

Limits of Semistatic Trading Strategies

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Abstract

We show that pointwise limits of semistatic trading strategies in discrete time are again semistatic strategies. The analysis is carried out in full generality for a two-period model, and under a probabilistic condition for multi-period, multi-stock models. Our result contrasts with a counterexample of Acciaio, Larsson and Schachermayer, and shows that their observation is due to a failure of integrability rather than instability of the semistatic form. Mathematically, our results relate to the decomposability of functions as studied in the context of Schrödinger bridges.

Keywords Semistatic Trading; Borwein–Lewis theorem

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1 Introduction

Closedness properties of trading strategies play a pivotal role in mathematical finance. They are at the heart of the separation arguments underlying the Fundamental Theorem of Asset Pricing, the superhedging duality, the existence of optimal portfolios for utility maximization problems, and other key results (see, e.g., [7, 8] and the references therein). In the most classical setting, strategies refer to dynamic trading in a stock or other liquidly traded securities. Closedness refers to the limit of a sequence of final outcomes from self-financing trading again being an outcome of some admissible trading strategy, or at least being dominated by one.

In addition to dynamic trading in a stock (X_t), a semistatic strategy allows for buy-and-hold trading in options written on X , usually European

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options maturing at the time horizon T of the model. The static nature of this position reflects the increased trading cost relative to the stock. By combining options such as calls with different strikes, the trader can approximate the payoff $g(X_T)$ for an arbitrary function g . A linear pricing rule for all such payoffs is equivalent to fixing the risk-neutral distribution of X_T [6]. Taking this distribution as a primitive, a large body of literature starting with [14] developed the theory of model-free or robust finance, where the distribution of X is taken to be unknown or partially unknown, respectively, up to a no-arbitrage condition (see [2, 5, 9, 15], among many others).

In the classical setting where X is modeled on a given probability space, semistatic trading is equally natural but the mathematical foundations of the theory are not well developed. We might expect this setting to lie between the usual one (with a fixed reference measure but without options) and the model-free one (with options but without reference measure). Maybe surprisingly, this is not the case: it had been observed for some time that the standard arguments for closure and separation do not apply in a straightforward way. The explanation is provided by [1] whose main result consists of two counterexamples, one in discrete and one in continuous time, stating that the space of (final outcomes from) semistatic trading strategies is not closed in the sense defined there. This clearly creates an obstacle to developing the theory along the usual lines. In this paper, we focus on the discrete-time setting and provide *positive* closedness results. In particular, we illuminate what goes wrong in the example of [1]. Our results open the door to developing other aspects of mathematical finance in the semistatic setting. One such aspect, the existence of an optimal portfolio for the exponential utility maximization problem, is treated in a companion paper [18] which takes our result as its starting point. A related question comes up in [11, 12], where trading also involves the volatility index VIX and it is postulated that a certain optimal log-density has the form of a semistatic portfolio. The arguments of the present work should be useful to deduce this from first principles.

The aforementioned discrete-time example of [1] considers a two-period model of dynamic trading in a stock $(X_t)_{t=0,1,2}$ and static trading in options $g(X_2)$, where g is integrable under the law μ of X_2 . The physical measure P of the model is taken to be a risk-neutral measure, thus excluding issues related to arbitrage. The authors exhibit a sequence of bounded strategies with nonnegative outcomes converging in L^p for any finite p and prove that the limit of the outcomes is not the outcome of an admissible strategy (not even dominated by one). The proof is based on a clever contradiction argument which circumvents studying the limiting random variable in detail.

Our first main result below focuses on a two-period model and a sequence of semistatic strategies converging pointwise. In a non-probabilistic setting, it shows that the limiting outcome is again a semistatic strategy. This result implies stability under almost-sure convergence in a probabilistic setting by application to a set of measure one. As there are no restrictions on the probability measure, the result holds regardless of arbitrage considerations or other probabilistic assumptions. Returning to the example of [1], our result indicates that the main failure relates to the integrability of the limiting option position rather than the semistatic functional form: the limit is still a sum of dynamic trading and an option; however, the integrability of the option can fail. (This is by no means a negligible issue—it removes the obvious way of assigning a price to the option.) Of course, one may hope that *specific* limiting portfolios nevertheless enjoy good integrability properties. In the companion paper [18], we prove this for the optimal portfolio of exponential utility maximization.

The proof of our two-period, single stock result involves analyzing some finer algebraic structures of the set where convergence takes place. While our experience suggests that the result may extend to more general models, the complexity of our analysis grows rather quickly, possibly to the extent of becoming infeasible. In our second main result, we aim to exhibit a reasonably weak financial/probabilistic condition that eliminates some of the subtleties and enables a general closedness result by a more generic analysis. Indeed, our result covers the standard setting with any (finite) number of stocks and periods, and options on all individual stocks. The analysis immediately extends to trading constraints such as no-shorting, and with modest efforts it could be adapted for options at intermediate dates, options only on some of the stocks, or similar scenarios. The proposed financial condition is that the reference measure P of the model be equivalent to (i.e., have the same nullsets as) a product measure; namely, the product of all the marginals of the individual stocks at the individual dates. Intuitively, this means that none of the (a priori possible) future stock prices becomes completely impossible given an intermediate state of the prices. We remark that this interpretation is similar to the “conditional full support” assumption known in the theory of transaction costs; see [10] and the literature thereafter.

Remarkably, we have not been able to use the classical arguments of mathematical finance, nor the techniques known from the dual problem of martingale optimal transport [3] in this work, despite the setting lying between those two. Instead, we draw inspiration from the literature on the dual of the Schrödinger bridge problem, especially [4, 19]. See also [16, 17]

for general introductions. We shall see that in the present context, the analysis is significantly more involved as the increments of the stock create an interaction between the variables. Nevertheless, notions such as the connectedness defined in [4] are crucially helpful. The observation that semistatic trading is related to a martingale version of the Schrödinger bridge problem goes back to [13] which discusses a continuous-time problem.

The remainder of this paper is organized as follows. Section 2 contains a complete analysis of the two-period, single-stock model. Building on its insights, Section 3 proposes a probabilistic condition enabling the analysis of a multi-period, multi-stock model: we show that the problem of closedness reduces to a problem of linear algebra. The latter consists in establishing that a certain matrix has full rank, which is the content of Section 4.

2 Two-Period Model

In this section, we provide a pointwise convergence analysis for a two-period model with a single stock. In our formulation, the variable x represents the stock price at date 1 and y the price at date 2. Equivalently, the stock price process is given by the canonical process (X, Y) on \mathbb{R}^2 . Throughout the section, we fix $E \subset \mathbb{R}^2$, interpreted as the set of possible states for (x, y) .

Definition 2.1. A function $v : E \rightarrow \mathbb{R}$ is a semistatic strategy if

$$v(x, y) = h(x)(y - x) + g(y), \quad (x, y) \in E$$

for some functions $h, g : \mathbb{R} \rightarrow \mathbb{R}$. We refer to h as a stock position and g as an option position.

In general, h and g are not uniquely determined by v , hence one should think of v as a function $E \rightarrow \mathbb{R}$ rather than consisting of the pair h, g . We note that trading between the initial date 0 and date 1 is not represented explicitly. This entails no loss of generality: if the initial state is deterministic, an expression of the form $h_0(x - x_0) + h(x)(y - x) + g(y)$ can always be rewritten as a semistatic strategy in the above sense.

Theorem 2.2. *The set of semistatic strategies is closed under pointwise convergence.*

While the theorem states that a limit of semistatic strategies v_n is again a semistatic strategy, we emphasize that the corresponding stock and option positions do not converge in general. Clearly the theorem entails its probabilistic counterparts: if convergence holds P -a.s. under some measure P , we

can apply the above with E being the set of full measure where convergence holds. Even if P satisfies a no-arbitrage condition, the structure of E can be quite complicated in this context, hence the importance of leaving E general in the theorem.

The proof of Theorem 2.2 is stated at the end of Section 2.1, after an analysis which also provides detailed insight about what happens with the stock and option positions in the passage to the limit.

2.1 Analysis of the Two-Period Model

We divide the given set $E \subset \mathbb{R}^2$ into its set $E_d := \{(x, y) \in E : x = y\}$ of diagonal points and the complement $E_o := E \setminus E_d$ of off-diagonal points. Consider a semistatic strategy $v(x, y) = h(x)(y - x) + g(y)$ at a point $(x, y) \in E_o$, then as $y - x \neq 0$, the value $h(x)$ uniquely determines $g(y)$, and vice versa. Similarly, for a sequence $v_n(x, y) = h_n(x)(y - x) + g_n(y)$ where $v_n(x, y)$ converges, the value $h_n(x)$ converges if and only if $g_n(y)$ does. For $(x, y) \in E_d$, the situation is quite different: $v(x, y) = h(x)(y - x) + g(y) = g(y)$, so that the option position is determined (or convergent, respectively), whereas $h(x)(y - x) = 0$ irrespectively of the stock position $h(x)$ at x . This does not mean that $h(x)$ can be ignored—if $(x, y') \in E$ for some $y' \neq y$, the value $h(x)$ is nevertheless relevant for the strategy.

