The Bellman Equation for Power Utility Maximization with Semimartingales

Marcel Nutz
ETH Zurich, Department of Mathematics, 8092 Zurich, Switzerland
marcel.nutz@math.ethz.ch
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Abstract

We study utility maximization for power utility random fields with and without intermediate consumption in a general semimartingale model with closed portfolio constraints. We show that any optimal strategy leads to a solution of the corresponding Bellman equation. The optimal strategies are described pointwise in terms of the opportunity process, which is characterized as the minimal solution of the Bellman equation. We also give verification theorems for this equation.

Keywords power utility, Bellman equation, opportunity process, semimartingale characteristics, BSDE.
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1 Introduction

A classical problem of mathematical finance is the maximization of expected utility obtained from consumption or from terminal wealth. This paper focuses on power utility functions and presents the corresponding dynamic programming in a general constrained semimartingale framework. The homogeneity of these utility functions leads to a factorization of the value process into a part depending on the current wealth and the so-called opportunity process L. In our setting, the Bellman equation describes the drift rate of L and clarifies the local structure of our problem. Finding an optimal strategy boils down to maximizing a random function $y \mapsto g(\omega, t, y)$ on $\mathbb{R}^d$ for every state ω and date t. This function is given in terms of the semimartingale characteristics of L as well as the asset returns, and its maximum yields the drift rate of L. The role of the opportunity process is to augment the information contained in the return characteristics in order to have a local sufficient statistic for the global optimization problem.

We present three main results. First, we show that if there exists an optimal strategy for the utility maximization problem, *the opportunity process L solves the Bellman equation* and we provide a local description of the optimal strategies. We state the Bellman equation in two forms, as an identity for the drift rate of L and as a backward stochastic differential equation (BSDE) for L. Second, we characterize the opportunity process as the *minimal solution* of this equation. Finally, given some solution and an associated strategy, one can ask whether the strategy is optimal and the solution is the opportunity process. We present two different approaches which lead to *verification theorems* not comparable in strength unless the constraints are convex.

The present dynamic programming approach should be seen as complementary to convex duality, which remains the only method to obtain *existence* of optimal strategies in general models; see Kramkov and Schachermayer [21], Karatzas and Žitković [20], Karatzas and Kardaras [19]. However, convex duality alone offers limited insight into the optimal strategies for incomplete markets. In some cases, the Bellman equation can be solved directly by analytic methods; e.g., in the setting of Example 5.8 with continuous asset prices or in the Lévy process setting of Nutz [25]. In addition to the existence, one then obtains a way to compute the optimal strategies (at least numerically) and study their properties.

This paper is organized as follows. The next section specifies the optimization problem in detail, recalls the opportunity process and the martingale optimality principle, and fixes the notation for the characteristics. We also introduce set-valued processes describing the budget condition and state the assumptions on the portfolio constraints. Section 3 derives the Bellman equation, first as a drift condition and then as a BSDE. It becomes more explicit as we specialize to the case of continuous asset prices. The definition of a solution of the Bellman equation is given in Section 4, where we show the minimality of the opportunity process. Section 5 deals with the verification problem, which is converse to the derivation of the Bellman equation since it requires the passage from the local maximization to the global optimization problem. We present an approach via the value process and a second approach via a deflator, which corresponds to the dual problem in a suitable setting. Appendix A belongs to Section 3 and contains the measurable selections for the construction of the Bellman equation. It is complemented by Appendix B, where we construct an alternative parametrization of the market model by representative portfolios.

2 Preliminaries

The following notation is used. If $x, y \in \mathbb{R}$, we denote $x^+ = \max\{x, 0\}$ and $x \wedge y = \min\{x, y\}$. We set $1/0 := \infty$ where necessary. If $z \in \mathbb{R}^d$ is a d-dimensional vector, z^i is its ith coordinate, $z^\top$ its transpose, and $|z| = (z^\top z)^{1/2}$ the Euclidean norm. If X is an $\mathbb{R}^d$-valued semimartingale and π is an $\mathbb{R}^d$-valued predictable integrand, the vector stochastic integral is a scalar semimartingale with initial value zero and denoted by $\int \pi \, dX$ or by $\pi \bullet X$. The quadratic variation is the $d \times d$-matrix $[X] := [X, X]$ and if Y is a scalar semimartingale, $[X, Y]$ is the d-vector with $[X, Y]^i := [X^i, Y]$. When the reference measure is understood, relations between measurable functions hold almost everywhere unless otherwise mentioned. Our reference for any unexplained notion from stochastic calculus is Jacod and Shiryaev [15].

2.1 The Optimization Problem

We fix the time horizon $T \in (0, \infty)$ and a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfies the usual assumptions of right continuity and completeness as well as $\mathcal{F}_0 = \{\emptyset, \Omega\}$ P-a.s. We consider an $\mathbb{R}^d$-valued càdlàg semimartingale R with $R_0 = 0$ representing the returns of d risky assets. Their discounted prices are given by the stochastic exponential $S = \mathcal{E}(R) = (\mathcal{E}(R^1), \dots, \mathcal{E}(R^d))$; in the financial application, the components of S are assumed to be positive. Our agent also has a bank account at his disposal; it does not pay interest.

The agent is endowed with a deterministic initial capital $x_0 > 0$. A *trading strategy* is a predictable R-integrable $\mathbb{R}^d$-valued process π, where π^i indicates the fraction of wealth (or the portfolio proportion) invested in the ith risky asset. A *consumption strategy* is a nonnegative optional process c such that $\int_0^T c_t \, dt < \infty$ P-a.s. We want to consider two cases. Either consumption occurs only at the terminal time T (utility from "terminal wealth" only); or there is intermediate consumption plus a bulk consumption at the time horizon. To unify the notation, we introduce the measure μ on $[0, T]$ by

$$\mu(dt) := \begin{cases} 0 & \text{in the case without intermediate consumption,} \\ dt & \text{in the case with intermediate consumption.} \end{cases}$$

Let also $\mu^\circ := \mu + \delta_{\{T\}}$, where $\delta_{\{T\}}$ is the unit Dirac measure at T. The *wealth process* $X(\pi, c)$ corresponding to a pair (π, c) is defined by the equation

$$X_t(\pi, c) = x_0 + \int_0^t X_{s-}(\pi, c) \pi_s \, dR_s - \int_0^t c_s \, \mu(ds), \quad 0 \le t \le T.$$

We define the set of trading and consumption pairs

$$\mathcal{A}^0(x_0) := \big\{(\pi, c) : \, X(\pi, c) > 0, \, X_-(\pi, c) > 0 \text{ and } c_T = X_T(\pi, c)\big\}.$$

These are the strategies that satisfy the budget constraint. The convention $c_T = X_T(\pi, c)$ means that all the remaining wealth is consumed at time T. We consider also exogenous constraints imposed on the agent. For each $(\omega, t) \in \Omega \times [0, T]$ we are given a set $\mathscr{C}_t(\omega) \subseteq \mathbb{R}^d$ which contains the origin. The set of (constrained) *admissible* strategies is

$$\mathcal{A}(x_0) := \big\{(\pi, c) \in \mathcal{A}^0(x_0) : \pi_t(\omega) \in \mathscr{C}_t(\omega) \text{ for all } (\omega, t)\big\};$$

it is nonempty as $0 \in \mathscr{C}_t(\omega)$. Further assumptions on the set-valued mapping $\mathscr{C}$ will be introduced in Section 2.4. We fix the initial capital x_0 and usually write $\mathcal{A}$ for $\mathcal{A}(x_0)$. Abusing the notation, we write $c \in \mathcal{A}$ and call c admissible if there exists π such that $(\pi, c) \in \mathcal{A}$; an analogous convention is used for similar expressions.

We will often parametrize the consumption strategies as a fraction of wealth. Let $(\pi, c) \in \mathcal{A}$ and $X = X(\pi, c)$. Then

$$\kappa := \frac{c}{X}$$

is called the *propensity to consume* corresponding to (π, c). This yields a one-to-one correspondence between the pairs $(\pi, c) \in \mathcal{A}$ and the pairs (π, κ) such that $\pi \in \mathcal{A}$ and κ is a nonnegative optional process satisfying $\int_0^T \kappa_s\, ds < \infty$ P-a.s. and $\kappa_T = 1$ (see Nutz [26, Remark 2.1] for details). We shall abuse the notation and identify a consumption strategy with the corresponding propensity to consume; e.g., we write $(\pi, \kappa) \in \mathcal{A}$. Note that

$$X(\pi, \kappa) = x_0 \mathcal{E}\big(\pi \bullet R - \kappa \bullet \mu\big).$$

This simplifies verifying that some pair (π, κ) is admissible as $X(\pi, \kappa) > 0$ implies $X_-(\pi, \kappa) > 0$ (cf. [15, II.8a]).

The preferences of the agent are modeled by a time-additive random utility function as follows. Let D be a càdlàg, adapted, strictly positive process such that $E\big[\int_0^T D_s\, \mu^\circ(ds)\big] < \infty$ and fix $p \in (-\infty, 0) \cup (0, 1)$. We define the power utility random field

$$U_t(x) := D_t \tfrac{1}{p} x^p, \quad x \in (0, \infty),\ t \in [0, T].$$

This is the general form of a *p-homogeneous* utility random field such that a constant consumption yields finite expected utility. Interpretations and applications for the process D are discussed in [26]. We denote by U^* the convex conjugate of $x \mapsto U_t(x)$,

$$U_t^*(y) = \sup_{x>0} \big\{U_t(x) - xy\big\} = -\tfrac{1}{q} y^q D_t^\beta; \tag{2.1}$$

here $q := \frac{p}{p-1} \in (-\infty, 0) \cup (0, 1)$ is the exponent conjugate to p and the constant $\beta := \frac{1}{1-p} > 0$ is the relative risk tolerance of U. Note that we exclude the well-studied logarithmic utility (e.g., Goll and Kallsen [11]) which corresponds to $p=0$.

The *expected utility* corresponding to a consumption strategy $c \in \mathcal{A}$ is $E\big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\big]$; i.e., either $E[U_T(c_T)]$ or $E[\int_0^T U_t(c_t)\,dt + U_T(c_T)]$. The (value of the) utility maximization problem is said to be *finite* if

$$u(x_0) := \sup_{c\in\mathcal{A}(x_0)} E\Big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\Big] < \infty. \tag{2.2}$$

Note that this condition is void if $p<0$ as then $U<0$. If (2.2) holds, a strategy $(\pi,c)\in\mathcal{A}(x_0)$ is called *optimal* if $E\big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\big] = u(x_0)$.

Finally, we introduce the following sets; they are of minor importance and used only in the case $p<0$:

$$\begin{aligned}\mathcal{A}^f &:= \big\{(\pi,c)\in\mathcal{A} :\ \textstyle\int_0^T U_t(c_t)\,\mu^\circ(dt) > -\infty\big\},\\ \mathcal{A}^{fE} &:= \big\{(\pi,c)\in\mathcal{A} :\ E\big[\textstyle\int_0^T U_t(c_t)\,\mu^\circ(dt)\big] > -\infty\big\}.\end{aligned}$$

Anticipating that (2.2) will be in force, the indices stand for "finite" and "finite expectation". Clearly $\mathcal{A}^{fE}\subseteq\mathcal{A}^f\subseteq\mathcal{A}$, and equality holds if $p\in(0,1)$.

2.2 Opportunity Process

We recall the opportunity process, a reduced form of the value process in the language of control theory. We assume (2.2) in this section, which ensures that the following process is finite. By [26, Proposition 3.1, Remark 3.7] there exists a unique càdlàg semimartingale L, called *opportunity process*, such that

$$L_t\,\tfrac{1}{p}\big(X_t(\pi,c)\big)^p = \operatorname*{ess\,sup}_{\tilde c\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde c_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \tag{2.3}$$

for any $(\pi,c)\in\mathcal{A}$, where $\mathcal{A}(\pi,c,t) := \big\{(\tilde\pi,\tilde c)\in\mathcal{A} :\ (\tilde\pi,\tilde c) = (\pi,c) \text{ on } [0,t]\big\}$. We note that $L_T = D_T$ and that $u(x_0) = L_0\frac{1}{p}x_0^p$ is the value function from (2.2). The following is contained in [26, Lemma 3.5].

Lemma 2.1. *L is a special semimartingale for all p. If $p\in(0,1)$, then $L, L_- > 0$ up to evanescence. If $p<0$, the same holds provided that an optimal strategy exists.*

Proposition 2.2 ([26, Proposition 3.4])**.** *Let $(\pi,c)\in\mathcal{A}^{fE}$. Then the process*

$$L_t\tfrac{1}{p}\big(X_t(\pi,c)\big)^p + \int_0^t U_s(c_s)\,\mu(ds), \quad t\in[0,T]$$

is a supermartingale; it is a martingale if and only if (π,c) is optimal.

This is the "martingale optimality principle". The expected terminal value of this process equals $E[\int_0^T U_t(c_t)\,\mu^\circ(dt)]$, hence the assertion fails for $(\pi,c)\in\mathcal{A}\setminus\mathcal{A}^{fE}$.

2.3 Semimartingale Characteristics

In the remainder of this section we introduce tools which are necessary to describe the optimization problem locally. The use of semimartingale characteristics and set-valued processes follows [11] and [19], which consider logarithmic utility and convex constraints. That problem differs from ours in that it is "myopic"; i.e., the characteristics of R are sufficient to describe the local problem and so there is no need for an opportunity process.

We refer to [15] for background regarding semimartingale characteristics and random measures. Let μ^R be the integer-valued random measure associated with the jumps of R and let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a cut-off function; i.e., h is bounded and $h(x) = x$ in a neighborhood of $x = 0$. Let (B^R, C^R, ν^R) be the predictable characteristics of R relative to h. The *canonical representation* of R (cf. [15, II.2.35]) is

$$R = B^R + R^c + h(x) * (\mu^R - \nu^R) + (x - h(x)) * \mu^R. \tag{2.4}$$

The finite variation process $(x - h(x)) * \mu^R$ contains essentially the "large" jumps of R. The rest is the canonical decomposition of the semimartingale $\bar{R} = R - (x - h(x)) * \mu^R$, which has bounded jumps: $B^R = B^R(h)$ is predictable of finite variation, R^c is a continuous local martingale, and $h(x) * (\mu^R - \nu^R)$ is a purely discontinuous local martingale.

