcal{K}$ be the collection contains all the $C_i, i \in J$. The intersection of finitely many elements in $\mathcal{K}$ is nonempty, because it is in the form of $\prod_{i \in J} U_i$, where $U_i = X_i$ for finitely many $i \in J$ and $U_i = Y_i$ for other $i \in J$, and similar to what we just argued above, we could simply choose $i \in Y_i$ from each Y_i. This means $\mathcal{K}$ is a collection of closed subsets in the compact space $\prod_{i \in J} Y_i$ and $\mathcal{K}$ has finite intersection property.

By theorem 9, we know the intersection of all elements in $\mathcal{K}$ is nonempty, that is $\bigcap_{i \in J} C_i \neq \varnothing$. Since the elements in $\bigcap_{i \in J} C_i \neq \varnothing$ are of the form $(x_i)_{i \in J}$ such that $x_i \in X_i$, we can conclude $\prod_{i \in J} X_i \neq \varnothing$. □

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