We first focus our analysis on E_o . The following notion was introduced by [4] in a different context.

Definition 2.3. Two points $(x, y), (x', y') \in E_o$ are *connected*, denoted $(x, y) \sim (x', y')$, if there exist $k \in \mathbb{N}_0$ and $(x_i, y_i)_{i=1}^k \in E_o^k$ such that the points

$$(x, y), (x_1, y), (x_1, y_1), (x_2, y_1), \dots, (x_k, y_k), (x', y_k), (x', y') \quad (2.1)$$

all belong to E_o . In that case, $(x_i, y_i)_{i=1}^k$ is called a *path* (in E_o) from (x, y) to (x', y') . A set $C \subset E_o$ is connected (in E_o) if any two points in C are connected.

For the list (2.1), the crucial property is that only one coordinate is changed in each step. In our notation, the first coordinate changes first, but because a point can be repeated in the list, this entails no loss of generality. We observe that \sim is an equivalence relation on E_o . The corresponding equivalence classes $\mathcal{C} = \{C_\gamma : \gamma \in \Gamma\}$ are called the connected components of E_o .

We say that *uniqueness of portfolio positions* holds at $(x, y) \in E$ if for any semistatic strategy $v : E \rightarrow \mathbb{R}$, the stock and option positions $h(x)$

and $g(y)$ are uniquely determined at (x, y) . In that situation, *convergence of portfolio positions* holds at $(x, y) \in E$ if for any semistatic strategies v_n converging pointwise, the positions $h_n(x)$ and $g_n(y)$ are also convergent.

Lemma 2.4. *Let $C \subseteq E_o$ be connected. If uniqueness (convergence) of portfolio positions holds at some point of C , then uniqueness (convergence) of portfolio positions holds at all points of C .*

Proof. Let uniqueness of portfolio positions hold at (x, y) and let $(x, y) \sim (x', y')$. Consider a path as in (2.1), then as discussed at the beginning of Section 2.1, uniqueness of the option position at y implies uniqueness of the stock position at x_1 which in turn implies uniqueness of the option position at y_1 , and so on, leading to uniqueness at (x, y) . Similarly for the convergence. \square

Definition 2.5. A path $(x_i, y_i)_{i=1}^k$ from $(x, y) \in E_o$ to itself is called a *cycle*. The cycle is *identifying* if

$$\prod_{i=1}^k (y_i - x_i) - \prod_{i=1}^k (y_i - x_{i+1}) \neq 0, \quad (2.2)$$

where we use the cyclical convention $x_{k+1} := x_1$.

The terminology is explained by the subsequent lemma: such a cycle uniquely “identifies” the portfolio positions along its points. We note that $k \geq 2$ must hold for any identifying cycle. If $((x_1, y_1), (x_2, y_2))$ is a cycle, then it is identifying if and only if $x_1 \neq x_2$ and $y_1 \neq y_2$, because (2.2) reduces to $(x_1 - x_2)(y_1 - y_2) \neq 0$. In particular, the cycle can be envisioned as a non-degenerate rectangle $\{x_1, x_2\} \times \{y_1, y_2\} \subset E_o$. This simple characterization does not extend to larger cycles.

Lemma 2.6. *Let $(x_i, y_i)_{i=1}^k$ be an identifying cycle. Then uniqueness and convergence of portfolio positions hold at (x_1, y_1) .*

Proof. Consider a semistatic strategy $v(x, y) = h(x)(y - x) + g(y)$ along the points (2.1),

$$v(x, y) = h(x)(y - x) + g(y) \quad \text{for } (x, y) \in \{(x_i, y_i), (x_{i+1}, y_i) : 1 \leq i \leq k\}.$$

(b) *Closedness of semistatic strategies holds on C . More precisely, let $v_n(x, y) = h_n(x)(y - x) + g_n(y)$ be semistatic strategies converging pointwise on C to $v : C \rightarrow \mathbb{R}$. Then there exist h'_n, g'_n such that*

$$v_n(x, y) = h'_n(x)(y - x) + g'_n(y), \quad (x, y) \in C$$

and their pointwise limits exist,

$$h' := \lim h'_n \quad \text{on } C^x, \quad g' := \lim g'_n \quad \text{on } C^y.$$

In particular, $v(x, y) = h'(x)(y - x) + g'(y)$ on C .

The positions h'_n, g'_n can be constructed as follows. Fix an arbitrary point $(x_0, y_0) \in C$. Then there exist unique functions $a, b : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ such that $a(x)b(y) = y - x$ on C and $a(x_0) = 1$. We can choose

$$h'_n(x) := h_n(x) - \frac{h_n(x_0)}{a(x)}, \quad g'_n(y) := g_n(y) + h_n(x_0)b(y).$$

Before stating the proof, we recall the Borwein–Lewis characterization for the decomposability of a function of two variables into a product of single-variable functions. More generally, this result applies to group-valued functions on arbitrary sets; see [4, Theorem 3.3].

Lemma 2.9. *Let $S \subset \mathbb{R} \times \mathbb{R}$ and $c : S \rightarrow \mathbb{R} \setminus \{0\}$. The following are equivalent:*

(i) *There exist $a, b : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ such that*

$$c(x, y) = a(x)b(y), \quad (x, y) \in S.$$

(ii) *For any cycle $(x_i, y_i)_{i=1}^k$ in S ,*

$$\prod_{j=1}^k c(x_j, y_j) = \prod_{j=1}^k c(x_{j+1}, y_j), \quad (2.3)$$

where $x_{k+1} := x_1$.

In that case, on each connected component of S , the functions a and b are unique up on a scalar multiple.

Proof of Proposition 2.8. The absence of identifying cycles means that (2.3) holds for the function $c(x, y) := y - x$ on C . Note that c is valued in $\mathbb{R} \setminus \{0\}$ due to $C \subset E_o$. Hence Lemma 2.9 implies that there exist $a, b : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ such that $a(x)b(y) = y - x$ on C , and these functions are uniquely determined by the normalization that $a(x_0) = 1$ for some fixed $x_0 \in C^x$.

Consider a semistatic strategy $v(x, y) = h(x)(y - x) + g(y)$ on C . Given $\alpha \in \mathbb{R}$, let $h_\alpha(x) := h(x) + \alpha/a(x)$ and $g_\alpha(y) := g(y) - \alpha b(y)$. Then

$$\begin{aligned} h_\alpha(x)(y - x) + g_\alpha(y) &= [h(x) + \alpha/a(x)]a(x)b(y) + g(y) - \alpha b(y) \\ &= h(x)a(x)b(y) + g(y) = v(x, y), \end{aligned}$$

showing that the portfolio positions inducing v include the one-parameter family $(h_\alpha, g_\alpha)_{\alpha \in \mathbb{R}}$. Conversely, by connectedness, we know that portfolio positions are uniquely determined as soon as the option position is determined at one point y_0 . Because $\alpha \mapsto g_\alpha(y_0) = g(y_0) - \alpha b(y_0)$ is surjective onto \mathbb{R} , this shows that $(h_\alpha, g_\alpha)_{\alpha \in \mathbb{R}}$ exhausts all portfolio positions inducing v .

Turning to the convergence, note that

$$\bar{v}_n(x, y) := \frac{v_n(x, y)}{b(y)} = h_n(x)a(x) + \frac{g_n(y)}{b(y)}$$

on C . Define

$$\bar{h}_n(x) := h_n(x)a(x) - h_n(x_0)a(x_0), \quad \bar{g}_n(y) := \frac{g_n(y)}{b(y)} + h_n(x_0)a(x_0),$$

so that

$$\bar{v}_n(x, y) = \bar{h}_n(x) + \bar{g}_n(y). \tag{2.4}$$

Clearly $\bar{v}_n(x, y)$ is convergent for all $(x, y) \in C$. Moreover, $\bar{h}_n(x_0) = 0$ for all n , so that $\bar{h}_n(x_0)$ is convergent. As C is connected, the additive decomposition (2.4) implies as in the proof of Lemma 2.4 (or [4]) that the separate limits $\bar{g}(y) := \lim_n \bar{g}_n(y)$ and $\bar{h}(x) := \lim_n \bar{h}_n(x)$ exist for all $(x, y) \in C$. It follows that

$$\begin{aligned} h'_n(x) = \bar{h}_n(x)/a(x) &\rightarrow \bar{h}(x)/a(x) = h'(x), \\ g'_n(y) = \bar{g}_n(y)b(y) &\rightarrow \bar{g}(y)b(y) = g'(y), \end{aligned}$$

completing the proof. □

We can now prove the main result.