As L is a special semimartingale (Lemma 2.1), it has a canonical decomposition $L = L_0 + A^L + M^L$. Here L_0 is constant, A^L is predictable of finite variation and also called the *drift* of L, M^L is a local martingale, and $A^L_0 = M^L_0 = 0$. Analogous notation will be used for other special semimartingales. It is then possible to consider the characteristics (A^L, C^L, ν^L) of L with respect to the identity instead of a cut-off function. Writing x' for the identity on $\mathbb{R}$, the canonical representation is

$$L = L_0 + A^L + L^c + x' * (\mu^L - \nu^L);$$

see [15, II.2.38]. It will be convenient to use the joint characteristics of the $\mathbb{R}^d \times \mathbb{R}$-valued process (R, L). We denote a generic point in $\mathbb{R}^d \times \mathbb{R}$ by (x, x') and let $(B^{R,L}, C^{R,L}, \nu^{R,L})$ be the characteristics of (R, L) with respect to the function $(x, x') \mapsto (h(x), x')$. More precisely, we choose "good" versions of the characteristics so that they satisfy the properties given in [15, II.2.9]. For the $(d+1)$-dimensional process (R, L) we have the canonical representation

$$\begin{pmatrix} R \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ L_0 \end{pmatrix} + \begin{pmatrix} B^R \\ A^L \end{pmatrix} + \begin{pmatrix} R^c \\ L^c \end{pmatrix} + \begin{pmatrix} h(x) \\ x' \end{pmatrix} * (\mu^{R,L} - \nu^{R,L}) + \begin{pmatrix} x - h(x) \\ 0 \end{pmatrix} * \mu^{R,L}.$$

We denote by $(b^{R,L}, c^{R,L}, F^{R,L}; A)$ the *differential* characteristics with respect to a predictable locally integrable increasing process A; e.g.,

$$A_t := t + \sum_i \mathrm{Var}(B^{RL,i})_t + \sum_{i,j} \mathrm{Var}(C^{RL,ij})_t + \big(|(x, x')|^2 \wedge 1\big) * \nu^{R,L}_t.$$

Then $b^{R,L} \bullet A = B^{R,L}$, $c^{R,L} \bullet A = C^{R,L}$, and $F^{R,L} \bullet A = \nu^{R,L}$. We write $b^{R,L} = (b^R, a^L)^\top$ and $c^{R,L} = \left(\begin{smallmatrix} c^R & c^{RL} \\ (c^{RL})^\top & c^L \end{smallmatrix}\right)$, i.e., c^{RL} is a d-vector satisfying $(c^{RL}) \bullet A = \langle R^c, L^c \rangle$. We will often use that

$$\int_{\mathbb{R}^d \times \mathbb{R}} (|x|^2 + |x'|^2) \wedge (1 + |x'|)\; F^{R,L}(d(x,x')) < \infty \tag{2.5}$$

because L is a special semimartingale (cf. [15, II.2.29]). Let Y be any scalar semimartingale with differential characteristics (b^Y, c^Y, F^Y) relative to A and a cut-off function $\bar{h}$. We call

$$a^Y := b^Y + \int \big(x - \bar{h}(x)\big)\, F^Y(dx)$$

the *drift rate* of Y whenever the integral is well defined with values in $[-\infty, \infty]$, even if it is not finite. Note that a^Y does not depend on the choice of $\bar{h}$. If Y is special, the drift rate is finite and even A-integrable (and vice versa). As an example, a^L is the drift rate of L and $a^L \bullet A = A^L$ yields the drift.

Remark 2.3. Assume Y is a *nonpositive* scalar semimartingale. Then its drift rate a^Y is well defined with values in $[-\infty, \infty)$. Indeed, the fact that $Y = Y_- + \Delta Y \leq 0$ implies that $x \leq -Y_-$, $F^Y(dx)$-a.e.

If Y is a scalar semimartingale with drift rate $a^Y \in [-\infty, 0]$, we call Y a *semimartingale with nonpositive drift rate*. Here a^Y need not be finite, as in the case of a compound Poisson process with negative, non-integrable jumps. We refer to Kallsen [17] for the concept of σ-localization. Denoting by $L(A)$ the set of A-integrable processes and recalling that $\mathcal{F}_0$ is trivial, we conclude the following, e.g., from [19, Appendix 3].

Lemma 2.4. *Let Y be a semimartingale with nonpositive drift rate.*

(i) Y is a σ-supermartingale $\Leftrightarrow$ a^Y is finite $\Leftrightarrow$ Y is σ-locally of class (D).

(ii) Y is a local supermartingale $\Leftrightarrow$ $a^Y \in L(A)$ $\Leftrightarrow$ Y is locally of class (D).

(iii) If Y is uniformly bounded from below, it is a supermartingale.

2.4 Constraints and Degeneracies

We introduce some set-valued processes that will be used in the sequel, that is, for each (ω, t) they describe a subset of $\mathbb{R}^d$. We refer to Rockafellar [28] and Aliprantis and Border [1, §18] for background.

We start by expressing the budget constraint in this fashion. The process

$$\mathscr{C}^0_t(\omega) := \Big\{ y \in \mathbb{R}^d : F^R_t(\omega)\big\{x \in \mathbb{R}^d : y^\top x < -1\big\} = 0 \Big\}$$

was called the *natural constraints* in [19]. Clearly $\mathscr{C}^0$ is closed, convex, and contains the origin. Moreover, one can check (see [19, §3.3]) that it is *predictable* in the sense that for each closed $G \subseteq \mathbb{R}^d$, the lower inverse image $(\mathscr{C}^0)^{-1}(G) = \{(\omega,t) : \mathscr{C}_t(\omega) \cap G \neq \emptyset\}$ is predictable. (Here one can replace closed by compact or by open; see [28, 1A].) A statement such as "$\mathscr{C}^0$ is closed" means that $\mathscr{C}^0_t(\omega)$ is closed for all (ω,t); moreover, we will often omit the arguments (ω,t). We also consider the slightly smaller set-valued process

$$\mathscr{C}^{0,*} := \Big\{ y \in \mathbb{R}^d : F^R\big\{x \in \mathbb{R}^d : y^\top x \leq -1\big\} = 0 \Big\}.$$

These processes relate to the budget constraint as follows.

Lemma 2.5. *A process $\pi \in L(R)$ satisfies $\mathcal{E}(\pi \bullet R) \geq 0\,(>0)$ up to evanescence if and only if $\pi \in \mathscr{C}^0$ $(\mathscr{C}^{0,*})$ $P \otimes A$-a.e.*

Proof. Recall that $\mathcal{E}(\pi \bullet R) > 0$ if and only if $1 + \pi^\top \Delta R > 0$ ([15, II.8a]). Writing $V(x) = 1_{\{x:\, 1+\pi^\top x \leq 0\}}(x)$, we have that $(P \otimes A)\{\pi \notin \mathscr{C}^{0,*}\} = E[V(x) * \nu^R_T] = E[V(x) * \mu^R_T] = E\big[\sum_{s \leq T} 1_{\{x:\, 1+\pi_s^\top \Delta R_s \leq 0\}}\big]$. For the equivalence with $\mathscr{C}^0$, interchange strict and non-strict inequality signs. □

The process $\mathscr{C}^{0,*}$ is not closed in general (nor relatively open). Clearly $\mathscr{C}^{0,*} \subseteq \mathscr{C}^0$, and in fact $\mathscr{C}^0$ is the closure of $\mathscr{C}^{0,*}$: for $y \in \mathscr{C}^0_t(\omega)$, the sequence $\{(1-1/n)y\}_{n\geq 1}$ is in $\mathscr{C}^{0,*}_t(\omega)$ and converges to y. This implies that $\mathscr{C}^{0,*}$ is predictable; cf. [1, 18.3]. We will not be able to work directly with $\mathscr{C}^{0,*}$ because closedness is essential for the measurable selection arguments that will be used.

We turn to the exogenous portfolio constraints; i.e., the set-valued process $\mathscr{C}$ containing the origin. We consider the following conditions:

(C1) $\mathscr{C}$ is predictable.

(C2) $\mathscr{C}$ is closed.

(C3) *If* $p \in (0,1)$: There exists a $(0,1)$-valued process $\underline{\eta}$ such that
$$y \in (\mathscr{C} \cap \mathscr{C}^0) \setminus \mathscr{C}^{0,*} \implies \eta y \in \mathscr{C} \text{ for all } \eta \in (\underline{\eta}, 1),\ P \otimes A\text{-a.e.}$$

Condition (C3) is clearly satisfied if $\mathscr{C} \cap \mathscr{C}^0 \subseteq \mathscr{C}^{0,*}$, which includes the case of a continuous process R, and it is always satisfied if $\mathscr{C}$ is convex or, more generally, star-shaped with respect to the origin. If $p < 0$, (C3) should be read as always being satisfied.

We require (C3) to exclude a degenerate situation where, despite the Inada condition $U'(0) = \infty$, it is actually desirable for the agent to have a wealth process that vanishes in some states. That situation, illustrated in the subsequent example, would necessitate a more complicated notation while it can arise only in cases that are of minor interest.

Example 2.6. We assume that there is no intermediate consumption and $x_0 = 1$. Consider the one-period binomial model of a financial market; i.e., $S = \mathcal{E}(R)$ is a scalar process which is constant up to time T, where it has a single jump; say, $P[\Delta R_T = -1] = p_0$ and $P[\Delta R_T = K] = 1 - p_0$, where

$K > 0$ is a constant and $p_0 \in (0,1)$. The filtration is generated by R and we consider $\mathscr{C} \equiv \{0\} \cup \{1\}$. Then $E[U(X_T(\pi))] = U(1)$ if $\pi_T = 0$ and $E[U(X_T(\pi))] = p_0 U(0) + (1-p_0)U(1+K)$ if $\pi_T = 1$. If $U(0) > -\infty$, and if K is large enough, $\pi_T = 1$ performs better despite the fact that its *terminal wealth vanishes* with probability $p_0 > 0$. Of course, this cannot happen if $U(0) = -\infty$, i.e., $p < 0$.

By adjusting the constants in the example, one can also see that under non-convex constraints, there is in general *no uniqueness* for the optimal wealth processes (even if they are positive).

The final set-valued process is related to linear dependencies of the assets. As in [19], the predictable process of *null-investments* is

$$\mathscr{N} := \big\{ y \in \mathbb{R}^d : \, y^\top b^R = 0, \, y^\top c^R = 0, \, F^R\{x : \, y^\top x \neq 0\} = 0 \big\}.$$

Its values are linear subspaces of $\mathbb{R}^d$, hence closed, and provide the pointwise description of the null-space of $H \mapsto H \bullet R$. That is, $H \in L(R)$ satisfies $H \bullet R \equiv 0$ if and only if $H \in \mathscr{N}$ $P \otimes A$-a.e. An investment with values in $\mathscr{N}$ has no effect on the wealth process.

3 The Bellman Equation

We have now introduced the necessary notation to formulate our first main result. Two special cases of our Bellman equation can be found in the pioneering work of Mania and Tevzadze [23] and Hu *et al.* [14]. These articles consider models with continuous asset prices and we shall indicate the connections as we specialize to that case in Section 3.3. A related equation also arises in the study of mean-variance hedging by Černý and Kallsen [5] in the context of locally square-integrable semimartingales, although they do not use dynamic programming explicitly. Due to the quadratic setting, that equation is more explicit than ours and the mathematical treatment is quite different. Czichowsky and Schweizer [7] study a cone-constrained version of the related Markowitz problem and there the equation is no longer explicit.

The Bellman equation highlights the local structure of our utility maximization problem. In addition, it has two main benefits. First, it can be used as an abstract tool to derive properties of the optimal strategies and the opportunity process (e.g., Nutz [27]). Second, one can try to solve the equation directly in a given model and to deduce the optimal strategies. This is the point of view taken in Section 5 and obviously requires the precise form of the equation.

The following assumptions are in force for the entire Section 3.

Assumptions 3.1. The value of the utility maximization problem is finite, there exists an optimal strategy $(\hat{\pi}, \hat{c}) \in \mathcal{A}$, and $\mathscr{C}$ satisfies (C1)-(C3).

3.1 Bellman Equation in Joint Characteristics

Our first main result is the Bellman equation stated as a description of the drift rate of the opportunity process. We recall the conjugate function $U_t^*(y) = -\frac{1}{q}y^q D_t^\beta$.

Theorem 3.2. *The drift rate a^L of the opportunity process satisfies*

$$-p^{-1}a^L = U^*(L_-)\,\tfrac{d\mu}{dA} + \max_{y\in\mathscr{C}\cap\mathscr{C}^0} g(y), \tag{3.1}$$

where g is the predictable random function

$$\begin{aligned} g(y) := L_- y^\top\Big(b^R + \tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2}c^R y\Big) + \int_{\mathbb{R}^d\times\mathbb{R}} x' y^\top h(x)\, F^{R,L}(d(x,x')) \\ + \int_{\mathbb{R}^d\times\mathbb{R}} (L_- + x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^{R,L}(d(x,x')). \end{aligned} \tag{3.2}$$

The unique ($P\otimes\mu^\circ$-a.e.) optimal propensity to consume is

$$\hat\kappa = \Big(\frac{D}{L}\Big)^{\frac{1}{1-p}}. \tag{3.3}$$

Any optimal trading strategy π^* satisfies

$$\pi^* \in \arg\max_{\mathscr{C}\cap\mathscr{C}^0} g \tag{3.4}$$

and the corresponding optimal wealth process and consumption are given by

$$X^* = x_0\mathcal{E}\big(\pi^*\bullet R - \hat\kappa\bullet\mu\big);\quad c^* = X^*\hat\kappa.$$

We shall see in the proof that the maximization in (3.1) can be understood as a local version of the optimization problem. Indeed, recalling (2.1), the right hand side of (3.1) is the maximum of a single function over certain points $(k,y)\in\mathbb{R}_+\times\mathbb{R}^d$ that correspond to the admissible controls (κ,π). Moreover, optimal controls are related to maximizers of this function, a characteristic feature of any dynamic programming equation. The maximum of g is not explicit due to the jumps of R; this simplifies in the continuous case considered in Section 3.3 below. Some mathematical comments are also in order.

Remark 3.3. (i) The random function g is well defined on $\mathscr{C}^0$ in the extended sense (see Lemma A.2) and it does not depend on the choice of the cut-off function h by [15, II.2.25].

(ii) For $p<0$ we have a more precise statement: Given $\pi^*\in L(R)$ and $\hat\kappa$ as in (3.3), $(\pi^*,\hat\kappa)$ is optimal *if and only if* π^* takes values in $\mathscr{C}\cap\mathscr{C}^0$ and maximizes g. This will follow from Corollary 5.4 applied to the triplet $(L,\pi^*,\hat\kappa)$.

(iii) For $p \in (0,1)$, partial results in this direction follow from Section 5. The question is trivial for convex $\mathscr{C}$ by the next item.

(iv) If $\mathscr{C}$ is convex, $\arg\max_{\mathscr{C}\cap\mathscr{C}^0} g$ is unique in the sense that the difference of any two elements lies in $\mathscr{N}$ (see Lemma A.3).

We split the proof of Theorem 3.2 into several steps; the plan is as follows. Let $(\pi,\kappa) \in \mathcal{A}^{fE}$ and denote $X = X(\pi,\kappa)$. We recall from Proposition 2.2 that

$$Z(\pi,\kappa) := L\tfrac{1}{p}X^p + \int U_s(\kappa_s X_s)\,\mu(ds)$$

is a supermartingale, and a martingale if and only if (π,κ) is optimal. Hence we shall calculate its drift rate and then maximize over (π,κ); the maximum will be attained at any optimal strategy. This is fairly straightforward and essentially the content of Lemma 3.7 below. In the Bellman equation, we maximize over a subset of $\mathbb{R}^d$ for each (ω,t) and not over a set of strategies. This final step is a measurable selection problem and its solution will be the second part of the proof.

Lemma 3.4. *Let $(\pi,\kappa) \in \mathcal{A}^f$. The drift rate of $Z(\pi,\kappa)$ is*

$$a^{Z(\pi,\kappa)} = X(\pi,\kappa)_-^p \big(p^{-1}a^L + f(\kappa)\tfrac{d\mu}{dA} + g(\pi)\big) \in [-\infty,\infty),$$

where $f_t(k) := U_t(k) - L_{t-}k$ and g is given by (3.2)*. Moreover, $a^{Z(\hat\pi,\hat\kappa)} = 0$, and $a^{Z(\pi,\kappa)} \in (-\infty,0]$ for $(\pi,\kappa) \in \mathcal{A}^{fE}$.*

Proof. We can assume that the initial capital is $x_0 = 1$. Let $(\pi,\kappa) \in \mathcal{A}^f$, then in particular $Z := Z(\pi,\kappa)$ is finite. We also set $X := X(\pi,\kappa)$. By Itô's formula, we have $X^p = \mathcal{E}(\pi \bullet R - \kappa \bullet \mu)^p = \mathcal{E}(Y)$ with

$$Y = p(\pi \bullet R - \kappa \bullet \mu) + \tfrac{p(p-1)}{2}\pi^\top c^R \pi \bullet A + \big\{(1+\pi^\top x)^p - 1 - p\pi^\top x\big\} * \mu^R.$$

Integrating by parts in the definition of Z and using $X_s = X_{s-}$ $\mu(ds)$-a.e. (path-by-path), we have $X_-^{-p} \bullet Z = p^{-1}(L - L_0 + L_- \bullet Y + [L,Y]) + U(\kappa) \bullet \mu$. Here

$$\begin{aligned}[L,Y] &= [L^c,Y^c] + \sum \Delta L \Delta Y\\ &= p\pi^\top c^{RL} \bullet A + px'\pi^\top x * \mu^{R,L} + x'\big\{(1+\pi^\top x)^p - 1 - p\pi^\top x\big\} * \mu^{R,L}.\end{aligned}$$

Thus $X_-^{-p} \bullet Z$ equals

$$\begin{aligned}&p^{-1}(L - L_0) + L_-\pi \bullet R + f(\kappa) \bullet \mu + L_-\tfrac{(p-1)}{2}\pi^\top c^R \pi \bullet A + \pi^\top c^{RL} \bullet A\\ &+ x'\pi^\top x * \mu^{R,L} + (L_- + x')\big\{p^{-1}(1+\pi^\top x)^p - p^{-1} - \pi^\top x\big\} * \mu^{R,L}.\end{aligned}$$

Writing $x = h(x) + x - h(x)$ and $\bar R = R - (x - h(x)) * \mu^R$ as in (2.4),

$$\begin{aligned}&X_-^{-p} \bullet Z = \qquad\qquad (3.5)\\ &p^{-1}(L - L_0) + L_-\pi \bullet \bar R + f(\kappa) \bullet \mu + L_-\pi^\top\big(\tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2}c^R\pi\big) \bullet A\\ &+ x'\pi^\top h(x) * \mu^{R,L} + (L_- + x')\big\{p^{-1}(1+\pi^\top x)^p - p^{-1} - \pi^\top h(x)\big\} * \mu^{R,L}.\end{aligned}$$

Since π need not be locally bounded, we use from now on a predictable cut-off function h such that $\pi^\top h(x)$ is bounded; e.g., $h(x) = x1_{\{|x|\leq 1\}\cap\{|\pi^\top x|\leq 1\}}$. Then the compensator of $x'\pi^\top h(x) * \mu^{R,L}$ exists, since L is special.

Let $(\pi,\kappa) \in \mathcal{A}^{fE}$. Then the compensator of the last integral in the right hand side of (3.5) also exists; indeed, all other terms in that equality are special, since Z is a supermartingale. The drift rate can now be read from (3.5) and (2.4), and it is nonpositive by the supermartingale property. The drift rate vanishes for the optimal $(\hat{\pi},\hat{\kappa})$ by the martingale condition from Proposition 2.2.

Now consider $(\pi,\kappa) \in \mathcal{A}^f \setminus \mathcal{A}^{fE}$. Note that necessarily $p<0$ (otherwise $\mathcal{A}^f = \mathcal{A}^{fE}$). Thus $Z \leq 0$, so by Remark 2.3 the drift rate a^Z is well defined with values in $[-\infty,\infty)$—alternatively, this can also be read from the integrals in (3.5) via (2.5). Using directly the definition of a^Z, we find the same formula for a^Z is as above. □

We do not have the supermartingale property for $(\pi,\kappa) \in \mathcal{A}^f \setminus \mathcal{A}^{fE}$, so it is not evident that $a^{Z(\pi,\kappa)} \leq 0$ in that case. However, we have the following

Lemma 3.5. *Let $(\pi,\kappa) \in \mathcal{A}^f$. Then $a^Z(\pi,\kappa) \in [0,\infty]$ implies $a^Z(\pi,\kappa) = 0$.*

Proof. Denote $Z = Z(\pi,\kappa)$. For $p>0$ we have $\mathcal{A}^f = \mathcal{A}^{fE}$ and the claim is immediate from Lemma 3.4. Let $p<0$. Then $Z \leq 0$ and by Lemma 2.4(iii), $a^Z \in [0,\infty]$ implies that Z is a submartingale . Hence $E[Z_T] = E\big[\int_0^T U_t(\kappa_t X_t(\pi,\kappa))\,\mu^\circ(dt)\big] > -\infty$, that is, $(\pi,\kappa) \in \mathcal{A}^{fE}$. Now Lemma 3.4 yields $a^Z(\pi,\kappa) \leq 0$. □

We observe in Lemma 3.4 that the drift rate splits into separate functions involving κ and π, respectively. For this reason, we can single out the

Proof of the consumption formula (3.3). Let $(\pi,\kappa) \in \mathcal{A}$. Note the following feature of our parametrization: we have $(\pi,\kappa^*) \in \mathcal{A}$ for *any* nonnegative optional process κ^* such that $\int_0^T \kappa^*_s\,\mu(ds) < \infty$ and $\kappa^*_T = 1$. Indeed, $X(\pi,\kappa) = x_0\mathcal{E}(\pi \bullet R - \kappa \bullet \mu)$ is positive by assumption. As μ is continuous, $X(\pi,\kappa^*) = x_0\mathcal{E}(\pi \bullet R - \kappa^* \bullet \mu)$ is also positive.

In particular, let $(\hat{\pi},\hat{\kappa})$ be optimal, $\beta = (1-p)^{-1}$ and $\kappa^* = (D/L)^\beta$; then $(\hat{\pi},\kappa^*) \in \mathcal{A}$. In fact the paths of $U(\kappa^* X(\hat{\pi},\kappa^*)) = p^{-1}D^{\beta p+1}X(\hat{\pi},\kappa^*)^p L^{-\beta p}$ are bounded P-a.s. (because the processes are càdlàg; $L, L_- > 0$ and $\beta p+1 = \beta > 0$) so that $(\hat{\pi},\kappa^*) \in \mathcal{A}^f$.

Note that $P \otimes \mu$-a.e., we have $\kappa^* = (D/L_-)^\beta = \arg\max_{k\geq 0} f(k)$, hence $f(\kappa^*) \geq f(\hat{\kappa})$. Suppose $(P \otimes \mu)\{f(\kappa^*) > f(\hat{\kappa})\} > 0$, then the formula from Lemma 3.4 and $a^{Z(\hat{\pi},\hat{\kappa})} = 0$ imply $a^{Z(\hat{\pi},\kappa^*)} \geq 0$ and $(P \otimes A)\{a^{Z(\hat{\pi},\kappa^*)} > 0\} > 0$, a contradiction to Lemma 3.5. It follows that $\hat{\kappa} = \kappa^*$ $P \otimes \mu$-a.e. since f has a unique maximum. □

Remark 3.6. The previous proof does not use the assumptions (C1)-(C3).

Lemma 3.7. *Let π be a predictable process with values in $\mathscr{C} \cap \mathscr{C}^{0,*}$. Then*

$$(P \otimes A)\big\{g(\hat{\pi}) < g(\pi)\big\} = 0.$$

Proof. We argue by contradiction and assume $(P \otimes A)\{g(\hat{\pi}) < g(\pi)\} > 0$. By redefining π, we may assume that $\pi = \hat{\pi}$ on the complement of this predictable set. Then

$$g(\hat{\pi}) \leq g(\pi) \quad \text{and} \quad (P \otimes A)\{g(\hat{\pi}) < g(\pi)\} > 0. \tag{3.6}$$

As π is σ-bounded, we can find a constant $C > 0$ such that the process $\tilde{\pi} := \pi 1_{|\pi| \leq C} + \hat{\pi} 1_{|\pi| > C}$ again satisfies (3.6); that is, we may assume that π is R-integrable. Since $\pi \in \mathscr{C} \cap \mathscr{C}^{0,*}$, this implies $(\pi, \hat{\kappa}) \in \mathcal{A}$ (as observed above, the consumption $\hat{\kappa}$ plays no role here). The contradiction follows as in the previous proof. $\square$

In view of Lemma 3.7, the main task will be to construct a *measurable* maximizing sequence for g.

Lemma 3.8. *Under Assumptions 3.1, there exists a sequence (π^n) of predictable $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued processes such that*

$$\limsup_n g(\pi^n) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g \quad P \otimes A\text{-a.e.}$$

We defer the proof of this lemma to Appendix A, together with the study of the properties of g. The theorem can then be proved as follows.

Proof of Theorem 3.2. Let π^n be as in Lemma 3.8. Then Lemma 3.7 with $\pi = \pi^n$ yields $g(\hat{\pi}) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g$, which is (3.4). By Lemma 3.4 we have $0 = a^{Z(\hat{\pi}, \hat{\kappa})} = p^{-1} a^L + f(\hat{\kappa}) \frac{d\mu}{dA} + g(\hat{\pi})$. This is (3.1) as $f(\hat{\kappa}) = U^*(L_-)$ $P \otimes \mu$-a.e. due to (3.3). $\square$

3.2 Bellman Equation as BSDE

In this section we express the Bellman equation as a BSDE. The unique orthogonal decomposition of the local martingale M^L with respect to R (cf. [15, III.4.24]) leads to the representation

$$L = L_0 + A^L + \varphi^L \bullet R^c + W^L * (\mu^R - \nu^R) + N^L, \tag{3.7}$$

where, using the notation of [15], $\varphi^L \in L^2_{loc}(R^c)$, $W^L \in G_{loc}(\mu^R)$, and N^L is a local martingale such that $\langle (N^L)^c, R^c \rangle = 0$ and $M^P_{\mu^R}(\Delta N^L | \widetilde{\mathcal{P}}) = 0$. The last statement means that $E[(V \Delta N^L) * \mu^R_T] = 0$ for any sufficiently integrable predictable function $V = V(\omega, t, x)$. We also introduce

$$\widehat{W}^L_t := \int_{\mathbb{R}^d} W^L(t, x)\, \nu^R(\{t\} \times dx),$$

then $\Delta\big(W^L * (\mu^R - \nu^R)\big) = W^L(\Delta R)1_{\{\Delta R \neq 0\}} - \widehat{W}^L$ by definition of the purely discontinuous local martingale $W^L * (\mu^R - \nu^R)$ and we can write

$$\Delta L = \Delta A^L + W^L(\Delta R)1_{\{\Delta R \neq 0\}} - \widehat{W}^L + \Delta N^L.$$

We recall that Assumptions 3.1 are in force. Now (3.1) can be restated as follows, the random function g being the same as before but in new notation.

Corollary 3.9. *The opportunity process L and the processes defined by* (3.7) *satisfy the BSDE*

$$L = L_0 - pU^*(L_-) \bullet \mu - p \max_{y \in \mathscr{C} \cap \mathscr{C}^0} g(y) \bullet A + \varphi^L \bullet R^c + W^L * (\mu^R - \nu^R) + N^L \tag{3.8}$$

with terminal condition $L_T = D_T$, where g is given by

$$\begin{aligned} &g(y) := \\ &L_- y^\top \Big(b^R + c^R\big(\tfrac{\varphi^L}{L_-} + \tfrac{(p-1)}{2} y\big)\Big) + \int_{\mathbb{R}^d} \big(\Delta A^L + W^L(x) - \widehat{W}^L\big) y^\top h(x)\, F^R(dx) \\ &+ \int_{\mathbb{R}^d} \big(L_- + \Delta A^L + W^L(x) - \widehat{W}^L\big)\big\{p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx). \end{aligned}$$

We observe that the orthogonal part N^L does not appear in the definition of g. In a suitable setting, it is linked to the "dual problem"; see Remark 5.18.

It is possible (but notationally more cumbersome) to prove a version of Lemma 3.4 using g as in Corollary 3.9 and the decomposition (3.7), thus involving only the characteristics of R instead of the joint characteristics of (R, L). Using this approach, we see that the increasing process A in the BSDE can be chosen based on R and without reference to L. This is desirable if we want to consider other solutions of the equation, as in Section 4. One consequence is that A can be chosen to be continuous if and only if R is quasi left continuous (cf. [15, II.2.9]). Since $p^{-1}A^L = -f(\hat{\kappa}) \bullet \mu - g(\hat{\pi}) \bullet A$, $\mathrm{Var}(A^L)$ is absolutely continuous with respect to $A + \mu$, and we conclude:

Remark 3.10. If R is quasi left continuous, A^L is continuous.

If R is quasi left continuous, $\nu^R(\{t\} \times \mathbb{R}^d) = 0$ for all t by [15, II.1.19], hence $\widehat{W}^L = 0$ and we have the simpler formula

$$\begin{aligned} g(y) = L_- y^\top \Big(b^R + c^R\big(\tfrac{\varphi^L}{L_-} + \tfrac{(p-1)}{2} y\big)\Big) + \int_{\mathbb{R}^d} W^L(x) y^\top h(x)\, F^R(dx) \\ + \int_{\mathbb{R}^d} \big(L_- + W^L(x)\big)\big\{p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx). \end{aligned}$$

3.3 The Case of Continuous Prices

In this section we specialize the previous results to the case where R is a continuous semimartingale and mild additional conditions are satisfied. As usual in this setting, the martingale part of R will be denoted by M rather than R^c. In addition to Assumptions 3.1, the following conditions are in force for the present Section 3.3.

Assumptions 3.11.

(i) R is continuous,

(ii) $R = M + \int d\langle M\rangle\lambda$ for some $\lambda \in L^2_{loc}(M)$ (*structure condition*),

(iii) the orthogonal projection of $\mathscr{C}$ onto $\mathscr{N}^\perp$ is closed.

Note that $\mathscr{C}^{0,*} = \mathbb{R}^d$ due to (i), in particular (C3) is void. When R is continuous, it necessarily satisfies (ii) when a no-arbitrage property holds; see Schweizer [29]. By (i) and (ii) we can write the differential characteristics of R with respect to, e.g., $A_t := t + \sum_{i=1}^d \langle M^i\rangle_t$. It will be convenient to factorize $c^R = \sigma\sigma^\top$, where σ is a predictable matrix-valued process; hence $\sigma\sigma^\top\, dA = d\langle M\rangle$. Then (ii) implies $\mathscr{N} = \ker \sigma^\top$ because $\sigma\sigma^\top y = 0$ implies $(\sigma^\top y)^\top(\sigma^\top y) = 0$. Since $\sigma^\top : \ker(\sigma^\top)^\perp \to \sigma^\top\mathbb{R}^d$ is a homeomorphism, we see that (iii) is equivalent to

$$\sigma^\top\mathscr{C} \quad \text{is closed.}$$

This condition depends on the semimartingale R. It is equivalent to the closedness of $\mathscr{C}$ itself if σ has full rank. For certain constraint sets (e.g., closed polyhedral or compact) the condition is satisfied for *all* matrices σ, but not so, e.g., for non-polyhedral cone constraints. We mention that violation of (iii) leads to nonexistence of optimal strategies in simple examples (cf. [25, Example 3.5]) and we refer to Czichowsky and Schweizer [8] for background.

Under (i), (3.7) is the more usual Kunita-Watanabe decomposition

$$L = L_0 + A^L + \varphi^L \bullet M + N^L,$$

where $\varphi^L \in L^2_{loc}(M)$ and N^L is a local martingale such that $[M, N^L] = 0$; see Ansel and Stricker [2, *cas* 3]. If $\emptyset \neq K \subseteq \mathbb{R}^d$ is a closed set, we denote the Euclidean distance to K by $d_K(x) = \min\{|x - y| : y \in K\}$, and d_K^2 is the squared distance. We also define the (set-valued) projection Π^K which maps $x \in \mathbb{R}^d$ to the points in K with minimal distance to x,

$$\Pi^K(x) = \big\{y \in K : |x - y| = d_K(x)\big\} \neq \emptyset.$$

If K is convex, Π^K is the usual (single-valued) Euclidean projection. In the present continuous setting, the random function g simplifies considerably:

$$g(y) = L_- y^\top \sigma\sigma^\top \Big(\lambda + \frac{\varphi^L}{L_-} + \tfrac{p-1}{2} y\Big) \tag{3.9}$$

and so the Bellman BSDE becomes more explicit.