Proof of Theorem 2.2. Let $v_n(x, y) = h_n(x)(y - x) + g_n(y)$ be semistatic strategies converging pointwise on E to $v : E \rightarrow \mathbb{R}$. We shall construct $h : E^x \rightarrow \mathbb{R}$ and $g : E^y \rightarrow \mathbb{R}$ such that $v(x, y) = h(x)(y - x) + g(y)$ on E .

Recall the partition $(C_\gamma)_{\gamma \in \Gamma}$ of E_o into connected components. The definition of \sim implies that for each $\gamma \in \Gamma$,

$$C_\gamma = (C_\gamma^x \times C_\gamma^y) \cap E_o \subset (C_\gamma^x \times C_\gamma^y) \cap E;$$

the last inclusion can be strict as $C_\gamma^x \times C_\gamma^y$ can contain points from the diagonal E_d . Conversely, some points from the diagonal may not pertain to $C_\gamma^x \times C_\gamma^y$ for any $\gamma \in \Gamma$; these points form the set

$$N := E \setminus \bigcup_{\gamma \in \Gamma} (C_\gamma^x \times C_\gamma^y) \subseteq E_d.$$

Let $(D_j)_{j \in J}$ be the collection consisting of all singletons $\{(x, y)\}$ with $(x, y) \in N$ as well as the sets $(C_\gamma^x \times C_\gamma^y) \cap E$, $\gamma \in \Gamma$. Note that two points in E_o sharing one coordinate are necessarily connected, and two points in E_d sharing one coordinate must coincide. This implies that $(D_j)_{j \in J}$ is a partition of E with the following property: If $(x, y), (x, y') \in E$, then (x, y) and (x, y') belong to the same component D_j . Similarly, if $(x, y), (x', y) \in E$, then (x, y) and (x', y) belong to the same component D_j . As a consequence, we may construct the positions h, g separately on each D_j without danger of creating any inconsistencies. (This is not true for $\{C_\gamma, E_d\}$, whence the need for yet another collection.)

- (i) Let $D_j = (C_\gamma^x \times C_\gamma^y) \cap E$ for some $\gamma \in \Gamma$, where C_γ contains an identifying cycle. Then we can choose $(h, g) := \lim(h_n, g_n)$ on (C_γ^x, C_γ^y) according to Corollary 2.7.
- (ii) Let $D_j = (C_\gamma^x \times C_\gamma^y) \cap E$ for some $\gamma \in \Gamma$, where C_γ does not contain an identifying cycle. Then we can choose (h, g) on (C_γ^x, C_γ^y) according to Proposition 2.8.
- (iii) Let $D_j = \{(x, y)\}$ for some $(x, y) \in N \subseteq E_d$. Then we can define $h(x) = 0$ and $g(y) = v(x, y)$. In fact, as $y - x = 0$, any choice for $h(x)$ will do. \square

The preceding analysis also quantifies the non-uniqueness for portfolio positions; that is, the exact number of degrees of freedom in choosing the portfolio positions for any given semistatic strategy.

Remark 2.10. Given any semistatic strategy v on E , the set of all portfolio positions (h, g) with $v(x, y) = h(x)(y - x) + g(y)$ for all $(x, y) \in E$ is a k -parameter family, where

$$k = \text{card} (N \cup \{\gamma \in \Gamma : C_\gamma \text{ contains no identifying cycle}\}).$$

3 General Probabilistic Model

Given integers $d \geq 1$ and $T \geq 2$, we denote by $X = (X_t)_{t=1}^T$ the canonical process on $(\mathbb{R}^d)^T$, where $X_t = (X_{t,j})_{j=1}^d$ are interpreted as the prices of d stocks at date t . We also fix a probability measure P on $(\mathbb{R}^d)^T$; only the nullsets of P will matter for our results. For the purposes of this section, a *semistatic strategy* is a random variable V satisfying

$$V = \sum_{t=1}^{T-1} \sum_{j=1}^d \hat{h}_{t,j}(X_1, \dots, X_t) (X_{t+1,j} - X_{t,j}) + \sum_{j=1}^d \hat{g}_j(X_{T,j}) \quad (3.1)$$

P -a.s. for some real-valued measurable functions $\hat{h}_{t,j}$ and \hat{g}_j . It will be notationally convenient to work instead with the random variables

$$h_{t,j} := \hat{h}_{t,j}(X_1, \dots, X_t), \quad g_j := \hat{g}_j(X_{T,j}).$$

We call $h = (h_{t,j})$ the stock position and $g = (g_j)$ the option position, respectively. Together, they form the portfolio position (h, g) of V . The portfolio position is not uniquely determined by V in general: it is clearly possible to add a constant to g_1 and subtract the same from g_j for any $j \neq 1$, without affecting V . These $d - 1$ degrees of freedom are easily removed by fixing an ‘‘anchor’’ point $x^0 \in (\mathbb{R}^d)^T$ and normalizing

$$g_j(x^0) = 0, \quad j = 2, \dots, d. \quad (3.2)$$

In the context of Theorem 3.1 below, it will be shown that the anchor point can be chosen arbitrarily outside a certain nullset.

In this probabilistic setting, we say that *uniqueness of portfolio positions* holds if for any semistatic strategy V , after a normalization of the form (3.2), the portfolio position (h, g) is uniquely determined P -a.s. *Convergence of portfolio positions* holds if, after a normalization of the form (3.2), for any semistatic strategies $V^{(n)}$ converging P -a.s., the corresponding positions $(h^{(n)}, g^{(n)})$ also converge P -a.s.

The aim of this section is to exhibit a probabilistic condition circumventing some of the complications highlighted in Section 2. To that end, let $\mu_{t,j}$

be the law of $X_{t,j}$, or equivalently, the one-dimensional marginal law of P on the component (t, j) . We assume throughout that $\mu_{t,i}$ is not a Dirac measure for any t, j . (This serves to simplify the exposition; while the results and arguments could be generalized, the degenerate case is not relevant financially and hence omitted.) The key condition for our result is that P be measure-theoretically equivalent to the product of its marginals.

Theorem 3.1. *Suppose that $P \sim \otimes_{t=1}^T \otimes_{j=1}^d \mu_{t,j}$. Then the set of semistatic strategies is closed under P -a.s. convergence. Moreover, uniqueness and convergence of portfolio positions hold.*

As mentioned in the Introduction, $P \sim \otimes_{t=1}^T \otimes_{j=1}^d \mu_{t,j}$ intuitively means that none of the (a priori possible) future stock prices becomes completely impossible given an intermediate state of the prices, similarly as in the condition of conditional full support [10]. Technically, we shall see that for such P , any set of full measure contains an abundance of cuboids that will play the role of identifying cycles (cf. Section 2). Indeed, fix $x^0, x^1 \in (\mathbb{R}^d)^T$ such that $x_{t,j}^0 \neq x_{t,j}^1$ for all t, j and consider the cuboid D generated by their components,

$$D := \prod_{t=1}^T \prod_{j=1}^d \{x_{t,j}^0, x_{t,j}^1\} \subseteq (\mathbb{R}^d)^T. \quad (3.3)$$

That is, each point in D is a matrix $(x_{t,j}^{\varepsilon_{t,j}})_{t,j}$ where $\varepsilon_{t,j} \in \{0, 1\}$.

Proposition 3.2. *Let V and $V^{(n)}$ be semistatic strategies with portfolio positions (h, g) and $(h^{(n)}, g^{(n)})$, respectively, such that (3.1) and (3.2) hold on D . Then (h, g) are uniquely determined on D . Moreover, $V^{(n)} \rightarrow V$ pointwise on D implies that $(h^{(n)}, g^{(n)}) \rightarrow (h, g)$ pointwise on D .*

The proof is lengthy and deferred to Section 4. In a nutshell, we view (3.1) as linear system where the values of h and g at points in D are the variables; each equation of the system corresponds to evaluating V at a point in D . We prove that the (finite-dimensional) linear map associated with the system is injective, hence admits a continuous inverse. As a result, the portfolio position (h, g) is a continuous function of the strategy V .

The next lemma is a general measure-theoretic fact; it formalizes the claim that there is an abundance of cuboids in any set of full P -measure.