Corollary 3.12. *Any optimal trading strategy* π^* *satisfies*

$$\sigma^\top \pi^* \in \Pi^{\sigma^\top \mathscr{C}} \Big\{ \sigma^\top (1-p)^{-1} \Big(\lambda + \frac{\varphi^L}{L_-} \Big) \Big\}.$$

The opportunity process satisfies the BSDE

$$L = L_0 - pU^*(L_-) \bullet \mu + F(L_-, \varphi^L) \bullet A + \varphi^L \bullet M + N^L; \quad L_T = D_T,$$

where

$$F(L_-, \varphi^L) = \\ \frac{1}{2} L_- \Big\{ p(1-p) d^2_{\sigma^\top \mathscr{C}} \Big(\sigma^\top (1-p)^{-1} \Big(\lambda + \frac{\varphi^L}{L_-} \Big) \Big) + \tfrac{p}{p-1} \Big| \sigma^\top \Big(\lambda + \frac{\varphi^L}{L_-} \Big) \Big|^2 \Big\}.$$

If $\mathscr{C}$ *is a convex cone,* $F(L_-, \varphi^L) = \frac{p}{2(p-1)} L_- \big| \Pi^{\sigma^\top \mathscr{C}} \big\{ \sigma^\top \big(\lambda + \frac{\varphi^L}{L_-} \big) \big\} \big|^2$. *If* $\mathscr{C} = \mathbb{R}^d$*, then* $F(L_-, \varphi^L) \bullet A = \frac{p}{2(p-1)} \int L_- \big(\lambda + \frac{\varphi^L}{L_-} \big)^\top d\langle M \rangle \, \big(\lambda + \frac{\varphi^L}{L_-} \big)$ *and the unique (mod.* $\mathscr{N}$*) optimal trading strategy is* $\pi^* = (1-p)^{-1} \big(\lambda + \frac{\varphi^L}{L_-} \big)$.

Proof. Let $\beta = (1-p)^{-1}$. We obtain $\sigma^\top (\arg\max_{\mathscr{C}} g) = \Pi^{\sigma^\top \mathscr{C}} \big\{ \sigma^\top \beta \big(\lambda + \frac{\varphi^L}{L_-} \big) \big\}$ by completing the square in (3.9), moreover, for any $\pi^* \in \arg\max_{\mathscr{C}} g$,

$$g(\pi^*) = \tfrac{1}{2} L_- \Big\{ \beta \Big(\lambda + \frac{\varphi^L}{L_-} \Big)^\top \sigma \sigma^\top \Big(\lambda + \frac{\varphi^L}{L_-} \Big) - \beta^{-1} d^2_{\sigma^\top \mathscr{C}} \Big(\sigma^\top \beta \Big(\lambda + \frac{\varphi^L}{L_-} \Big) \Big) \Big\}.$$

In the case where $\mathscr{C}$, and hence $\sigma^\top \mathscr{C}$, is a convex cone, $\Pi := \Pi^{\sigma^\top \mathscr{C}}$ is single-valued, positively homogeneous, and Πx is orthogonal to $x - \Pi x$ for any $x \in \mathbb{R}^d$. Writing $\Psi := \sigma^\top \big(\lambda + \frac{\varphi^L}{L_-} \big)$ we get $g(\pi^*) = L_- \beta (\Pi \Psi)^\top (\Psi - \frac{1}{2} \Pi \Psi) = L_- \frac{1}{2} \beta \big(\Pi \Psi \big)^\top \big(\Pi \Psi \big)$. Finally, $\Pi \Psi = \Psi$ if $\mathscr{C} = \mathbb{R}^d$. The result follows from Corollary 3.9. □

Of course the consumption formula (3.3) and Remark 3.3 still apply. We remark that the BSDE for the unconstrained case $\mathscr{C} = \mathbb{R}^d$ (and $\mu = 0$, $D = 1$) was previously obtained in [23] in a similar spirit. A variant of the constrained BSDE for an Itô process model (and $\mu = 0$, $D = 1$) appears in [14], where a converse approach is taken: the equation is derived only formally and then existence results for BSDEs are employed together with a verification argument. We shall extend that result in Section 5 (Example 5.8) when we study verification.

If L is continuous, the BSDE of Corollary 3.12 simplifies if it is stated for $\log(L)$ rather than L, but in general the given form is more convenient as the jumps are "hidden" in N^L.

Remark 3.13. (i) Continuity of R does not imply that L is continuous. For instance, in the Itô process model of Barndorff-Nielsen and Shephard [3] with Lévy driven coefficients, the opportunity process is not continuous. See, e.g., Theorem 3.3 and the subsequent remark in Kallsen and Muhle-Karbe [18]. If R satisfies the structure condition and the filtration $\mathbb{F}$ is continuous, it clearly follows that L is continuous. Here $\mathbb{F}$ is called continuous if all $\mathbb{F}$-martingales are continuous, as, e.g., for the Brownian filtration. In general, L is related to the predictable characteristics of the asset returns rather than their levels. As an example, Lévy models have jumps but constant characteristics; here L turns out to be a smooth function (see [25]).

(ii) In the present setting we see that F has quadratic growth in φ^L, so that the Bellman equation is a "quadratic BSDE" (see also Example 5.8). In general, F does not satisfy the bounds which are usually assumed in the theory of such BSDEs. Together with existence results for the utility maximization problem (see the citations from the introduction), the Bellman equation yields various examples of BSDEs with the opportunity process as a solution. This includes terminal conditions D_T which are integrable and unbounded (see also [26, Remark 2.4]).

4 Minimality of the Opportunity Process

This section considers the Bellman equation as such, having possibly many solutions, and we characterize the opportunity process as the minimal solution. As mentioned above, it seems more natural to use the BSDE formulation for this purpose (but see Remark 4.4). We first have to clarify what we mean by a solution of the BSDE. We consider R and A as given. Since the finite variation part in the BSDE is predictable, a solution will certainly be a special semimartingale. If ℓ is any special semimartingale, there exists a unique orthogonal decomposition [15, III.4.24]

$$\ell = \ell_0 + A^\ell + \varphi^\ell \bullet R^c + W^\ell * (\mu^R - \nu^R) + N^\ell, \tag{4.1}$$

using the same notation as in (3.7). These processes are essentially unique, and so it suffices to consider the left hand side of the BSDE for the notion of a solution. (In BSDE theory, a solution would be, at least, a quadruple.) We define the random function g^ℓ as in Corollary 3.9, with L replaced by ℓ. Since ℓ is special, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}} (|x|^2 + |x'|^2) \wedge (1 + |x'|)\ F^{R,\ell}(d(x,x')) < \infty \tag{4.2}$$

and the arguments from Lemma A.2 show that g^ℓ is well defined on $\mathscr{C}^0$ with values in $\mathbb{R} \cup \{\operatorname{sign}(p)\infty\}$. Hence we can consider (formally at first) the BSDE (3.8) with L replaced by ℓ; i.e.,

$$\ell = \ell_0 - pU^*(\ell_-) \bullet \mu - p \max_{y \in \mathscr{C} \cap \mathscr{C}^0} g^\ell(y) \bullet A + \varphi^\ell \bullet R^c + W^\ell * (\mu^R - \nu^R) + N^\ell \tag{4.3}$$

with terminal condition $\ell_T = D_T$.

Definition 4.1. A càdlàg special semimartingale ℓ is called a *solution of the Bellman equation* (4.3) if

- $\ell, \ell_- > 0$,
- there exists a $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process $\check{\pi} \in L(R)$ such that $g^\ell(\check{\pi}) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g^\ell < \infty$,
- ℓ and the processes from (4.1) satisfy (4.3) with $\ell_T = D_T$.

Moreover, we define $\check{\kappa} := (D/\ell)^\beta$, where $\beta = (1-p)^{-1}$. We call $(\check{\pi}, \check{\kappa})$ the strategy associated with ℓ, and for brevity, we also call $(\ell, \check{\pi}, \check{\kappa})$ a solution.

If the process $\check{\pi}$ is not unique, we choose and fix one. The assumption $\ell > 0$ excludes pathological cases where ℓ jumps to zero and becomes positive immediately afterwards and thereby ensures that $\check{\kappa}$ is admissible. More precisely, the following holds.

Remark 4.2. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation.

(i) $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^{fE}$.

(ii) $\sup_{\mathscr{C} \cap \mathscr{C}^0} g^\ell$ is a predictable, A-integrable process.

(iii) If $p \in (0,1)$, g^ℓ is finite on $\mathscr{C} \cap \mathscr{C}^0$.

(iv) The condition $\ell > 0$ is automatically satisfied if (a) $p \in (0,1)$ or if (b) $p < 0$ and there is no intermediate consumption and Assumptions 3.1 are satisfied.

Proof. (i) We have $\int_0^T \check{\kappa}_s \, \mu(ds) < \infty$ P-a.s. since the paths of ℓ are bounded away from zero. Moreover, $\int_0^T U_t(\check{\kappa}_t X_t(\check{\pi}, \check{\kappa})) \, \mu(dt) < \infty$ as in the proof of (3.3) (stated after Lemma 3.5). This shows $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^f$. The fact that $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^{fE}$ is contained in the proof of Lemma 4.9 below.

(ii) We have $0 = g^\ell(0) \leq \sup_{\mathscr{C} \cap \mathscr{C}^0} g^\ell = g^\ell(\check{\pi})$. Hence $\sup_{\mathscr{C} \cap \mathscr{C}^0} g^\ell \bullet A$ is well defined, and it is finite because otherwise (4.3) could not hold.

(iii) Note that $p > 0$ implies $g^\ell > -\infty$ by its definition and (4.2), while $g^\ell < \infty$ by assumption.

(iv) If $p > 0$, (4.3) states that A^ℓ is decreasing. As $\ell_- > 0$ implies $\ell \geq 0$, ℓ is a supermartingale by Lemma 2.4. Since $\ell_T = D_T > 0$, the minimum principle for nonnegative supermartingales shows $\ell > 0$. Under (b) the assertion is a consequence of Theorem 4.5 below (which shows $\ell \geq L > 0$) upon noting that the condition $\ell > 0$ is not used in its proof when there is no intermediate consumption. $\square$

It may seem debatable to make existence of the maximizer $\check{\pi}$ part of the definition of a solution. However, associating a control with the solution is crucial for the following theory. Some justification is given by the following result for the continuous case (where $\mathscr{C}^{0,*} = \mathbb{R}^d$).

Proposition 4.3. *Let ℓ be any càdlàg special semimartingale such that $\ell, \ell_- > 0$. Under Assumptions 3.11, (C1) and (C2), there exists a $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued predictable process $\check{\pi}$ such that $g^\ell(\check{\pi}) = \sup_{\mathscr{C}\cap\mathscr{C}^0} g^\ell < \infty$, and any such process is R-integrable.*

Proof. As g^ℓ is analogous to (3.9), it is continuous and its supremum over $\mathbb{R}^d$ is finite. By continuity of R and the structure condition, $\pi \in L(R)$ if and only if $\int_0^T \pi^\top d\langle M\rangle \pi = \int_0^T |\sigma^\top \pi|^2 \, dA < \infty$ P-a.s.

Assume first that $\mathscr{C}$ is compact, then Lemma A.4 yields a measurable selector π for $\arg\max_{\mathscr{C}} g$. As in the proof of Corollary 3.12, $\sigma^\top \pi \in \Pi^{\sigma^\top \mathscr{C}} \sigma^\top \psi$ for $\psi := \beta\big(\lambda + \frac{\varphi^\ell}{\ell_-}\big)$, which satisfies $\int_0^T |\sigma^\top \psi|^2 \, dA < \infty$ by definition of λ and φ^ℓ. We note that $|\sigma^\top \pi| \leq |\sigma^\top \psi| + |\sigma^\top \pi - \sigma^\top \psi| \leq 2|\sigma^\top \psi|$ due to the definition of the projection and $0 \in \mathscr{C}$.

In the general case we approximate $\mathscr{C}$ by a sequence of compact constraints $\mathscr{C}^n := \mathscr{C} \cap \{x \in \mathbb{R}^d : |x| \leq n\}$, each of which yields a selector π^n for $\arg\max_{\mathscr{C}^n} g$. By the above, $|\sigma^\top \pi^n| \leq 2|\sigma^\top \psi|$, so the sequence $(\sigma^\top \pi^n)_n$ is bounded for fixed (ω, t). A random index argument as in the proof of Lemma A.4 yields a selector ϑ for a cluster point of this sequence. We have $\vartheta \in \sigma^\top \mathscr{C}$ by closedness of this set and we find a selector $\check{\pi}$ for the preimage $((\sigma^\top)^{-1}\vartheta) \cap \mathscr{C}$ using [28, 1Q]. We have $\check{\pi} \in \arg\max_{\mathscr{C}} g$ as the sets $\mathscr{C}^n$ increase to $\mathscr{C}$, and $\int_0^T |\sigma^\top \check{\pi}|^2 \, dA \leq 2 \int_0^T |\sigma^\top \psi|^2 \, dA < \infty$ shows $\check{\pi} \in L(R)$. $\square$

Another example for the construction of $\check{\pi}$ is given in [25, §5]. In general, two ingredients are needed: Existence of a maximizer for fixed (ω, t) will typically require a compactness condition in the form of a no-arbitrage assumption (in the previous proof, this is the structure condition). Moreover, a measurable selection is required; here the techniques from the appendices may be useful.

Remark 4.4. The BSDE formulation of the Bellman equation has the advantage that we can choose A based on R and speak about the class of all solutions. However, we do not want to write proofs in this cumbersome notation. Once we fix a solution ℓ (and maybe L, and finitely many other semimartingales), we can choose a new reference process $\tilde{A} = A + A'$ (where A' is increasing), with respect to which our semimartingales admit differential characteristics; in particular we can use the joint characteristics $(b^{R,\ell}, c^{R,\ell}, F^{R,\ell}; \tilde{A})$. As we change A, all drift rates change in that they are multiplied by $d\tilde{A}/dA$, so any (in)equalities between them are preserved. With this in mind, we shall use the joint characteristics of (R, ℓ) in the sequel without further comment and treat the two formulations of the Bellman equation as equivalent.

Our definition of a solution of the Bellman equation is loose in terms of integrability assumptions. Even in the continuous case, it is unclear "how many" solutions exist. The next result shows that we can always identify L by taking the smallest one; i.e., $L \leq \ell$ for any solution ℓ.

Theorem 4.5. *Under Assumptions 3.1, the opportunity process L is characterized as the minimal solution of the Bellman equation.*

Remark 4.6. As a consequence, the Bellman equation has a bounded solution if and only if the opportunity process is bounded (and similarly for other integrability properties). In conjunction with [26, §4.2] this yields examples of quadratic BSDEs which have bounded terminal value (for D_T bounded), but no bounded solution.

The proof of Theorem 4.5 is based on the following result; it is the fundamental property of any Bellman equation.

Proposition 4.7. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. For any $(\pi, \kappa) \in \mathcal{A}^f$,*

$$Z(\pi,\kappa) := \ell \tfrac{1}{p}\big(X(\pi,\kappa)\big)^p + \int U_s\big(\kappa_s X_s(\pi,\kappa)\big)\,\mu(ds) \tag{4.4}$$

is a semimartingale with nonpositive drift rate. Moreover, $Z(\check{\pi}, \check{\kappa})$ is a local martingale.

Proof. Let $(\pi, \kappa) \in \mathcal{A}^f$. Note that $Z := Z(\pi, \kappa)$ satisfies $\operatorname{sign}(p)Z \geq 0$, hence has a well defined drift rate a^Z by Remark 2.3. The drift rate can be calculated as in Lemma 3.4: If f^ℓ is defined similarly to the function f in that lemma but with L replaced by ℓ, then

$$\begin{aligned} a^Z &= X(\pi,\kappa)^p_- \big\{p^{-1}a^\ell + f^\ell(\kappa)\,\tfrac{d\mu}{dA} + g^\ell(\pi)\big\} \\ &= X(\pi,\kappa)^p_- \big\{\big(f^\ell(\kappa) - f^\ell(\check{\kappa})\big)\,\tfrac{d\mu}{dA} + g^\ell(\pi) - g^\ell(\check{\pi})\big\}. \end{aligned}$$

This is nonpositive because $\check{\kappa}$ and $\check{\pi}$ maximize f^ℓ and g^ℓ. For the special case $(\pi, \kappa) := (\check{\pi}, \check{\kappa})$ we have $a^Z = 0$ and so Z is a σ-martingale, thus a local martingale as $\operatorname{sign}(p)Z \geq 0$. $\square$

Remark 4.8. In Proposition 4.7, "semimartingale with nonpositive drift rate" can be replaced by "σ-supermartingale" if g^ℓ is finite on $\mathscr{C} \cap \mathscr{C}^0$.

Theorem 4.5 follows from the next lemma (which is actually stronger). We recall that for $p < 0$ the opportunity process L can be defined without further assumptions.

Lemma 4.9. *Let ℓ be a solution of the Bellman equation. If $p < 0$, then $L \leq \ell$. For $p \in (0, 1)$, the same holds if (2.2) is satisfied and there exists an optimal strategy.*

Proof. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution and define $Z(\pi, \kappa)$ as in (4.4).

Case $p < 0$: We choose $(\pi, \kappa) := (\check{\pi}, \check{\kappa})$. As $Z(\check{\pi}, \check{\kappa})$ is a negative local martingale by Proposition 4.7, it is a submartingale. In particular, $E[Z_T(\check{\pi}, \check{\kappa})] > -\infty$, and using $L_T = D_T$, this is the statement that the expected utility is finite, i.e., $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^{fE}$—this completes the proof of Remark 4.2(i). Recall that $\mu^\circ = \mu + \delta_{\{T\}}$. With $\check{X} := X(\check{\pi}, \check{\kappa})$ and $\check{c} := \check{\kappa}\check{X}$, and using $\ell_T = D_T = L_T$, we deduce

$$\begin{aligned}\ell_t \tfrac{1}{p} \check{X}_t^p + \int_0^t U_s(\check{c}_s)\, \mu(ds) &= Z_t(\check{\pi}, \check{\kappa}) \leq E\big[Z_T(\check{\pi}, \check{\kappa})\big|\mathcal{F}_t\big] \\ &\leq \operatorname{ess\,sup}_{\tilde{c} \in \mathcal{A}(\check{\pi}, \check{c}, t)} E\Big[\int_t^T U_s(\tilde{c}_s)\, \mu^\circ(ds)\Big|\mathcal{F}_t\Big] + \int_0^t U_s(\check{c}_s)\, \mu(ds) \\ &= L_t \tfrac{1}{p} \check{X}_t^p + \int_0^t U_s(\check{c}_s)\, \mu(ds),\end{aligned}$$

where the last equality holds by (2.3). As $\frac{1}{p}\check{X}_t^p < 0$, we have $\ell_t \geq L_t$.

Case $p \in (0, 1)$: We choose $(\pi, \kappa) := (\hat{\pi}, \hat{\kappa})$ to be an optimal strategy. Then $Z(\hat{\pi}, \hat{\kappa}) \geq 0$ is a supermartingale by Proposition 4.7 and Lemma 2.4(iii), and we obtain

$$\begin{aligned}\ell_t \tfrac{1}{p} \widehat{X}_t^p + \int_0^t U_s(\hat{c}_s)\, \mu(ds) &= Z_t(\hat{\pi}, \hat{\kappa}) \geq E\big[Z_T(\hat{\pi}, \hat{\kappa})\big|\mathcal{F}_t\big] \\ &= E\Big[\int_0^T U_s(\hat{c}_s)\, \mu^\circ(ds)\Big|\mathcal{F}_t\Big] = L_t \tfrac{1}{p} \widehat{X}_t^p + \int_0^t U_s(\hat{c}_s)\, \mu(ds)\end{aligned}$$

by the optimality of $(\hat{\pi}, \hat{\kappa})$ and (2.3). More precisely, we have used the fact that $(\hat{\pi}, \hat{\kappa})$ is also conditionally optimal (see [26, Remark 3.3]). As $\frac{1}{p}\widehat{X}_t^p > 0$, we conclude $\ell_t \geq L_t$. □

5 Verification

Suppose that we have found a solution of the Bellman equation; then we want to know whether it is the opportunity process and whether the associated strategy is optimal. In applications, it might not be clear *a priori* that an optimal strategy exists or even that the utility maximization problem is finite. Therefore, we stress that in this section these properties are not assumed. Also, we do not need the assumptions on $\mathscr{C}$ made in Section 2.4—they are not necessary because we start with a given solution.

Generally speaking, verification involves the candidate for an optimal control, $(\check{\pi}, \check{\kappa})$ in our case, and all the competing ones. It is often very difficult to check a condition involving all these controls, so it is desirable to have a verification theorem whose assumptions involve only $(\check{\pi}, \check{\kappa})$.

We present two verification approaches. The first one is via the value process and is classical for general dynamic programming: it uses little structure of the given problem. For $p \in (0,1)$, it yields the desired result. However, in a general setting, this is not the case for $p < 0$. The second approach uses the concavity of the utility function. To fully exploit this and make the verification conditions necessary, we will assume that $\mathscr{C}$ is convex. In this case, we shall obtain the desired verification theorem for all values of p.

5.1 Verification via the Value Process

The basis of this approach is the following simple result; we state it separately for better comparison with Lemma 5.10 below. In the entire section, $Z(\pi,\kappa)$ is defined by (4.4) whenever ℓ is given.

Lemma 5.1. *Let ℓ be any positive càdlàg semimartingale with $\ell_T = D_T$ and let $(\check{\pi},\check{\kappa}) \in \mathcal{A}$. Assume that for all $(\pi,\kappa) \in \mathcal{A}^{fE}$, the process $Z(\pi,\kappa)$ is a supermartingale. Then $Z(\check{\pi},\check{\kappa})$ is a martingale if and only if* (2.2) *holds and $(\check{\pi},\check{\kappa})$ is optimal and $\ell = L$.*

Proof. "$\Rightarrow$": Recall that $Z_0(\pi,\kappa) = \ell_0 \frac{1}{p} x_0^p$ does not depend on (π,κ) and that $E[Z_T(\pi,\kappa)] = E[\int_0^T U_t(\kappa_t(X_t(\pi,\kappa)))\,\mu^\circ(dt)]$ is the expected utility corresponding to (π,κ). With $\check{X} := X(\check{\pi},\check{\kappa})$, the (super)martingale condition implies that $E[\int_0^T U_t(\check{\kappa}_t\check{X}_t)\,\mu^\circ(dt)] \geq E[\int_0^T U_t(\kappa_t X_t(\pi,\kappa))\,\mu^\circ(dt)]$ for all $(\pi,\kappa) \in \mathcal{A}^{fE}$. Since for $(\pi,\kappa) \in \mathcal{A} \setminus \mathcal{A}^{fE}$ the expected utility is $-\infty$, this shows that $(\check{\pi},\check{\kappa})$ is optimal with $E[Z_T(\check{\pi},\check{\kappa})] = Z_0(\check{\pi},\check{\kappa}) = \ell_0 \frac{1}{p} x_0^p < \infty$. In particular, the opportunity process L is well defined. By Proposition 2.2, $L\frac{1}{p}\check{X}^p + \int U_s(\check{c}_s)\,\mu(ds)$ is a martingale, and as its terminal value equals $Z_T(\check{\pi},\check{\kappa})$, we deduce $\ell = L$ by comparison with (4.4), using $\check{X} > 0$.

The converse is contained in Proposition 2.2. $\square$

We can now state our first verification theorem.

Theorem 5.2. *Let $(\ell,\check{\pi},\check{\kappa})$ be a solution of the Bellman equation.*

(i) If $p \in (0,1)$, the following are equivalent:

- *(a) $Z(\check{\pi},\check{\kappa})$ is of class (D),*
- *(b) $Z(\check{\pi},\check{\kappa})$ is a martingale,*
- *(c)* (2.2) *holds and $(\check{\pi},\check{\kappa})$ is optimal and $\ell = L$.*

(ii) If $p < 0$, the following are equivalent:

- *(a) $Z(\pi,\kappa)$ is of class (D) for all $(\pi,\kappa) \in \mathcal{A}^{fE}$,*
- *(b) $Z(\pi,\kappa)$ is a supermartingale for all $(\pi,\kappa) \in \mathcal{A}^{fE}$,*
- *(c) $(\check{\pi},\check{\kappa})$ is optimal and $\ell = L$.*

Proof. When $p > 0$ and $(\pi,\kappa) \in \mathcal{A}^f$, $Z(\pi,\kappa)$ is positive and $a^{Z(\pi,\kappa)} \leq 0$ by Proposition 4.7, hence $Z(\pi,\kappa)$ is a supermartingale according to Lemma 2.4. By Proposition 4.7, $Z(\check{\pi},\check{\kappa})$ is a local martingale, so it is a martingale if and only if it is of class (D). Lemma 5.1 implies the result.

If $p < 0$, $Z(\pi, \kappa)$ is negative. Thus the local martingale $Z(\check{\pi}, \check{\kappa})$ is a submartingale, and a martingale if and only if it is also a supermartingale. Note that a class (D) semimartingale with nonpositive drift rate is a supermartingale. Conversely, any negative supermartingale Z is of class (D) due to the bounds $0 \geq Z \geq E[Z_T|\mathbb{F}]$. Lemma 5.1 implies the result after noting that if $\ell = L$, then Proposition 2.2 yields (b). $\square$

Theorem 5.2 is "as good as it gets" for $p > 0$, but as announced, the result for $p < 0$ is not satisfactory. In particular settings, this can be improved.

Remark 5.3 ($p < 0$)**.** (i) Assume we know *a priori* that *if* there is an optimal strategy $(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}$, then

$$(\hat{\pi}, \hat{\kappa}) \in \mathcal{A}^{(D)} := \big\{(\pi, \kappa) \in \mathcal{A} : \, X(\pi, \kappa)^p \text{ is of class (D)}\big\}.$$

In this case we can reduce our optimization problem to the class $\mathcal{A}^{(D)}$. If furthermore ℓ is bounded (which is not a strong assumption when $p < 0$), the class (D) condition in Theorem 5.2(ii) is automatically satisfied for $(\pi, \kappa) \in \mathcal{A}^{(D)}$. The verification then reduces to checking that $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^{(D)}$.

(ii) How can we establish the condition needed for (i)? One possibility is to show that L is uniformly bounded away from zero; then the condition follows (see the argument in the next proof). Of course, L is not known when we try to apply this. However, [26, §4.2] gives verifiable conditions for L to be (bounded and) bounded away from zero. They are stated for the unconstrained case $\mathscr{C} = \mathbb{R}^d$, but can be used nevertheless: if $L^{\mathbb{R}^d}$ is the opportunity process corresponding to $\mathscr{C} = \mathbb{R}^d$, the actual L satisfies $L \geq L^{\mathbb{R}^d}$ because the supremum in (2.3) is taken over a smaller set in the constrained case.

In the situation where ℓ and L^{-1} are bounded, we can also use the following result. Note also its use in Remark 3.3(ii) and recall that $1/0 := \infty$.

Corollary 5.4. *Let $p < 0$ and let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. Let L be the opportunity process and assume that ℓ/L is uniformly bounded. Then $(\check{\pi}, \check{\kappa})$ is optimal and $\ell = L$.*

Proof. Fix arbitrary $(\pi, \kappa) \in \mathcal{A}^{fE}$ and let $X = X(\pi, \kappa)$. The process $L\frac{1}{p}\big(X(\pi, \kappa)\big)^p + \int U_s(\kappa_s X_s)\,\mu(ds)$ is a negative supermartingale by Proposition 2.2, hence of class (D). Since $\int U_s(\kappa_s X_s)\,\mu(ds)$ is decreasing and its terminal value is integrable (definition of $\mathcal{A}^{fE}$), $L\frac{1}{p}X^p$ is also of class (D). The assumption yields that $\ell\frac{1}{p}X^p$ is of class (D), and then so is $Z(\pi, \kappa)$. $\square$

As bounded solutions are of special interest in BSDE theory, let us note the following consequence.

Corollary 5.5. *Let $p < 0$. Under Assumptions 3.1 the following are equivalent:*

(i) L is bounded and bounded away from zero,

(ii) there exists a unique bounded solution of the Bellman equation, and this solution is bounded away from zero.

One can note that in the setting of [26, §4.2], these conditions are further equivalent to a reverse Hölder inequality for the market model.

We give an illustration of Theorem 5.2 also for the case $p \in (0,1)$. Thus far, we have considered only the given exponent p and assumed (2.2). In many situations, there will exist some $p_0 \in (p,1)$ such that, if we consider the exponent p_0 instead of p, the utility maximization problem is still finite. Note that by Jensen's inequality this is a stronger assumption. We define for $q_0 \geq 1$ the class of semimartingales ℓ bounded in $L^{q_0}(P)$,

$$\mathbf{B}(q_0) := \{\ell : \sup_\tau \|\ell_\tau\|_{L^{q_0}(P)} < \infty\},$$

where the supremum ranges over all stopping times τ.

Corollary 5.6. *Let $p \in (0,1)$ and let there be a constant $k_1 > 0$ such that $D \geq k_1$. Assume that the utility maximization problem is finite for some $p_0 \in (p,1)$ and let $q_0 \geq 1$ be such that $q_0 > p_0/(p_0 - p)$. If $(\ell, \check{\pi}, \check{\kappa})$ is a solution of the Bellman equation (for p) with $\ell \in \mathbf{B}(q_0)$, then $\ell = L$ and $(\check{\pi}, \check{\kappa})$ is optimal.*

Proof. Let $\ell \in \mathbf{B}(q_0)$ be a solution, $(\check{\pi}, \check{\kappa})$ the associated strategy, and $\check{X} = X(\check{\pi}, \check{\kappa})$. By Theorem 5.2 and an argument as in the previous proof, it suffices to show that $\ell \check{X}^p$ is of class (D). Let $\delta > 1$ be such that $\delta/q_0 + \delta p/p_0 = 1$. For every stopping time τ, Hölder's inequality yields

$$E[(\ell_\tau \check{X}_\tau^p)^\delta] = E[(\ell_\tau^{q_0})^{\delta/q_0} (\check{X}_\tau^{p_0})^{\delta p/p_0}] \leq E[\ell_\tau^{q_0}]^{\delta/q_0} E[\check{X}_\tau^{p_0}]^{\delta p/p_0}.$$

We show that this is bounded uniformly in τ; then $\{\ell_\tau \check{X}_\tau^p : \tau \text{ stopping time}\}$ is bounded in $L^\delta(P)$ and hence uniformly integrable. Indeed, $E[\ell_\tau^{q_0}]$ is bounded by assumption. The set of wealth processes corresponding to admissible strategies is stable under stopping. Therefore $E[D_T \frac{1}{p_0} \check{X}_\tau^{p_0}] \leq u^{(p_0)}(x_0)$, the value function for the utility maximization problem with exponent p_0. The result follows as $D_T \geq k_1$. □

Remark 5.7. In [26, Example 4.6] we give a condition which implies that the utility maximization problem is finite for *all* $p_0 \in (0,1)$. Conversely, given such a $p_0 \in (p,1)$, one can show that $L \in \mathbf{B}(p_0/p)$ if D is uniformly bounded from above (see [27, Corollary 4.2]).

Example 5.8. We apply our results in an Itô model with bounded mean variance tradeoff process together with an existence result for BSDEs. For the case of utility from terminal wealth only, we retrieve (a minor generalization of) the pioneering result of [14, §3]; the case with intermediate consumption is new. Let W be an m-dimensional standard Brownian motion ($m \geq d$) and assume that $\mathbb{F}$ is generated by W. We consider

$$dR_t = b_t\,dt + \sigma_t\,dW_t,$$

where b is predictable $\mathbb{R}^d$-valued and σ is predictable $\mathbb{R}^{d\times m}$-valued with everywhere full rank; moreover, we consider constraints $\mathscr{C}$ satisfying (C1) and (C2). We are in the situation of Assumptions 3.3 with $dM = \sigma\,dW$ and $\lambda = (\sigma\sigma^\top)^{-1}b$. The process $\theta := \sigma^\top\lambda$ is called *market price of risk*. We assume that there are constants $k_i > 0$ such that

$$0 < k_1 \leq D \leq k_2 \quad \text{and} \quad \int_0^T |\theta_s|^2\,ds \leq k_3.$$

The latter condition is called *bounded mean-variance tradeoff*. Note that $dQ/dP = \mathcal{E}(-\lambda \bullet M)_T = \mathcal{E}(-\theta \bullet W)_T$ defines a local martingale measure for $\mathcal{E}(R)$. By [26, §4.2] the utility maximization problem is finite for all p and the opportunity process L is bounded and bounded away from zero. It is continuous due to Remark 3.13(i).