Lemma 3.3. *Consider probability spaces $(\Omega_i, \mathcal{F}_i, \mu_i)_{i=1}^n$ and their product $(\Omega, \mathcal{F}, \mu)$ given by $\Omega = \prod_{i=1}^n \Omega_i$, $\mathcal{F} = \otimes_{i=1}^n \mathcal{F}_i$ and $\mu = \otimes_{i=1}^n \mu_i$. If $A \in \mathcal{F}$*

satisfies $\mu(A) = 1$, then μ -a.e. $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in A$ satisfies

$$\mu \left\{ x \in A : \prod_{i=1}^n \{x_i^0, x_i\} \subseteq A \right\} = 1.$$

Proof. Let Ω^0, Ω^1 be two copies of Ω with components denoted Ω_i^j . Consider the product space $\hat{\Omega} = \prod_{i=1}^n (\Omega_i^0 \times \Omega_i^1)$ endowed with the product σ -field $\hat{\mathcal{F}}$ and the product measure $\hat{\mu} = \otimes_{i=1}^n (\mu_i \otimes \mu_i)$. For each multi-index $J = (j_1, j_2, \dots, j_n) \in \{0, 1\}^n$, define the projection $\pi_J : \hat{\Omega} \rightarrow \prod_{i=1}^n \Omega_i^{j_i}$ by

$$(x_1^0, x_1^1, x_2^0, x_2^1, \dots, x_n^0, x_n^1) \mapsto (x_1^{j_1}, x_2^{j_2}, \dots, x_n^{j_n}).$$

Clearly the law of π_J under $\hat{\mu}$ is μ , so that $\hat{\mu}(\pi_J \in A) = \mu(A) = 1$. Defining $S := \cap \{\pi_J \in A\}$ as the intersection over all $J \in \{0, 1\}^n$, it follows that $\hat{\mu}(S) = 1$. Denote by S_{x^0} the section of S at $x^0 \in \Omega^0$. In view of $\hat{\mu}(S) = 1$, Fubini's theorem implies

$$\mu \{x^0 \in \Omega^0 : \mu(S_{x^0}) = 1\} = 1.$$

The desired result follows once we observe that $(x_1^0, x_1, x_2^0, x_2, \dots, x_n^0, x_n) \in S$ if and only if $\prod_{i=1}^n \{x_i^0, x_i\} \subseteq A$. \square

We can now deduce the main result.

Proof of Theorem 3.1. Suppose that $V^{(n)}$ is of the form (3.1) with portfolio position $(h^{(n)}, g^{(n)})$ and that $V^{(n)} \rightarrow V$ on $A \subseteq (\mathbb{R}^d)^T$ with $P(A) = 1$. In view of $P \sim \otimes_{t=1}^T \otimes_{j=1}^d \mu_{t,j}$, Lemma 3.3 implies that there exists $x^0 \in A$ such that

$$B := \left\{ x \in A : \prod_{t=1}^T \prod_{j=1}^d \{x_{t,j}^0, x_{t,j}\} \subseteq A \right\}$$

satisfies $P(B) = 1$. (In fact, P -almost any $x^0 \in A$ will do.) We use this point x^0 for the normalization (3.2).

We claim that the limit $(h(x), g(x)) := \lim_n (h^{(n)}(x), g^{(n)}(x))$ exists for all $x \in B$. To prove this, let $x^1 \in B$ satisfy $x_{t,j}^1 \neq x_{t,j}^0$ for all t, j and consider the cuboid D determined by x^0 and x^1 ; cf. (3.3). Then we see from Proposition 3.2 that $(h^{(n)}, g^{(n)})$ converges to some (h, g) on D , and that (h, g) is uniquely determined on D . In particular, the limit exists at $x := x^0$ and at $x := x^1$. Given an arbitrary $x \in B$, as $\mu_{t,j}$ is not a Dirac measure, we can find $x^1 \in B$ such that $x_{t,j}^1 = x_{t,j}$ if $x_{t,j} \neq x_{t,j}^0$ and $x_{t,j}^1 \neq x_{t,j}^0$ for all t, j . Applying the above to x^0 and x^1 , we see that the limit exists at x . The same argument also establishes the uniqueness and convergence of portfolio positions. \square

4 Proof of Proposition 3.2

Suppose that $V : D \rightarrow \mathbb{R}$ is of the form (3.1) with portfolio position (h, g) , where g satisfies (3.2) at x^0 . This induces a linear system with the values of h and g at the points in D as variables and the price increments $\Delta_{t+1,j} = X_{t+1,j} - X_{t,j}$ as coefficients. To start with the simplest example, consider $T = 2$ and $d = 1$, so that $V = h_{1,1}(X_{2,1} - X_{1,1}) + g_1$ and $D = \{(x_{1,1}^0, x_{2,1}^0), (x_{1,1}^0, x_{2,1}^1), (x_{1,1}^1, x_{2,1}^0), (x_{1,1}^1, x_{2,1}^1)\}$. This corresponds to 4 equations and can be cast as a 4×4 linear system

$$\begin{bmatrix} x_{2,1}^0 - x_{1,1}^0 & & & 1 \\ & x_{2,1}^1 - x_{1,1}^0 & & 1 \\ & & x_{2,1}^0 - x_{1,1}^1 & 1 \\ & & x_{2,1}^1 - x_{1,1}^1 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_{1,1}(x_{1,1}^0) \\ \hat{h}_{1,1}(x_{1,1}^1) \\ \hat{g}_1(x_{2,1}^0) \\ \hat{g}_1(x_{2,1}^1) \end{bmatrix} = \begin{bmatrix} V(x_{1,1}^0, x_{2,1}^0) \\ V(x_{1,1}^0, x_{2,1}^1) \\ V(x_{1,1}^1, x_{2,1}^0) \\ V(x_{1,1}^1, x_{2,1}^1) \end{bmatrix}. \quad (4.1)$$

In this example, the condition (3.2) is vacuous as $d = 1$.

In the general case, (3.1) on D can be viewed as a $N_r \times N_c$ linear system that we describe next. For $1 \leq t \leq T$, consider the binary vector $\varepsilon_t = (\varepsilon_{t,1}, \varepsilon_{t,2}, \dots, \varepsilon_{t,d}) \in \{0, 1\}^d$. We view

$$\vec{\varepsilon}_t := (\varepsilon_s)_{s=1}^t := (\varepsilon_{1,1}, \dots, \varepsilon_{1,d}, \varepsilon_{2,1}, \dots, \varepsilon_{2,d}, \dots, \varepsilon_{t,1}, \dots, \varepsilon_{t,d}) \in \{0, 1\}^{dt}$$

as a multi-index of length dt and denote the collection of all such $\vec{\varepsilon}_t$ by \mathcal{E}_t . There is a one-to-one correspondence between \mathcal{E}_T and D via $\vec{\varepsilon}_T \mapsto x^{\vec{\varepsilon}_T} := (x_{t,j}^{\varepsilon_{t,j}})_{t,j}$. More generally, for $1 \leq t \leq T$, the set \mathcal{E}_t corresponds to the set $\prod_{s=1}^t \prod_{j=1}^d \{x_{s,j}^0, x_{s,j}^1\}$ via $\vec{\varepsilon}_t \mapsto x^{\vec{\varepsilon}_t} := (x_{s,j}^{\varepsilon_{s,j}})_{s,j}$, where $1 \leq s \leq t$ and $1 \leq j \leq d$.

Every point $x^{\vec{\varepsilon}_T} \in D$ gives rise to an equation as we evaluate V at $x^{\vec{\varepsilon}_T}$. Hence the number of rows in our system is $N_r = 2^{dT}$, the cardinality of \mathcal{E}_T . On the other hand, for each $1 \leq t \leq T-1$ and each $1 \leq j \leq d$, $\hat{h}_{t,j}$ gives rise to 2^{dt} variables, namely $\hat{h}_{t,j}(x^{\vec{\varepsilon}_t})$ for $\vec{\varepsilon}_t \in \mathcal{E}_t$. In addition, recalling (3.2), \hat{g} gives rise to $d+1$ variables, namely $\hat{g}_1(x_{T,1}^0)$ and $\hat{g}_j(x_{T,j}^1)$ for $1 \leq j \leq d$. As a result, the total number of variables (hence columns in the system) is

$$N_c = d \sum_{t=1}^{T-1} 2^{dt} + (d+1) = d \frac{2^{dT} - 1}{2^d - 1} + 1.$$

As in (4.1), the coefficients of the matrix are given by stock price increments $(x_{t+1,j}^{\varepsilon_{t+1,j}} - x_{t,j}^{\varepsilon_{t,j}})_{t,j}$ and binary entries related to the options. To unambiguously determine the matrix, we need to specify an order for the

rows and columns; in fact, we tailor our order to facilitate the exposition below. For $1 \leq t \leq T$, consider the natural lexicographic order on \mathcal{E}_t and equip $(x^{\vec{\varepsilon}_t})_{\vec{\varepsilon}_t \in \mathcal{E}_t}$ with the induced order; that is, $x^{\vec{\varepsilon}_t} \leq x^{\vec{\eta}_t}$ if and only if $\vec{\varepsilon}_t \leq \vec{\eta}_t$. In particular, the order on \mathcal{E}_T induces an order on $D = (x^{\vec{\varepsilon}_T})_{\vec{\varepsilon}_T \in \mathcal{E}_T}$. The row ordering is set by evaluating (3.1) on D in that order. The column ordering is given by the following (top-to-bottom) hierarchy:

- (i) The variables from \hat{h} appear before those from \hat{g} .
- (ii) The $d \sum_{t=1}^{T-1} 2^{dt}$ variables $\hat{h}_{t,j}(x^{\vec{\varepsilon}_t})$ are sorted
 - (a) in *descending* order of $t \in \{T-1, T-2, \dots, 1\}$,
 - (b) then in ascending order of $\vec{\varepsilon}_t \in \mathcal{E}_t$, and
 - (c) lastly in ascending order of $j \in \{1, 2, \dots, d\}$.
- (iii) The $d+1$ variables from \hat{g} are ordered as

$$\hat{g}_1(x_{T,1}^0), \hat{g}_1(x_{T,1}^1), \hat{g}_2(x_{T,2}^1), \hat{g}_3(x_{T,3}^1), \dots, \hat{g}_d(x_{T,d}^1).$$

The reader can verify that the matrix in (4.1) follows the desired ordering; a more advanced example can be found in (4.6) below. For the general case $T \geq 2$ and $d \geq 1$, the above convention uniquely determines a matrix L_T which will be shown to have the following property.

Lemma 4.1. *The matrix L_T has full column rank.*

Remark 4.2. In the case $d = 1$ of a single stock, L_T is a square matrix and

$$\det L_T = (-1)^{T-1} \left[\prod_{t=1}^{T-1} (x_{t,1}^0 - x_{t,1}^1)^{2^{t-1}} \right] (x_{T,1}^0 - x_{T,1}^1)^{2^{T-1}-1} \neq 0.$$

The proof uses arguments similar to the proof of Lemma 4.1 below.

Once the lemma is established, Proposition 3.2 is a direct consequence:

Proof of Proposition 3.2. By Lemma 4.1, the linear map associated with L_T is injective. Its inverse is continuous as a linear finite-dimensional map, showing that the portfolio position is a continuous function of the strategy. \square

4.1 Proof of Lemma 4.1

Next, we introduce some additional notation for the proof of Lemma 4.1. Given $1 \leq t \leq T - 1$ and $\vec{\varepsilon}_t \in \mathcal{E}_t$, we define $D^{\vec{\varepsilon}_t} \subseteq D$ by

$$D^{\vec{\varepsilon}_t} := \{x^{\vec{\varepsilon}_t}\} \times \left(\prod_{s=t+1}^T \prod_{j=1}^d \{x_{s,j}^0, x_{s,j}^1\} \right). \quad (4.2)$$

That is, $D^{\vec{\varepsilon}_t}$ consists of the $2^{d(T-t)}$ points in D that share the vector $x^{\vec{\varepsilon}_t}$ in their first dt coordinates. Note that for each $1 \leq t \leq T - 1$, $\{D^{\vec{\varepsilon}_t} : \vec{\varepsilon}_t \in \mathcal{E}_t\}$ forms a partition of D .

For each $1 \leq t \leq T - 1$, $1 \leq j \leq d$ and $(k, l) \in \{0, 1\}^2$, we introduce a shorthand for the corresponding stock price increment

$$\Delta_{t+1,j}^{k,l} := x_{t+1,j}^k - x_{t,j}^l, \quad (4.3)$$

as well as the stock price difference

$$\delta_{t,j} := x_{t,j}^0 - x_{t,j}^1 \quad (4.4)$$

which is nonzero by assumption. We record two identities for later use,

$$\Delta_{t+1,j}^{0,l} - \Delta_{t+1,j}^{1,l} = \delta_{t+1,j} \quad \text{and} \quad \Delta_{t+1,j}^{l,0} - \Delta_{t+1,j}^{l,1} = -\delta_{t,j}, \quad (4.5)$$

where the right-hand sides do not depend on $l \in \{0, 1\}$.

As the proof of Lemma 4.1 is somewhat involved, we first state an example to illustrate some of the arguments.

Example 4.3 ($T = 2$ and $d = 2$). Note that (3.1) reads

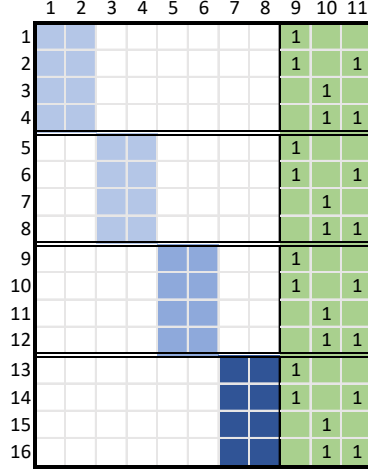
$$V = h_{1,1}(X_{2,1} - X_{1,1}) + h_{1,2}(X_{2,2} - X_{1,2}) + g_1 + g_2.$$

The matrix L_2 has 16 rows and 11 columns; see Figure 1(a). Since $\mathcal{E}_1 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, the column ordering corresponds to the vector of 11 variables

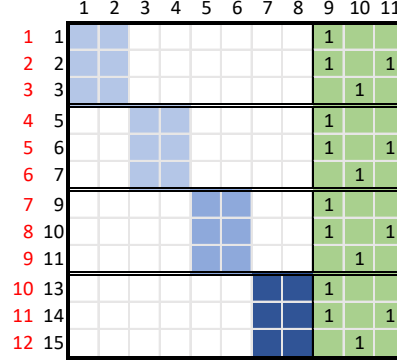
$$\left[\hat{h}_{1,1}(x^{\vec{\varepsilon}_1}), \hat{h}_{1,2}(x^{\vec{\varepsilon}_1}) \text{ for } \vec{\varepsilon}_1 \in \mathcal{E}_1 \right], \hat{g}_1(x_{2,1}^0), \hat{g}_1(x_{2,1}^1), \hat{g}_2(x_{2,2}^0), \hat{g}_2(x_{2,2}^1). \quad (4.6)$$

To show that L_2 has full column rank, we proceed in four steps.

Step 1. Our goal is to divide L_2 into 4 submatrices $(L_2^{\vec{\varepsilon}_1})_{\vec{\varepsilon}_1 \in \mathcal{E}_1}$, each of which has 4 rows and 11 columns. We will identify a linearly dependent row in each $L_2^{\vec{\varepsilon}_1}$.



(a) L_2 in Example 4.3



(b) L'_2 in Example 4.3

Figure 1: The blue and green blocks are coefficients associated with \hat{h} and \hat{g} , respectively. The white blocks are identically zero. Panel (a): L_2 is divided by the double border into 4 submatrices $(L_2^{\vec{\varepsilon}_1})_{\vec{\varepsilon}_1 \in \mathcal{E}_1}$, each of which has 4 rows and 11 columns (where only 5 columns are nonzero). Panel (b): L'_2 is formed by vertically stacking the 3×11 submatrices $(L_2^{\vec{\varepsilon}_1})'$ for all $\vec{\varepsilon}_1 \in \mathcal{E}_1$. The old and new row numbers are printed in black and red, respectively.

Consider (4.2) with $T = 2$. For each $\vec{\varepsilon}_1 \in \mathcal{E}_1$, evaluating V on $D^{\vec{\varepsilon}_1}$ gives rise to a 4×11 submatrix $L_2^{\vec{\varepsilon}_1}$ of L_2 . It has 4 rows as $|D^{\vec{\varepsilon}_1}| = 4$; there are in total 4 submatrices as $|\mathcal{E}_1| = 4$. The coefficients across these submatrices follow a similar pattern. For concreteness, consider $\vec{\varepsilon}_1 = (0, 0) \in \mathcal{E}_1$. Recalling the definition of price increments from (4.3), evaluating V on $D^{\vec{\varepsilon}_1}$ gives

$$\begin{bmatrix} \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,0} & \cdots & 1 & & \\ \Delta_{2,1}^{0,0} & \Delta_{2,2}^{1,0} & \cdots & 1 & & 1 \\ \Delta_{2,1}^{1,0} & \Delta_{2,2}^{0,0} & \cdots & & 1 & \\ \Delta_{2,1}^{1,0} & \Delta_{2,2}^{1,0} & \cdots & & 1 & 1 \end{bmatrix} \begin{bmatrix} h_{1,1}(x^{(0,0)}) \\ h_{1,2}(x^{(0,0)}) \\ \vdots \\ g_1(x_{2,1}^0) \\ g_1(x_{2,1}^1) \\ g_2(x_{2,2}^1) \end{bmatrix} = \begin{bmatrix} v(x^{(0,0)}, x_{2,1}^0, x_{2,2}^0) \\ v(x^{(0,0)}, x_{2,1}^0, x_{2,2}^1) \\ v(x^{(0,0)}, x_{2,1}^1, x_{2,2}^0) \\ v(x^{(0,0)}, x_{2,1}^1, x_{2,2}^1) \end{bmatrix}, \quad (4.7)$$

where the omitted entries (including 6 entire columns) are zero. Denote the matrix in (4.7) by $L_2^{\vec{\varepsilon}_1}$ and let R_i stand for its Row i . Note that $R_1 - R_2 =$

$R_3 - R_4$. Since R_4 is a linear combination of the other rows, dropping it from $L_2^{\vec{\varepsilon}_1}$ does not alter the matrix rank. This argument applies to all $\vec{\varepsilon}_1 \in \mathcal{E}_1$. Denote the remaining matrix by $(L_2^{\vec{\varepsilon}_1})'$. We form the matrix L_2' by vertically stacking $(L_2^{\vec{\varepsilon}_1})'$ for all $\vec{\varepsilon}_1 \in \mathcal{E}_1$; see Figure 1(b). It follows that L_2' has the same rank as L_2 .