As suggested above, we write the Bellman BSDE for $Y := \log(L)$ rather than L in this setting. If $Y = A^Y + \varphi^Y \bullet M + N^Y$ is the Kunita-Watanabe decomposition, we write $Z := \sigma^\top\varphi^Y$ and choose $Z^\perp$ such that $Z^\perp \bullet W = N^Y$ by Brownian representation. The orthogonality of the decomposition implies $\sigma^\top Z^\perp = 0$ and that $Z^\top Z^\perp = 0$. We write $\delta = 1$ if there is intermediate consumption and $\delta = 0$ otherwise. Then Itô's formula and Corollary 3.12 (with $A_t := t$) yield the BSDE

$$dY = f(Y, Z, Z^\perp)\,dt + (Z + Z^\perp)\,dW; \quad Y_T = \log(D_T) \tag{5.1}$$

with

$$\begin{aligned} f(Y, Z, Z^\perp) = {}& \tfrac{1}{2}p(1-p)\,d^2_{\sigma^\top\mathscr{C}}\big(\beta(\theta + Z)\big) + \tfrac{q}{2}|\theta + Z|^2 \\ &+ \delta(p-1)D^\beta \exp\big((q-1)Y\big) - \tfrac{1}{2}(|Z|^2 + |Z^\perp|^2). \end{aligned}$$

Here $\beta = (1-p)^{-1}$ and $q = p/(p-1)$; the dependence on (ω, t) is suppressed in the notation. Using the orthogonality relations and $p(1-p)\beta^2 = -q$, one can check that $f(Y, Z, Z^\perp) = f(Y, Z + Z^\perp, 0) =: f(Y, \widetilde{Z})$, where $\widetilde{Z} := Z + Z^\perp$. As $0 \in \mathscr{C}$, we have $d^2_{\sigma^\top\mathscr{C}}(x) \leq |x|^2$. Hence there exist a constant $C > 0$ and an increasing continuous function ϕ such that

$$|f(y, \tilde{z})| \leq C\big(|\theta|^2 + \phi(y) + |\tilde{z}|^2\big).$$

The following monotonicity property handles the exponential nonlinearity caused by the consumption: As $p - 1 < 0$ and $q - 1 < 0$,

$$-y\big[f(y, \tilde{z}) - f(0, \tilde{z})\big] \leq 0.$$

Thus we have Briand and Hu's [4, Condition (A.1)] after noting that they call $-f$ what we call f, and [4, Lemma 2] states the existence of a bounded solution Y to the BSDE (5.1). Let us check that $\ell := \exp(Y)$ is the opportunity process. We define an associated strategy $(\check{\pi}, \check{\kappa})$ by $\check{\kappa} := (D/\ell)^{\beta}$ and Proposition 4.3; then we have a solution $(\ell, \check{\pi}, \check{\kappa})$ of the Bellman equation in the sense of Definition 4.1. For $p < 0$ ($p \in (0,1)$), Corollary 5.4 (Corollary 5.6) yields $\ell = L$ and the optimality of $(\check{\pi}, \check{\kappa})$. In fact, the same verification argument applies if we replace $\check{\pi}$ by any other predictable $\mathscr{C}$-valued π^* such that $\sigma^\top \pi^* \in \Pi^{\sigma^\top \mathscr{C}}\{\beta(\theta + Z)\}$; recall from Proposition 4.3 that $\pi^* \in L(R)$ holds automatically. To conclude: we have that

$$L = \exp(Y) \text{ is the opportunity process}$$

and the set of optimal strategies equals the set of all $(\pi^*, \hat{\kappa})$ such that

- $\hat{\kappa} = (D/L)^{\beta}$ μ°-a.e.,
- π^* is predictable, $\mathscr{C}$-valued and $\sigma^\top \pi^* \in \Pi^{\sigma^\top \mathscr{C}}\{\beta(\theta + Z)\}$ $P \otimes dt$-a.e.

One can remark that the previous arguments show $Y' = \log(L)$ whenever Y' is a solution of the BSDE (5.1) which is uniformly bounded from above. Hence we have proved uniqueness for (5.1) in this class of solutions, which is not immediate from BSDE theory. One can also note that, in contrast to [14], we did not use the theory of BMO martingales in this example. Finally, we remark that the existence of an optimal strategy can also be obtained by convex duality, under the additional assumption that $\mathscr{C}$ is convex.

We close this section with a formula intended for future applications.

Remark 5.9. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. Sometimes exponential formulas can be used to verify that $Z(\check{\pi}, \check{\kappa})$ is of class (D).

Let h be a predictable cut-off function such that $\check{\pi}^\top h(x)$ is bounded, e.g., $h(x) = x 1_{\{|x| \le 1\} \cap \{|\check{\pi}^\top x| \le 1\}}$, and define Ψ to be the local martingale

$$\ell_-^{-1} \bullet M^\ell + p\check{\pi} \bullet R^c + p\check{\pi}^\top h(x) * (\mu^R - \nu^R) + p(x'/\ell_-)\check{\pi}^\top h(x) * (\mu^{R,\ell} - \nu^{R,\ell})$$
$$+ (1 + x'/\ell_-)\big\{(1 + \check{\pi}^\top x)^p - 1 - p\check{\pi}^\top h(x)\big\} * (\mu^{R,\ell} - \nu^{R,\ell}).$$

Then $\mathcal{E}(\Psi) > 0$, and if $\mathcal{E}(\Psi)$ is of class (D), then $Z(\check{\pi}, \check{\kappa})$ is also of class (D).

Proof. Let $Z = Z(\check{\pi}, \check{\kappa})$. By a calculation as in the proof of Lemma 3.4 and the local martingale condition from Proposition 4.7, $(\frac{1}{p}\check{X}_-^p)^{-1} \bullet Z = \ell_- \bullet \Psi$. Hence $Z = Z_0 \mathcal{E}(\Psi)$ in the case without intermediate consumption. For the general case, we have seen in the proof of Corollary 5.4 that Z is of class (D) whenever $\ell \frac{1}{p}\check{X}^p$ is. Writing the definition of $\check{\kappa}$ as $\check{\kappa}^{p-1} = \ell_-/D$ μ-a.e., we have $\ell \frac{1}{p}\check{X}^p = Z - \int \check{\kappa}\ell_- \frac{1}{p}\check{X}^p \, d\mu = (\ell_- \frac{1}{p}\check{X}_-^p) \bullet (\Psi - \check{\kappa} \bullet \mu)$, hence $\ell \frac{1}{p}\check{X}^p = Z_0 \mathcal{E}(\Psi - \check{\kappa} \bullet \mu) = Z_0 \mathcal{E}(\Psi) \exp(-\check{\kappa} \bullet \mu)$. It remains to note that $\exp(-\check{\kappa} \bullet \mu) \le 1$. ∎

5.2 Verification via Deflator

The goal of this section is a verification theorem which involves only the candidate for the optimal strategy and holds for general semimartingale models. Our plan is as follows. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation and assume for the moment that $\mathscr{C}$ is convex. As the concave function g^ℓ has a maximum at $\check{\pi}$, the directional derivatives at $\check{\pi}$ in all directions should be nonpositive (if they can be defined). A calculation will show that, at the level of processes, this yields a supermartingale property which is well known from duality theory and allows for verification. In the case of non-convex constraints, the directional derivatives need not be defined in any sense. Nevertheless, the formally corresponding quantities yield the expected result. To make the first order conditions necessary, we later specialize to convex $\mathscr{C}$. As in the previous section, we first state a basic result; it is essentially classical.

Lemma 5.10. *Let ℓ be any positive càdlàg semimartingale with $\ell_T = D_T$. Suppose there exists $(\check{\pi}, \check{\kappa}) \in \mathcal{A}$ with $\check{\kappa} = (D/\ell)^\beta$ and let $\check{X} := X(\check{\pi}, \check{\kappa})$. Assume $Y := \ell \check{X}^{p-1}$ has the property that for all $(\pi, \kappa) \in \mathcal{A}$,*

$$\Gamma(\pi, \kappa) := X(\pi, \kappa) Y + \int \kappa_s X_s(\pi, \kappa) Y_s \, \mu(ds)$$

is a supermartingale. Then $\Gamma(\check{\pi}, \check{\kappa})$ is a martingale if and only if (2.2) holds and $(\check{\pi}, \check{\kappa})$ is optimal and $\ell = L$.

Proof. "$\Rightarrow$": Let $(\pi, \kappa) \in \mathcal{A}$ and denote $c = \kappa X(\pi, \kappa)$ and $\check{c} = \check{\kappa}\check{X}$. Note the partial derivative $\partial U(\check{c}) = D \check{\kappa}^{p-1} \check{X}^{p-1} = \ell \check{X}^{p-1} = Y$. Concavity of U implies $U(c) - U(\check{c}) \leq \partial U(\check{c})(c - \check{c}) = Y(c - \check{c})$, hence

$$\begin{aligned} E\Big[\int_0^T U_s(c_s)\, \mu^\circ(ds)\Big] - E\Big[\int_0^T U_s(\check{c}_s)\, \mu^\circ(ds)\Big] &\leq E\Big[\int_0^T Y_s(c_s - \check{c}_s)\, \mu^\circ(ds)\Big] \\ &= E[\Gamma_T(\pi, \kappa)] - E[\Gamma_T(\check{\pi}, \check{\kappa})]. \end{aligned}$$

Let $\Gamma(\check{\pi}, \check{\kappa})$ be a martingale; then $\Gamma_0(\pi, \kappa) = \Gamma_0(\check{\pi}, \check{\kappa})$ and the supermartingale property imply that the last line is nonpositive. As (π, κ) was arbitrary, $(\check{\pi}, \check{\kappa})$ is optimal with expected utility $E\big[\int_0^T U_s(\check{c}_s)\, \mu^\circ(ds)\big] = E[\frac{1}{p}\Gamma_T(\check{\pi}, \check{\kappa})] = \frac{1}{p}\Gamma_0(\check{\pi}, \check{\kappa}) = \frac{1}{p} x_0^p \ell_0 < \infty$. The rest is as in the proof of Lemma 5.1. $\square$

The process Y is a supermartingale deflator in the language of [19]. We refer to [26] for the connection of the opportunity process with convex duality, which in fact suggests Lemma 5.10. Note that unlike $Z(\pi, \kappa)$ from the previous section, $\Gamma(\pi, \kappa)$ is positive for all values of p.

Our next goal is to link the supermartingale property to local first order conditions. Let $y, \check{y} \in \mathscr{C} \cap \mathscr{C}^0$ (we will plug in $\check{\pi}$ for $\check{y}$). The formal directional derivative of g^ℓ at $\check{y}$ in the direction of y is $(y - \check{y})^\top \nabla g^\ell(\check{y}) = G^\ell(y, \check{y})$, where, by formal differentiation under the integral sign (cf. (3.2))

$$G^\ell(y,\check{y}) := \tag{5.2}$$
$$\ell_-(y-\check{y})^\top\Big(b^R+\tfrac{c^{R\ell}}{\ell_-}+(p-1)c^R\check{y}\Big)+\int_{\mathbb{R}^d\times\mathbb{R}}(y-\check{y})^\top x' h(x)\,F^{R,\ell}(d(x,x'))$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}}(\ell_-+x')\big\{(1+\check{y}^\top x)^{p-1}(y-\check{y})^\top x-(y-\check{y})^\top h(x)\big\}\,F^{R,\ell}(d(x,x')).$$

We take this expression as the definition of $G^\ell(y,\check{y})$ whenever the last integral is well defined (the first one is finite by (4.2)). The differentiation cannot be justified in general, but see the subsequent section.

Lemma 5.11. *Let $y\in\mathscr{C}^0$ and $\check{y}\in\mathscr{C}^{0,*}\cap\{g^\ell>-\infty\}$. Then $G^\ell(y,\check{y})$ is well defined with values in $(-\infty,\infty]$ and $G^\ell(\cdot,\check{y})$ is lower semicontinuous on $\mathscr{C}^0$.*

Proof. Writing $(y-\check{y})^\top x=1+y^\top x-(1+\check{y}^\top x)$, we can express $G^\ell(y,\check{y})$ as

$$\ell_-(y-\check{y})^\top\Big(b^R+\tfrac{c^{R\ell}}{\ell_-}+(p-1)c^R\check{y}\Big)+\int_{\mathbb{R}^d\times\mathbb{R}}(y-\check{y})^\top x' h(x)\,F^{R,\ell}(d(x,x'))$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}}(\ell_-+x')\Big\{\frac{1+y^\top x}{(1+\check{y}^\top x)^{1-p}}-1-(y+(p-1)\check{y})^\top h(x)\Big\}\,F^{R,\ell}(d(x,x'))$$
$$-\int_{\mathbb{R}^d\times\mathbb{R}}(\ell_-+x')\big\{(1+\check{y}^\top x)^p-1-p\check{y}^\top h(x)\big\}\,F^{R,\ell}(d(x,x')).$$

The first integral is finite and continuous in y by (4.2). The last integral above occurs in the definition of $g^\ell(\check{y})$, cf. (3.2), and it is finite if $g^\ell(\check{y})>-\infty$ and equals $+\infty$ otherwise. Finally, consider the second integral above and call its integrand $\psi=\psi(y,\check{y},x,x')$. The Taylor expansion $\frac{1+y^\top x}{(1+\check{y}^\top x)^{1-p}}=1+(y+(p-1)\check{y})^\top x+\frac{(p-1)}{2}\big(2y+(p-2)\check{y}\big)^\top x\;x^\top\check{y}+o(|x|^3)$ shows that $\int_{\{|x|+|x'|\leq 1\}}\psi\,dF^{R,\ell}$ is well defined and finite. It also shows that given a compact $K\subset\mathbb{R}^d$, there is $\varepsilon>0$ such that $\int_{\{|x|+|x'|\leq\varepsilon\}}\psi\,dF^{R,\ell}$ is continuous in $y\in K$ (and also in $\check{y}\in K$). The details are as in Lemma A.2. Moreover, for $y\in\mathscr{C}^0$ we have the lower bound $\psi\geq(\ell_-+x')\{-1-(y+(p-1)\check{y})^\top h(x)\}$, which is $F^{R,\ell}$-integrable on $\{|x|+|x'|>\varepsilon\}$ for any $\varepsilon>0$, again by (4.2). The result now follows by Fatou's lemma. □

We can now connect the local first order conditions for g^ℓ and the global supermartingale property: it turns out that the formal derivative G^ℓ determines the sign of the drift rate of Γ (cf. (5.3) below), which leads to the following proposition. Here and in the sequel, we denote $\check{X}=X(\check{\pi},\check{\kappa})$.