Step 2. Recall that elementary row and column operations preserve the matrix rank. In this step, we aim to bring L_2' to a block lower triangular matrix

$$\begin{bmatrix} L_2'' \\ B & C \end{bmatrix}, \quad \text{where } C = \begin{bmatrix} 1 & & \\ 1 & & 1 \\ & 1 & \end{bmatrix} \quad (4.8)$$

by suitable row operations. Indeed, let $\mathbf{1} = (1, 1) \in \mathcal{E}_1$ and $\vec{\varepsilon}_1 \neq \mathbf{1}$ in \mathcal{E}_1 . After subtracting $(L_2^{\mathbf{1}})'$ from $(L_2^{\vec{\varepsilon}_1})'$, we denote the resultant matrix by $(L_2^{\vec{\varepsilon}_1})''$. By applying this procedure to all $\vec{\varepsilon}_1 \neq \mathbf{1}$, we bring L_2' into the desired form, where

$$L_2'' = \begin{bmatrix} \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,0} & & & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{0,1} \\ \Delta_{2,1}^{0,0} & \Delta_{2,2}^{1,0} & & & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{1,1} \\ \Delta_{2,1}^{1,0} & \Delta_{2,2}^{0,0} & & & -\Delta_{2,1}^{1,1} & -\Delta_{2,2}^{0,1} \\ \hline & \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,1} & & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{0,1} \\ & \Delta_{2,1}^{0,0} & \Delta_{2,2}^{1,1} & & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{1,1} \\ & \Delta_{2,1}^{1,0} & \Delta_{2,2}^{0,1} & & -\Delta_{2,1}^{1,1} & -\Delta_{2,2}^{0,1} \\ \hline & & & \Delta_{2,1}^{0,1} & \Delta_{2,2}^{0,0} & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{0,1} \\ & & & \Delta_{2,1}^{0,1} & \Delta_{2,2}^{1,0} & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{1,1} \\ & & & \Delta_{2,1}^{1,1} & \Delta_{2,2}^{0,0} & -\Delta_{2,1}^{1,1} & -\Delta_{2,2}^{0,1} \end{bmatrix}. \quad (4.9)$$

Clearly C has full rank, and it follows that L_2 has full column rank if and only if L_2'' does.

Step 3. Recall the definition of stock price differences (4.4). Let $\vec{\varepsilon}_1 \neq \mathbf{1}$. In $(L_2^{\vec{\varepsilon}_1})''$, subtracting its Row 1 from all other rows leaves each of them with precisely two nonzero entries of the same magnitude, $\delta_{2,j}$, with the opposite

signs; cf. (4.5). Applying this procedure to all $\vec{\varepsilon}_1 \neq \mathbf{1}$ yields

$$\left[\begin{array}{cccccc} \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,0} & & & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{0,1} \\ & -\delta_{2,2} & & & & \delta_{2,2} \\ -\delta_{2,1} & & & & \delta_{2,1} & \\ \hline & & \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,1} & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{0,1} \\ & & & -\delta_{2,2} & & \delta_{2,2} \\ & -\delta_{2,1} & & & \delta_{2,1} & \\ \hline & & & \Delta_{2,1}^{0,1} & \Delta_{2,2}^{0,0} & -\Delta_{2,1}^{0,1} & -\Delta_{2,2}^{0,1} \\ & & & & -\delta_{2,2} & & \delta_{2,2} \\ & & & -\delta_{2,1} & & \delta_{2,1} & \end{array} \right]. \quad (4.10)$$

Recall that the odd and even numbered columns correspond to the variables $\hat{h}_{1,1}(x^{\vec{\varepsilon}_1})$ and $\hat{h}_{1,2}(x^{\vec{\varepsilon}_1})$ for $\vec{\varepsilon}_1 \in \mathcal{E}_1$, respectively. In particular, Column 7 corresponds to $\hat{h}_{1,1}(x^{\mathbf{1}})$ and Column 8 corresponds to $\hat{h}_{1,2}(x^{\mathbf{1}})$. We add Columns 1, 3, 5 to Column 7, and Columns 2, 4, 6 to Column 8:

$$\left[\begin{array}{cccccc} \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,0} & & & -\delta_{1,1} & -\delta_{1,2} \\ & -\delta_{2,2} & & & & \\ -\delta_{2,1} & & & & & \\ \hline & & \Delta_{2,1}^{0,0} & \Delta_{2,2}^{0,1} & & -\delta_{1,1} \\ & & & -\delta_{2,2} & & \\ & -\delta_{2,1} & & & & \\ \hline & & & \Delta_{2,1}^{0,1} & \Delta_{2,2}^{0,0} & & -\delta_{1,2} \\ & & & & -\delta_{2,2} & & \\ & & & -\delta_{2,1} & & & \end{array} \right]. \quad (4.11)$$

All rows except Rows 1, 4, 7 have exactly one nonzero entry, and none of these entries share a common column. (We remark that Rows 1, 4, 7 correspond to the multi-indices $(\vec{\varepsilon}_1, 0, 0)$ for $\vec{\varepsilon}_1 \neq \mathbf{1}$ in \mathcal{E}_1 .) Dropping these rows and columns, we denote the remaining 3×2 submatrix by M_2 :

$$M_2 = \begin{bmatrix} -\delta_{1,1} & -\delta_{1,2} \\ -\delta_{1,1} & \\ & -\delta_{1,2} \end{bmatrix}. \quad (4.12)$$

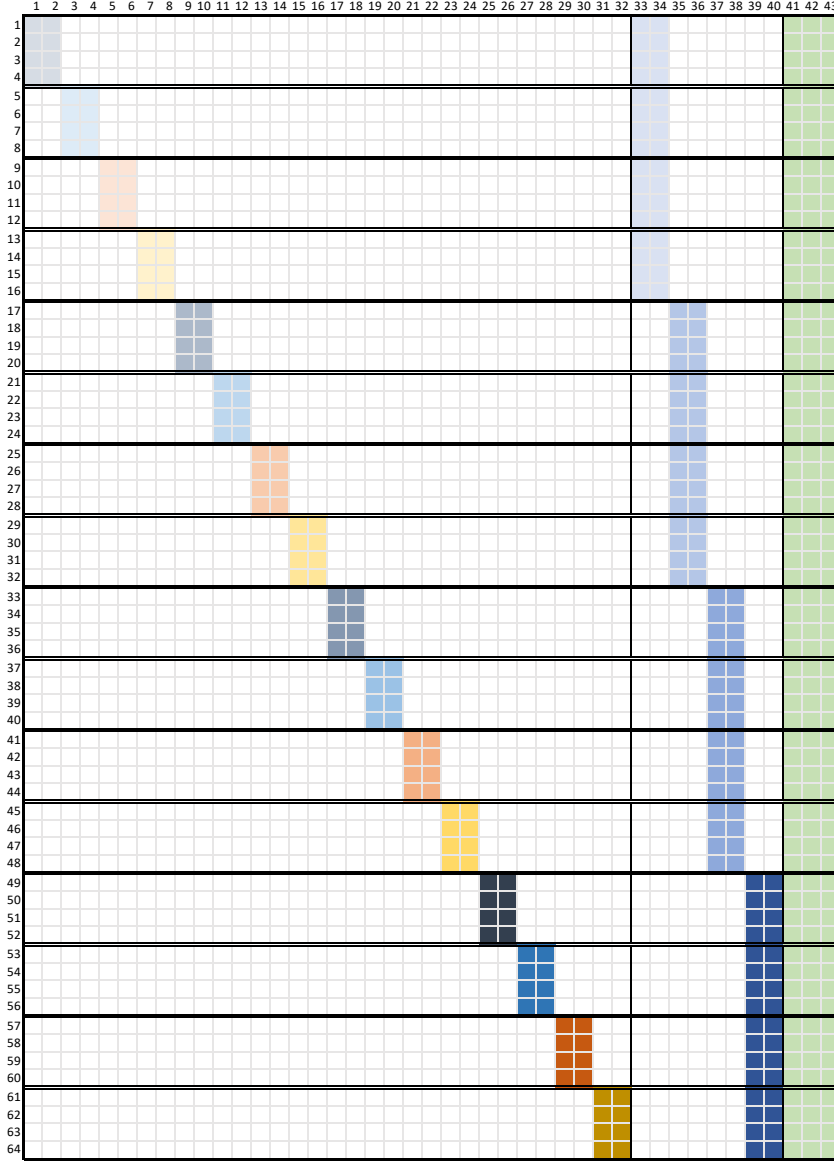


Figure 2: Illustration of L_T in the proof of Lemma 4.1 for $T = 3$ and $d = 2$. Columns 1–32 correspond to the variables $\hat{h}_{2,j}(x^{\vec{\varepsilon}_2})$ for $1 \leq j \leq 2$ and $\vec{\varepsilon}_2 \in \mathcal{E}_2$, Columns 33–40 correspond to the variables $\hat{h}_{1,j}(x^{\vec{\varepsilon}_1})$ for $1 \leq j \leq 2$ and $\vec{\varepsilon}_1 \in \mathcal{E}_1$, and lastly Columns 41–43 correspond to the variables $\hat{g}_1(x_{3,1}^0), \hat{g}_1(x_{3,1}^1), \hat{g}_2(x_{3,2}^1)$.

that for some fixed $1 \leq k \leq d$, we have $\eta_{t,j} = \rho_{t,j} = \varepsilon_{t,j}$ for all $1 \leq t \leq T-1$ and $1 \leq j \leq d$, $\eta_{T,j} = \rho_{T,j}$ for all $j \neq k$, and lastly $\eta_{T,k} = 0$ and $\rho_{T,k} = 1$. Then, it follows from (3.1) that

$$V(x^{\vec{\eta}_T}) - V(x^{\vec{\rho}_T}) = \hat{h}_{T-1,k}(x^{\vec{\varepsilon}_{T-1}})(x_{T,k}^0 - x_{T,k}^1) + \hat{g}_k(x_{T,k}^0) - \hat{g}_k(x_{T,k}^1), \quad (4.14)$$

where the right-hand side does not depend on any $x_{T,j}^{\eta_{T,j}} = x_{T,j}^{\rho_{T,j}}$ for $j \neq k$ (i.e., the shared stock prices at date T).

Let $\mathbf{0}$ and e_j , $1 \leq j \leq d$, be the zero vector and the unit vectors in $\{0, 1\}^d$. Note that $\vec{\varepsilon}_{T-1} \in \mathcal{E}_{T-1}$ is still fixed, and that $(\vec{\varepsilon}_{T-1}, \mathbf{0})$ and $(\vec{\varepsilon}_{T-1}, e_j)$, $1 \leq j \leq d$, are $d+1$ elements in \mathcal{E}_T . Let $(L_T^{\vec{\varepsilon}_{T-1}})'$ be the submatrix of $L_T^{\vec{\varepsilon}_{T-1}}$ whose rows are generated by these elements. By repeated applications of (4.14), it is not hard to see that all rows in $L_T^{\vec{\varepsilon}_{T-1}}$ are linear combinations of the rows in $(L_T^{\vec{\varepsilon}_{T-1}})'$. This argument applies to all $\vec{\varepsilon}_{T-1} \in \mathcal{E}_{T-1}$. We form the matrix L'_T by vertically stacking $(L_T^{\vec{\varepsilon}_{T-1}})'$ for all $\vec{\varepsilon}_{T-1} \in \mathcal{E}_{T-1}$. It follows that L'_T has the same rank as L_T .

As a remark for later use, we note that for each $1 \leq t \leq T-1$, coefficients of L_T (and L'_T) in the columns corresponding to $\hat{h}_{t,j}(x^{\vec{\varepsilon}_t})$ for $1 \leq j \leq d$ are nonzero if and only if they belong to the rows that arise from evaluating V on $D^{\vec{\varepsilon}_t}$. In other words, for each $1 \leq t \leq T-1$, the $d2^{dt}$ columns of L_T that correspond to $\hat{h}_{t,j}(x^{\vec{\varepsilon}_t})$ for $1 \leq j \leq d$ and $\vec{\varepsilon}_t \in \mathcal{E}_t$ form a block diagonal matrix, where each block has $2^{d(T-t)}$ rows and d columns.

Step 2. After some block-by-block row operations, we will bring L'_2 to a block lower triangular matrix

$$\begin{bmatrix} L''_T & \\ B & C \end{bmatrix}, \quad \text{with } C \text{ as in (4.13).}$$

For $1 \leq t \leq T-1$, we denote $(1, \dots, 1) \in \mathcal{E}_t$ by $\mathbf{1}_t$. Fix $\vec{\varepsilon}_{T-1} \neq \mathbf{1}_{T-1}$ in \mathcal{E}_{T-1} . After subtracting $(L_T^{\mathbf{1}_{T-1}})'$ from $(L_T^{\vec{\varepsilon}_{T-1}})'$, we denote the resultant matrix by $(L_T^{\vec{\varepsilon}_{T-1}})''$. Applying this procedure to all $\vec{\varepsilon}_{T-1} \neq \mathbf{1}_{T-1}$ brings L'_T into the form above, where the submatrix $[B \ C]$ is precisely $(L_T^{\mathbf{1}_{T-1}})'$. Our choice of $d+1$ rows in Step 1 guarantees that C is the matrix (4.13) and has full rank. Thus, it follows that L_T has full column rank if and only if L''_T does.

To understand the structure of L''_T , we need to recall the remark in Step 1 and the stock price difference (4.4). Note that the columns of L''_T that correspond to $\hat{h}_{t,j}(x^{\vec{\varepsilon}_t})$ for $1 \leq t \leq T-1$, $1 \leq j \leq d$ and $\vec{\varepsilon}_t \neq \mathbf{1}_t$ in \mathcal{E}_t remain the same as in L'_T , since these columns are identically zero in $(L_T^{\mathbf{1}_{T-1}})'$. On the other hand, the nonzero coefficients of $(L_T^{\mathbf{1}_{T-1}})'$ in the columns that

correspond to $\hat{h}_{t,j}(x^{\mathbf{1}^t})$ are subtracted from the coefficients of L'_T in these columns. We state two particular instances for later use:

- (i) The columns in L''_T that correspond to $\hat{h}_{T-1,j}(x^{\mathbf{1}^{T-1}})$ for $1 \leq j \leq d$ repeat the negative values of those columns in $(L_T^{\mathbf{1}^{T-1}})'$; cf. (4.9).
- (ii) For any $\varepsilon_{T-1} \in \{0,1\}^d$ and $1 \leq j \leq d$, the coefficient of L''_T in the row corresponding to $(\mathbf{1}_{T-2}, \varepsilon_{T-1}, \mathbf{0})$ and the column corresponding to $\hat{h}_{T-2,j}(x^{\mathbf{1}^{T-2}})$ is

$$(x_{T-1,j}^{\varepsilon_{T-1,j}} - x_{T-2,j}^{\mathbf{1}}) - (x_{T-1,j}^{\mathbf{1}} - x_{T-2,j}^{\mathbf{1}}) = \begin{cases} \delta_{T-1,j} & \text{if } \varepsilon_{T-1,j} = 0 \\ 0 & \text{if } \varepsilon_{T-1,j} = 1. \end{cases}$$

Analogous patterns hold in the columns corresponding to $\hat{h}_{t,j}(x^{\mathbf{1}^t})$ for $1 \leq t \leq T-3$.

Step 3. Note that L''_T has $(d+1)(2^{d(T-1)} - 1)$ rows and $d \sum_{t=1}^{T-1} 2^{dt}$ columns. Fix $\vec{\varepsilon}_{T-1} \neq \mathbf{1}_{T-1}$. In $(L_T^{\vec{\varepsilon}_{T-1}})''$, we subtract its Row 1 from all other rows. In view of (i) in Step 2, a variant of (4.14) implies that for $1 \leq j \leq d$, Row $(d+2-j)$ has two nonzero entries of the same magnitude, $\delta_{T,j}$, with the opposite signs. More precisely, $-\delta_{T,j}$ is in the column corresponding to $\hat{h}_{T-1,j}(x^{\vec{\varepsilon}_{T-1}})$ and $\delta_{T,j}$ is in the column corresponding to $\hat{h}_{T-1,j}(x^{\mathbf{1}^{T-1}})$; cf. (4.10).

For $1 \leq j \leq d$, we add the column corresponding to $\hat{h}_{T-1,j}(x^{\vec{\varepsilon}_{T-1}})$ to the column corresponding to $\hat{h}_{T-1,j}(x^{\mathbf{1}^{T-1}})$. As a result, for $1 \leq j \leq d$, Row $(d+2-j)$ now has precisely one nonzero entry, $-\delta_{T,j}$, and this entry is located in the column corresponding to $\hat{h}_{T-1,j}(x^{\vec{\varepsilon}_{T-1}})$; cf. (4.11). Moreover, for $1 \leq j \leq d$, the first coefficient of $(L_T^{\vec{\varepsilon}_{T-1}})''$ in the column corresponding to $\hat{h}_{T-1,j}(x^{\mathbf{1}^{T-1}})$ is

$$\begin{cases} -\delta_{T-1,j} & \text{if } \varepsilon_{T-1,j} = 0 \\ 0 & \text{if } \varepsilon_{T-1,j} = 1 \end{cases} \quad (4.15)$$

and all other coefficients are zero. We apply this procedure to all $\vec{\varepsilon}_{T-1} \neq \mathbf{1}_{T-1}$. As a result, $d(2^{d(T-1)} - 1)$ rows have only one nonzero entry each, and these entries respectively belong to the first $d(2^{d(T-1)} - 1)$ columns of L''_T . Dropping these rows and columns, we denote the remaining matrix by M_T . Thus, we have shown that L_T has full column rank if and only if M_T does.