Proposition 5.12. *Let $(\ell,\check{\pi},\check{\kappa})$ be a solution of the Bellman equation and $(\pi,\kappa)\in\mathcal{A}$. Then $\Gamma(\pi,\kappa):=\ell\check{X}^{p-1}X(\pi,\kappa)+\int\kappa_s\ell_s\check{X}_s^{p-1}X_s(\pi,\kappa)\,\mu(ds)$ is a supermartingale (local martingale) if and only if $G^\ell(\pi,\check{\pi})\leq 0$ $(=0)$.*

Proof. Define $\bar{R} = R - (x - h(x)) * \mu^R$ as in (2.4). We abbreviate $\bar{\pi} := (p-1)\check{\pi} + \pi$ and similarly $\bar{\kappa} := (p-1)\check{\kappa} + \kappa$. We defer to Lemma C.1 a calculation showing that $\big(\check{X}_-^{p-1}X_-(\pi,\kappa)\big)^{-1} \bullet \big(\ell\check{X}^{p-1}X(\pi,\kappa)\big)$ equals

$$\ell - \ell_0 + \ell_-\bar{\pi} \bullet \bar{R} - \ell_-\bar{\kappa} \bullet \mu + \ell_-(p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A + \bar{\pi}^\top c^{R\ell} \bullet A$$
$$+ \bar{\pi}^\top x' h(x) * \mu^{R,\ell} + (\ell_- + x')\big\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top h(x)\big\} * \mu^{R,\ell}.$$

Here we use a predictable cut-off function h such that $\bar{\pi}^\top h(x)$ is bounded; e.g., $h(x) = x1_{\{|x|\leq 1\}\cap\{|\bar{\pi}^\top x|\leq 1\}}$. Since $(\ell, \check{\pi}, \check{\kappa})$ is a solution, the drift of ℓ is

$$A^\ell = -pU^*(\ell_-) \bullet \mu - pg^\ell(\check{\pi}) \bullet A = (p-1)\ell_-\check{\kappa} \bullet \mu - pg^\ell(\check{\pi}) \bullet A.$$

By Remark 2.3, $\Gamma := \Gamma(\pi, \kappa)$ has a well defined drift rate a^Γ with values in $(-\infty, \infty]$. From the two formulas above and (2.4) we deduce

$$a^\Gamma = \check{X}_-^{p-1} X(\pi,\kappa)_- G^\ell(\pi, \check{\pi}). \tag{5.3}$$

Here $\check{X}_-^{p-1}X(\pi,\kappa)_- > 0$ by admissibility. If Γ is a supermartingale, then $a^\Gamma \leq 0$, and the converse holds by Lemma 2.4 in view of $\Gamma \geq 0$. ☐

We obtain our second verification theorem from Proposition 5.12 and Lemma 5.10.

Theorem 5.13. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. Assume that $P \otimes A$-a.e., $G^\ell(y, \check{\pi}) \in [-\infty, 0]$ for all $y \in \mathscr{C} \cap \mathscr{C}^{0,*}$. Then*

$$\Gamma(\check{\pi}, \check{\kappa}) := \ell\check{X}^p + \int \check{\kappa}_s \ell_s \check{X}_s^p \,\mu(ds)$$

is a local martingale. It is a martingale if and only if (2.2) holds and $(\check{\pi}, \check{\kappa})$ is optimal and $\ell = L$ is the opportunity process.

If $\mathscr{C}$ is not convex, one can imagine situations where the directional derivative of g^ℓ at the maximum is positive—i.e., the assumption on $G^\ell(y, \check{\pi})$ is sufficient but not necessary. This changes in the subsequent section.

5.2.1 The Convex-Constrained Case

We assume in this section that $\mathscr{C}$ is convex; then $\mathscr{C} \cap \mathscr{C}^0$ is also convex. Our aim is to show that the nonnegativity condition on G^ℓ in Theorem 5.13 is automatically satisfied in this case. We start with an elementary but crucial observation about "differentiation under the integral sign".

Lemma 5.14. *Consider two distinct points y_0 and $\check{y}$ in $\mathbb{R}^d$ and let $C = \{\eta y_0 + (1-\eta)\check{y} : 0 \leq \eta \leq 1\}$. Let ρ be a function on $\Sigma \times C$, where Σ is some Borel space with measure ν, such that $x \mapsto \rho(x, y)$ is ν-measurable,*

$\int \rho^+(x,\cdot)\,\nu(dx) < \infty$ on C, and $y \mapsto \rho(x,y)$ is concave. In particular, the directional derivative

$$D_{\check{y},y}\rho(x,\cdot) := \lim_{\varepsilon\to 0+} \frac{\rho\big(x, \check{y} + \varepsilon(y-\check{y})\big) - \rho(x,\check{y})}{\varepsilon}$$

exists in $(-\infty,\infty]$ for all $y \in C$. Let α be another concave function on C.

Define $\gamma(y) := \alpha(y) + \int \rho(x,y)\,\nu(dx)$ and assume that $\gamma(y_0) > -\infty$ and that $\gamma(\check{y}) = \max_C \gamma < \infty$. Then for all $y \in C$,

$$D_{\check{y},y}\gamma = D_{\check{y},y}\alpha + \int D_{\check{y},y}\rho(x,\cdot)\,\nu(dx) \;\in (-\infty, 0] \tag{5.4}$$

and in particular $D_{\check{y},y}\rho(x,\cdot) < \infty$ $\nu(dx)$-a.e.

Proof. Note that γ is concave, hence we also have $\gamma > -\infty$ on C. Let $v = (y-\check{y})$ and $\varepsilon > 0$, then $\frac{\gamma(\check{y}+\varepsilon v)-\gamma(\check{y})}{\varepsilon} = \frac{\alpha(\check{y}+\varepsilon v)-\alpha(\check{y})}{\varepsilon} + \int \frac{\rho(x,\check{y}+\varepsilon v)-\rho(x,\check{y})}{\varepsilon}\,\nu(dx)$. By concavity, these quotients increase monotonically as $\varepsilon \downarrow 0$, in particular their limits exist. The left hand side is nonpositive as $\check{y}$ is a maximum and monotone convergence yields (5.4). □

For completeness, let us mention that if $\gamma(y_0) = -\infty$, there are examples where the left hand side of (5.4) is $-\infty$ but the right hand side is finite; we shall deal with this case separately. We deduce the following version of Theorem 5.13; as discussed, it involves only the control $(\check{\pi},\check{\kappa})$.

Theorem 5.15. *Let $(\ell,\check{\pi},\check{\kappa})$ be a solution of the Bellman equation and assume that $\mathscr{C}$ is convex. Then $\Gamma(\check{\pi},\check{\kappa}) := \ell\check{X}^p + \int \check{\kappa}_s\ell_s\check{X}^p_s\,\mu(ds)$ is a local martingale. It is a martingale if and only if (2.2) holds and $(\check{\pi},\check{\kappa})$ is optimal and $\ell = L$.*

Proof. To apply Theorem 5.13, we have to check that $G^\ell(y,\check{\pi}) \in [-\infty,0]$ for $y \in \mathscr{C}\cap\mathscr{C}^{0,*}$. Recall that $\check{\pi}$ is a maximizer for g^ℓ and that G^ℓ was defined by differentiation under the integral sign. Lemma 5.14 yields $G^\ell(y,\check{\pi}) \leq 0$ whenever $y \in \{g^\ell > -\infty\}$. This ends the proof for $p \in (0,1)$ as g^ℓ is then finite. If $p < 0$, the definition of g^ℓ and Remark A.7 show that the set $\{g^\ell > -\infty\}$ contains the set $\bigcup_{\eta\in[0,1)} \eta(\mathscr{C}\cap\mathscr{C}^0)$ which, in turn, is clearly dense in $\mathscr{C}\cap\mathscr{C}^{0,*}$. Hence $\{g^\ell > -\infty\}$ is dense in $\mathscr{C}\cap\mathscr{C}^{0,*}$ and we obtain $G^\ell(y,\check{\pi}) \in [-\infty,0]$ for all $y \in \mathscr{C}\cap\mathscr{C}^{0,*}$ using the lower semicontinuity from Lemma 5.11. □

Remark 5.16. (i) We note that $\Gamma(\check{\pi},\check{\kappa}) = pZ(\check{\pi},\check{\kappa})$ if Z is defined as in (4.4). In particular, Remark 5.9 can be used also for $\Gamma(\check{\pi},\check{\kappa})$.

(ii) Muhle-Karbe [24] considers certain one-dimensional (unconstrained) affine models and introduces a sufficient optimality condition in the form of an algebraic inequality (see [24, Theorem 4.20(3)]). This condition can be seen as a special case of the statement that $G^L(y,\check{\pi}) \in [-\infty,0]$ for $y \in \mathscr{C}^{0,*}$; in particular, we have shown its necessity.

Of course, all our verification results can be seen as a uniqueness result for the Bellman equation. As an example, Theorem 5.15 yields:

Corollary 5.17. *If $\mathscr{C}$ is convex, there is at most one solution of the Bellman equation in the class of solutions $(\ell, \check{\pi}, \check{\kappa})$ such that $\Gamma(\check{\pi}, \check{\kappa})$ is of class (D).*

Similarly, one can give corollaries for the other results. We close with a comment concerning convex duality.

Remark 5.18. (i) A major insight in [21] was that the "dual domain" for utility maximization (here with $\mathscr{C} = \mathbb{R}^d$) should be a set of supermartingales rather than (local) martingales when the price process has jumps. A one-period example for log-utility [21, Example 5.1'] showed that the supermartingale solving the dual problem can indeed have nonvanishing drift. In that example it is clear that this arises when the budget constraint becomes binding. For general models and log-utility, [11] comments on this phenomenon. The calculations of this section yield an instructive "local" picture also for power utility.

Under Assumptions 3.1, the opportunity process L and the optimal strategy $(\hat{\pi}, \hat{\kappa})$ solve the Bellman equation. Assume that $\mathscr{C}$ is convex and let $\widehat{X} = X(\hat{\pi}, \hat{\kappa})$. Consider $\widehat{Y} = L\widehat{X}^{p-1}$, which was the solution to the dual problem in [26]. We have shown that $\widehat{Y}\mathcal{E}(\pi \bullet R)$ is a supermartingale for every $\pi \in \mathcal{A}$; i.e., $\widehat{Y}$ is a supermartingale deflator. Choosing $\pi = 0$, we see that $\widehat{Y}$ is itself a supermartingale, and by (5.3) its drift rate satisfies

$$a^{\widehat{Y}} = \widehat{X}_-^{p-1} G^L(0, \hat{\pi}) = -\widehat{X}_-^{p-1} \hat{\pi}^\top \nabla g(\hat{\pi}).$$

Hence $\widehat{Y}$ is a local martingale if and only if $\hat{\pi}^\top \nabla g(\hat{\pi}) = 0$. One can say that $-\hat{\pi}^\top \nabla g(\hat{\pi}) < 0$ means that the constraints are binding, whereas in an "unconstrained" case the gradient of g would vanish; i.e., $\widehat{Y}$ has nonvanishing drift rate at a given (ω, t) whenever the constraints are binding. Even if $\mathscr{C} = \mathbb{R}^d$, we still have the budget constraint $\mathscr{C}^0$ in the maximization of g. If in addition R is continuous, $\mathscr{C}^0 = \mathbb{R}^d$ and we are truly in an unconstrained situation. Then $\widehat{Y}$ is a local martingale; indeed, in the setting of Corollary 3.12 we calculate

$$\widehat{Y} = y_0 \mathcal{E}\Big(-\lambda \bullet M + \frac{1}{L_-} \bullet N^L \Big), \quad y_0 := L_0 x_0^{p-1}.$$

Note how N^L, the martingale part of L orthogonal to R, yields the solution to the dual problem.

(ii) From the proof of Proposition 5.12 we have that the general formula for the local martingale part of $\widehat{Y}$ is

$$M^{\widehat{Y}} = \widehat{X}_-^{p-1} \bullet \bigg(M^L + L_-(p-1)\hat{\pi} \bullet M^{\bar{R}} + (p-1)\hat{\pi}^\top x' h(x) * (\mu^{R,L} - \nu^{R,L})$$
$$+ (L_- + x')\big\{(1 + \hat{\pi}^\top x)^{p-1} - 1 - (p-1)\hat{\pi}^\top h(x)\big\} * (\mu^{R,L} - \nu^{R,L}) \bigg).$$

This is relevant in the problem of *q-optimal equivalent martingale measures*; cf. Goll and Rüschendorf [12] for a general perspective. Let $u(x_0) < \infty$, $D \equiv 1$, $\mu = 0$, $\mathscr{C} = \mathbb{R}^d$, and assume that the set $\mathscr{M}$ of equivalent local martingale measures for $S = \mathcal{E}(R)$ is nonempty. Given $q = p/(p-1) \in (-\infty, 0) \cup (0,1)$ conjugate to p, $Q \in \mathscr{M}$ is called *q-optimal* if $E[-q^{-1}(dQ/dP)^q]$ is finite and minimal over $\mathscr{M}$. If $q < 0$, i.e., $p \in (0,1)$, then $u(x_0) < \infty$ is equivalent to the existence of some $Q \in \mathscr{M}$ such that $E[-q^{-1}(dQ/dP)^q] < \infty$; moreover, Assumptions 3.1 are satisfied (see Kramkov and Schachermayer [21, 22]). Using [21, Theorem 2.2(iv)] we conclude that

(a) the q-optimal martingale measure exists if and only if $a^{\widehat{Y}} \equiv 0$ and $M^{\widehat{Y}}$ is a true martingale;

(b) in that case, $1 + y_0^{-1} M^{\widehat{Y}}$ is its P-density process.

This generalizes earlier results of [12] as well as of Grandits [13], Jeanblanc *et al.* [16] and Choulli and Stricker [6].

A Proof of Lemma 3.8: A Measurable Maximizing Sequence

The main goal of this appendix is to construct a measurable maximizing sequence for the random function g (cf. Lemma 3.8). The entire section is under Assumptions 3.1. Before beginning the proof, we discuss the properties of g; recall that

$$\begin{aligned} g(y) := L_- y^\top \Big(b^R + \tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2} c^R y\Big) + \int_{\mathbb{R}^d \times \mathbb{R}} x' y^\top h(x)\, F^{R,L}(d(x,x')) \\ + \int_{\mathbb{R}^d \times \mathbb{R}} (L_- + x') \big\{ p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x) \big\}\, F^{R,L}(d(x,x')). \end{aligned} \tag{A.1}$$

Lemma A.1. *$L_- + x'$ is strictly positive $F^L(dx')$-a.e.*

Proof. We have

$$\begin{aligned} (P \otimes \nu^L)\{L_- + x' \le 0\} &= E\big[1_{\{L_- + x' \le 0\}} * \nu^L_T\big] \\ &= E\big[1_{\{L_- + x' \le 0\}} * \mu^L_T\big] \\ &= E\Big[\sum_{s \le T} 1_{\{L_s \le 0\}} 1_{\{\Delta L_s \ne 0\}}\Big], \end{aligned}$$

which vanishes as $L > 0$ by Lemma 2.1. □

Fix (ω, t) and let $l := L_{t-}(\omega)$. Furthermore, let F be any Lévy measure on $\mathbb{R}^{d+1}$ which is equivalent to $F^{R,L}_t(\omega)$ and satisfies (2.5). Equivalence implies that $\mathscr{C}_t^0(\omega), \mathscr{C}_t^{0,*}(\omega)$, and $\mathscr{N}_t(\omega)$ are the same if defined with respect to F instead of F^R. Given $\varepsilon > 0$, let

$$I_\varepsilon^F(y) := \int_{\{|x|+|x'|\leq\varepsilon\}} (l+x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F(d(x,x')),$$
$$I_{>\varepsilon}^F(y) := \int_{\{|x|+|x'|>\varepsilon\}} (l+x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F(d(x,x')),$$

so that

$$I^F(y) := I_\varepsilon^F(y) + I_{>\varepsilon}^F(y)$$

is the last integral in (A.1) when $F = F_t^{R,L}(\omega)$. We know from the proof of Lemma 3.4 that $I^{F^{R,L}}(\pi)$ is well defined and finite for any $\pi \in \mathcal{A}^{fE}$ (of course, when $p > 0$, this is essentially due to the assumption (2.2)). For general F, I^F has the following properties.

Lemma A.2. *Consider a sequence $y_n \to y_\infty$ in $\mathscr{C}^0$.*

(i) For any $y \in \mathscr{C}^0$, the integral $I^F(y)$ is well defined in $\mathbb{R} \cup \{\operatorname{sign}(p)\infty\}$.

(ii) For $\varepsilon \leq (2\sup_n |y_n|)^{-1}$ we have $I_\varepsilon^F(y_n) \to I_\varepsilon^F(y_\infty)$.

(iii) If $p \in (0,1)$, I^F is l.s.c., that is, $\liminf_n I^F(y_n) \geq I^F(y_\infty)$.