Step 4. We now show that M_T indeed has full column rank. We argue by

induction on T ; the base case $T = 2$ was established in Remark 4.4. Note that M_T has $(2^{d(T-1)} - 1)$ rows and $d \sum_{t=0}^{T-2} 2^{dt}$ columns, where

- the rows correspond to $(\vec{\varepsilon}_{T-1}, \mathbf{0})$ for all $\vec{\varepsilon}_{T-1} \neq \mathbf{1}_{T-1}$ in \mathcal{E}_{T-1} ;
- the first d columns correspond to $\hat{h}_{T-1,j}(x^{\mathbf{1}_{T-1}})$ for $1 \leq j \leq d$; and
- the rest of the columns correspond to $\hat{h}_{t,j}(x^{\vec{\varepsilon}_t})$ for $t \in \{T-2, \dots, 1\}$, $\vec{\varepsilon}_t \in \mathcal{E}_t$ and $j \in \{1, \dots, d\}$.

The matrix in Figure 3(a) may serve as an illustration. As

$$\{(\vec{\varepsilon}_{T-1}, \mathbf{0}) : \vec{\varepsilon}_{T-1} \in \mathcal{E}_{T-1}\} = \{(\vec{\varepsilon}_{T-2}, \varepsilon_{T-1}, \mathbf{0}) : \vec{\varepsilon}_{T-2} \in \mathcal{E}_{T-2}, \varepsilon_{T-1} \in \{0, 1\}^d\}$$

we can divide M_T into submatrices $(M_T^{\vec{\varepsilon}_{T-2}})_{\vec{\varepsilon}_{T-2} \in \mathcal{E}_{T-2}}$ such that the rows of $M_T^{\vec{\varepsilon}_{T-2}}$ correspond to $(\vec{\varepsilon}_{T-2}, \varepsilon_{T-1}, \mathbf{0})$ for all $\varepsilon_{T-1} \in \{0, 1\}^d$. Each $M_T^{\vec{\varepsilon}_{T-2}}$ has 2^d rows, with the exception that $M_T^{\mathbf{1}_{T-2}}$ has $2^d - 1$ rows (since M_T does not contain the row corresponding to $\mathbf{1}_{T-1}$). For this reason, we need to treat the cases $d \geq 2$ and $d = 1$ separately.

(a) Case $d \geq 2$: Similarly to Step 1, the rows in M_T are linearly dependent, and we can reduce M_T to a matrix M'_T of $(d+1)2^{d(T-2)}$ rows without decreasing its rank. That is, M'_T consists of $2^{d(T-2)}$ submatrices $(M_T^{\vec{\varepsilon}_{T-2}})'$ of $d+1$ rows for $\vec{\varepsilon}_{T-2} \in \mathcal{E}_{T-2}$. As a consequence of (4.15), the first d columns of $(M_T^{\vec{\varepsilon}_{T-2}})'$ make up a $(d+1) \times d$ matrix

$$C'' := \begin{bmatrix} -\delta_{T-1,1} & -\delta_{T-1,2} & \dots & -\delta_{T-1,d-1} & -\delta_{T-1,d} \\ \vdots & \vdots & \dots & -\delta_{T-1,d-1} & 0 \\ \vdots & \vdots & \dots & 0 & -\delta_{T-1,d} \\ \vdots & \vdots & \dots & \ddots & -\delta_{T-1,d-1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & -\delta_{T-1,2} & \dots & \vdots & \vdots \\ -\delta_{T-1,1} & 0 & \dots & \vdots & \vdots \\ 0 & -\delta_{T-1,2} & \dots & -\delta_{T-1,d-1} & -\delta_{T-1,d} \end{bmatrix} \quad (4.16)$$

which does not depend on $\vec{\varepsilon}_{T-2}$ and has full column rank. Thus, the first d columns of M'_T are formed by vertically stacking C'' $2^{d(T-2)}$ times.

Recalling (ii) from Step 2, we note that the last $d + 1$ rows of M'_T correspond to $(\mathbb{1}_{T-2}, \mathbf{0}, \mathbf{0})$ and $(\mathbb{1}_{T-2}, \mathbf{e}_j, \mathbf{0})$ for $1 \leq j \leq d$. As a result, the submatrix of M'_T formed by taking the last $d + 1$ rows and the columns corresponding to $h_{T-2,j}(x^{\mathbb{1}_{T-2}})$ for $1 \leq j \leq d$ is precisely $-C''$. Now, adding the columns corresponding to $\hat{h}_{T-1,j}(x^{\mathbb{1}_{T-1}})$ to the columns corresponding to $\hat{h}_{T-2,j}(x^{\mathbb{1}_{T-2}})$ for $1 \leq j \leq d$ can bring M'_T to the block triangular form

$$\begin{bmatrix} A'' & M''_T \\ C'' & \end{bmatrix}.$$

(b) Case $d = 1$: There is no need to remove any rows from M_T . We set $M'_T = M_T$ and note that it has $2^{T-1} - 1$ rows. A subtle difference to (a) is that M'_T does not contain the row corresponding to $(\mathbb{1}_{T-1}, \mathbf{e}_1, \mathbf{0})$. (We remark that $\mathbf{e}_1 = 1$ and $\mathbf{0} = 0$ in the case $d = 1$.) The coefficients of M'_T in the first column alternate between $-\delta_{T-1,1}$ and 0. Also, the last row of M'_T has only one nonzero entry, $\delta_{T-1,1}$, in the column corresponding to

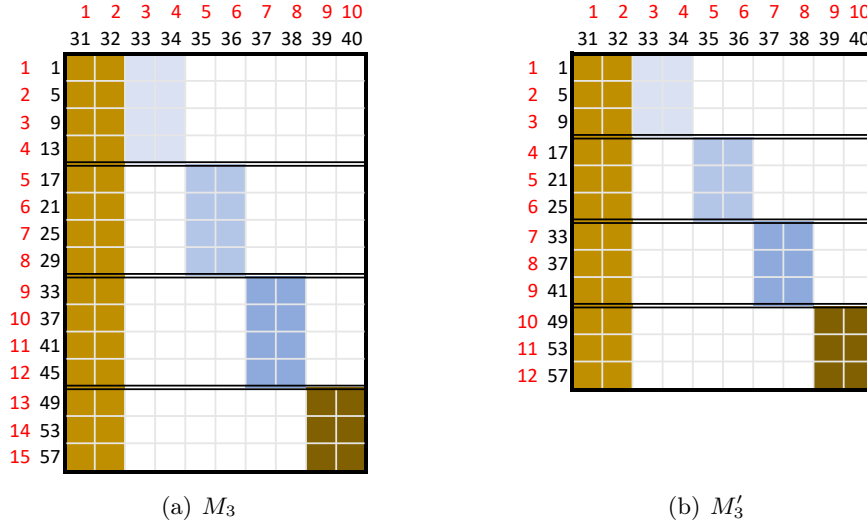


Figure 3: Illustration of Step 4 in the proof of Lemma 4.1 for $T = 3$ and $d = 2$. The row and column numbers in black are inherited from L_3 , and the new row and column numbers are printed in red. The first two columns correspond to $\hat{h}_{2,j}(x^{\mathbb{1}_2})$ for $j = 1, 2$; the rest of the columns correspond to $\hat{h}_{1,j}(x^{\varepsilon_1})$ for $j = 1, 2$ and $\varepsilon_1 \in \mathcal{E}_1$. Coefficients in each light gold block are the negatives of those in the dark gold block.

$h_{T-2,1}(x^{1^{T-2}})$. Adding the column corresponding to $\hat{h}_{T-1,1}(x^{1^{T-1}})$ to the column corresponding to $\hat{h}_{T-2,1}(x^{1^{T-2}})$ brings M_T' to the form above with the 1×1 matrix $C'' = [-\delta_{T-1,1}]$.

Therefore, in either case, M_T'' is a matrix with $(d+1)(2^{d(T-2)} - 1)$ rows and $d \sum_{t=1}^{T-2} 2^{dt}$ columns. It remains to show that it has full column rank. Applying the same procedure as in Step 3 to M_T'' , we see that $d(2^{d(T-2)} - 1)$ rows have only one nonzero entry each, and none of these entries share a common column. We then drop these rows and columns. One can check that the remaining matrix, of $(2^{d(T-2)} - 1)$ rows and $d \sum_{t=0}^{T-3} 2^{dt}$ columns, is precisely M_{T-1} ; we omit further details in the interest of brevity. The inductive hypothesis applies. \square

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