(iv) If $p < 0$, I^F is u.s.c., that is, $\limsup_n I^F(y_n) \leq I^F(y_\infty)$. Moreover, $y \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$ implies $I^F(y) = -\infty$.

Proof. The first item follows from the subsequent considerations.

(ii) We may assume that h is the identity on $\{|x| \leq \varepsilon\}$; then on this set $p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x) =: \psi(z)|_{z=y^\top x}$, where the function ψ is smooth on $\{|z| \leq 1/2\} \subseteq \mathbb{R}$ satisfying

$$\psi(z) = p^{-1}(1+z)^p - p^{-1} - z = \tfrac{p-1}{2} z^2 + o(|z|^3)$$

because $1+z$ is bounded away from 0. Thus $\psi(z) = z^2\tilde{\psi}(z)$ with a function $\tilde{\psi}$ that is continuous and in particular bounded on $\{|z| \leq 1/2\}$.

As a Lévy measure, F integrates $(|x'|^2 + |x|^2)$ on compacts; in particular, $G(d(x,x')) := |x|^2\, F(d(x,x'))$ defines a finite measure on $\{|x| + |x'| \leq \varepsilon\}$. Hence $I_\varepsilon^F(y)$ is well defined and finite for $|y| \leq (2\varepsilon)^{-1}$, and dominated convergence shows that $I_\varepsilon^F(y) = \int_{\{|x|+|x'|\leq\varepsilon\}} (l+x')\tilde{\psi}(y^\top x)\, G(d(x,x'))$ is continuous in y on $\{|y| \leq (2\varepsilon)^{-1}\}$.

(iii) For $|y|$ bounded by a constant C, the integrand in I^F is bounded from below by $C' + |x'|$ for some constant C' depending on y only through C. We choose ε as before. As $C' + |x'|$ is F-integrable on $\{|x| + |x'| > \varepsilon\}$ by (2.5), $I^F(y)$ is well defined in $\mathbb{R} \cup \{\infty\}$ and l.s.c. by Fatou's lemma.

(iv) The first part follows as in (iii), now the integrand is bounded from above by $C' + |x'|$. If $y \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$, Lemma A.1 shows that the integrand equals $-\infty$ on a set of positive F-measure. $\square$

Lemma A.3. *The function g is concave. If $\mathscr{C}$ is convex, g has at most one maximum on $\mathscr{C} \cap \mathscr{C}^0$, modulo $\mathscr{N}$.*

Proof. We first remark that the assertion is not trivial because g need not be strictly concave on $\mathscr{N}^\perp$, for example, the process $R_t = t(1, \dots, 1)^\top$ was not excluded.

Note that g is of the form $g(y) = Hy + J(y)$, where $Hy = L_- y^\top b^R + y^\top c^{RL} + \int x' y^\top h(x)\, F^{R,L}$ is linear and $J(y) = \frac{(p-1)}{2} L_- y^\top c^R y + I^{F^{R,L}}(y)$ is concave. We may assume that $h(x) = x 1_{\{|x| \leq 1\}}$.

Let $y_1, y_2 \in \mathscr{C} \cap \mathscr{C}^0$ be such that $g(y_1) = g(y_2) = \sup g =: g^* < \infty$, our aim is to show $y_1 - y_2 \in \mathscr{N}$. By concavity, $g^* = g((y_1 + y_2)/2)) = [g(y_1) + g(y_2)]/2$, which implies $J((y_1 + y_2)/2)) = [J(y_1) + J(y_2)]/2$ due to the linearity of H. Using the definition of J, this shows that J is constant on the line segment connecting y_1 and y_2. A first consequence is that $y_1 - y_2$ lies in the set $\{y : y^\top c^R = 0,\ F^R\{x : y^\top x \neq 0\} = 0\}$ and a second is that $Hy_1 = Hy_2$. It remains to show $(y_1 - y_2)^\top b^R = 0$ to have $y_1 - y_2 \in \mathscr{N}$.

Note that $F^R\{x : y^\top x \neq 0\} = 0$ implies $F^{R,L}\{x : y^\top h(x) \neq 0\} = 0$. Moreover, $y^\top c^R = 0$ implies $y^\top c^{RL} = 0$ due to the absolute continuity $\langle R^{c,i}, L^c \rangle \ll \langle R^{c,i} \rangle$ which follows from the Kunita-Watanabe inequality. Therefore, the first consequence above implies $\int x'(y_1 - y_2)^\top h(x)\, F^{R,L} = 0$ and $(y_1 - y_2)^\top c^{RL} = 0$, and now the second consequence and the definition of H yield $0 = H(y_1 - y_2) = L_-(y_1 - y_2)^\top b^R$. Thus $(y_1 - y_2)^\top b^R = 0$ as $L_- > 0$ and this ends the proof. □

We can now move toward the main goal of this section. Clearly we need some variant of the "Measurable Maximum Theorem" (see, e.g., [1, 18.19], [19, Theorem 9.5], [28, 2K]). We state a version that is tailored to our needs and has a simple proof; the technique is used also in Proposition 4.3.

Lemma A.4. *Let $\mathscr{D}$ be a predictable set-valued process with nonempty compact values in $2^{\mathbb{R}^d}$. Let $f(y) = f(\omega, t, y)$ be a proper function on $\mathscr{D}$ with values in $\mathbb{R} \cup \{-\infty\}$ such that*

(i) $f(\varphi)$ is predictable whenever φ is a $\mathscr{D}$-valued predictable process,

(ii) $y \mapsto f(y)$ is upper semicontinuous on $\mathscr{D}$ for fixed (ω, t).

Then there exists a $\mathscr{D}$-valued predictable process π such that $f(\pi) = \max_{\mathscr{D}} f$.

Proof. We start with the Castaing representation [28, 1B] of $\mathscr{D}$: there exist $\mathscr{D}$-valued predictable processes $(\varphi_n)_{n \geq 1}$ such that $\overline{\{\varphi_n : n \geq 1\}} = \mathscr{D}$ for each (ω, t). By (i), $f^* := \max_n f(\varphi_n)$ is predictable, and $f^* = \max_{\mathscr{D}} f$ by (ii). Fix $k \geq 1$ and let $\Lambda_n := \{f^* - f(\varphi_n) \leq 1/k\}$, $\Lambda^n := \Lambda_n \setminus (\Lambda_1 \cup \dots \cup \Lambda_{n-1})$. Define $\pi^k := \sum_n \varphi_n 1_{\Lambda^n}$, then $f^* - f(\pi^k) \leq 1/k$ and $\pi^k \in \mathscr{D}$.

It remains to select a cluster point: By compactness, $(\pi^k)_{k \geq 1}$ is bounded for each (ω, t), so there is a convergent subsequence along "random indices" τ_k. More precisely, there exists a strictly increasing sequence of integer-valued predictable processes $\tau_k = \{\tau_k(\omega, t)\}$ and a predictable process π^*

such that $\lim_k \pi_t^{\tau_k(\omega,t)}(\omega) = \pi_t^*(\omega)$ for all (ω, t). See, e.g., the proof of Föllmer and Schied [10, Lemma 1.63]. We have $f^* = f(\pi^*)$ by (ii). $\square$

Our random function g satisfies property (i) of Lemma A.4 because the characteristics are predictable (recall the definition [15, II.1.6]). We also note that the intersection of closed predictable processes is predictable [28, 1M]. The sign of p is important as it switches the semicontinuity of g; we start with the immediate case $p < 0$ and denote $B_r(\mathbb{R}^d) = \{x \in \mathbb{R}^d : |x| \leq r\}$.

Proof of Lemma 3.8 for $p < 0$. In this case g is u.s.c. on $\mathscr{C} \cap \mathscr{C}^0$ (Lemma A.2). Let $\mathscr{D}(n) := \mathscr{C} \cap \mathscr{C}^0 \cap B_n(\mathbb{R}^d)$. Lemma A.4 yields a predictable process $\pi^n \in \arg\max_{\mathscr{D}(n)} g$ for each $n \geq 1$, and clearly $\lim_n g(\pi^n) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g$. As $g(\pi^n) \geq g(0) = 0$, we have $\pi^n \in \mathscr{C}^{0,*}$ by Lemma A.2. $\square$

A.1 Measurable Maximizing Sequence for $p \in (0, 1)$

Fix $p \in (0, 1)$. Since the continuity properties of g are not clear, we will use an approximating sequence of continuous functions. (See also Appendix B, where an alternative approach is discussed and the continuity is clarified under an additional assumption on $\mathscr{C}$.) We will approximate g using Lévy measures with enhanced integrability, a method suggested by [19] in a similar problem. This preserves monotonicity properties that will be useful to pass to the limit.

All this is not necessary if R is locally bounded, or more generally if $F^{R,L}$ satisfies the following condition. We start with fixed (ω, t).

Definition A.5. Let F be a Lévy measure on $\mathbb{R}^{d+1}$ which is equivalent to $F^{R,L}$ and satisfies (2.5). (i) We say that F is *p-suitable* if

$$\int (1 + |x'|)(1 + |x|)^p 1_{\{|x|>1\}} \, F(d(x, x')) < \infty.$$

(ii) The *p-suitable approximating sequence for F* is the sequence $(F_n)_{n \geq 1}$ of Lévy measures defined by $dF_n/dF = f_n$, where

$$f_n(x) = 1_{\{|x| \leq 1\}} + e^{-|x|/n} 1_{\{|x|>1\}}.$$

It is easy to see that each F_n in (ii) shares the properties of F, while in addition being p-suitable because $(1 + |x|)^p e^{-|x|/n}$ is bounded. As the sequence f_n is increasing, monotone convergence shows that $\int V dF_n \uparrow \int V \, dF$ for any measurable function $V \geq 0$ on $\mathbb{R}^{d+1}$. We denote by g^F the function which is defined as in (A.1) but with $F^{R,L}$ replaced by F.

Lemma A.6. *If F is p-suitable, g^F is real-valued and continuous on $\mathscr{C}^0$.*

Proof. Pick $y_n \to y$ in $\mathscr{C}^0$. The only term in (A.1) for which continuity is not evident, is the integral $I^F = I^F_\varepsilon + I^F_{>\varepsilon}$, where we choose ε as in Lemma A.2. We have $I^F_\varepsilon(y_n) \to I^F_\varepsilon(y)$ by that lemma. When F is p-suitable, the continuity of $I^F_{>\varepsilon}$ follows from the dominated convergence theorem. $\square$

Remark A.7. Define the set

$$(\mathscr{C} \cap \mathscr{C}^0)^\diamond := \bigcup_{\eta \in [0,1)} \eta(\mathscr{C} \cap \mathscr{C}^0).$$

Its elements y have the property that $1 + y^\top x$ is $F^R(dx)$-essentially *bounded away* from zero. Indeed, $y = \eta y_0$ with $\eta \in [0,1)$ and $F^R\{y_0^\top x \geq -1\} = 0$, hence $1 + y^\top x \geq 1 - \eta$, F^R-a.e. In particular, $(\mathscr{C} \cap \mathscr{C}^0)^\diamond \subseteq \mathscr{C}^{0,*}$. If $\mathscr{C}$ is star-shaped with respect to the origin, we also have $(\mathscr{C} \cap \mathscr{C}^0)^\diamond \subseteq \mathscr{C}$.

We introduce the compact-valued process $\mathscr{D}(r) := \mathscr{C} \cap \mathscr{C}^0 \cap B_r(\mathbb{R}^d)$.

Lemma A.8. *Let F be p-suitable. Under (C3),* $\arg\max_{\mathscr{D}(r)} g^F \subseteq \mathscr{C}^{0,*}$.

More generally, this holds whenever F is a Lévy measure equivalent to $F^{R,L}$ satisfying (2.5) *and g^F is finite-valued.*

Proof. Assume that $\check{y} \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$ is a maximum of g^F. Let $\eta \in (\underline{\eta}, 1)$ be as in the definition of (C3) and $y_0 := \eta \check{y}$. By Lemma 5.14, the directional derivative $D_{\check{y},y_0} g$ can be calculated by differentiating under the integral sign. For the integrand of I^F we have

$$D_{\check{y},y_0}\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\} = (1-\eta)\big\{(1+\check{y}^\top x)^{p-1}\check{y}^\top x - \check{y}^\top h(x)\big\}.$$

But this is infinite on a set of positive measure as $\check{y} \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$ means that $F\{\check{y}^\top x = -1\} > 0$, contradicting the last assertion of Lemma 5.14. $\square$

Let F be a Lévy measure on $\mathbb{R}^{d+1}$ which is equivalent to $F^{R,L}$ and satisfies (2.5). The crucial step is

Lemma A.9. *Let (F_n) be the p-suitable approximating sequence for F and fix $r > 0$. For each n,* $\arg\max_{\mathscr{D}(r)} g^{F_n} \neq \emptyset$, *and for any $y_n^* \in \arg\max_{\mathscr{D}(r)} g^{F_n}$ it holds that* $\limsup_n g^F(y_n^*) = \sup_{\mathscr{D}(r)} g^F$.

Proof. We first show that

$$I^{F_n}(y) \to I^F(y) \quad \text{for any } y \in \mathscr{C}^0. \tag{A.2}$$

Recall that $I^{F_n}(y) = \int (l+x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\} f_n(x)\, F(d(x,x'))$, where f_n is nonnegative and increasing in n. As $f_n = 1$ in a neighborhood of the origin, we need to consider only $I^{F_n}_{>\varepsilon}$ (for $\varepsilon = 1$, say). Its integrand is bounded below, simultaneously for all n, by a negative constant times $(1 + |x'|)$, which is F-integrable on the relevant domain. As (f_n) is increasing, we can apply monotone convergence on the set $\big\{(x,x') : p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x) \geq 0\big\}$ and dominated convergence on the complement to deduce (A.2).

Existence of $y_n^* \in \arg\max_{\mathscr{D}(r)} g^{F_n}$ is clear by compactness of $\mathscr{D}(r)$ and continuity of g^{F_n} (Lemma A.6). Let $y \in \mathscr{D}(r)$ be arbitrary. By definition of y_n^* and (A.2),

$$\limsup_n g^{F_n}(y_n^*) \geq \limsup_n g^{F_n}(y) = g^F(y).$$

We show $\limsup_n g^F(y_n^*) \geq \limsup_n g^{F_n}(y_n^*)$. We can split the integral $I^{F_n}(y)$ into a sum of three terms: The integral over $\{|x| \leq 1\}$ is the same as for I^F, since $f_n = 1$ on this set. We can assume that the cut-off h vanishes outside $\{|x| \leq 1\}$. The second term is then

$$\int_{\{|x|>1\}} (l + x')p^{-1}(1 + y^\top x)^p f_n \, dF,$$

here the integrand is nonnegative and hence increasing in n, for all y; and the third term is

$$\int_{\{|x|>1\}} (l + x')(-p^{-1}) f_n \, dF,$$

which is decreasing in n but converges to $\int_{\{|x|>1\}} (l + x')(-p^{-1}) \, dF$. Thus

$$g^F(y_n^*) \geq g^{F_n}(y_n^*) - \varepsilon_n$$

with the sequence $\varepsilon_n := \int_{\{|x|>1\}} (l + x')(-p^{-1})(f_n - 1) \, dF \downarrow 0$. Together, we conclude $\sup_{\mathscr{D}(r)} g^F \geq \limsup_n g^F(y_n^*) \geq \limsup_n g^{F_n}(y_n^*) \geq \sup_{\mathscr{D}(r)} g^F$. $\square$

Proof of Lemma 3.8 for $p \in (0,1)$. Fix $r > 0$. By Lemma A.4 we can find measurable selectors $\pi^{n,r}$ for $\arg\max_{\mathscr{D}(r)} g^{F_n}$; i.e., $\pi_t^{n,r}(\omega)$ plays the role of y_n^* in Lemma A.9. Taking $\pi^n := \pi^{n,n}$ and noting $\mathscr{D}(n) \uparrow \mathscr{C} \cap \mathscr{C}^0$, Lemma A.9 shows that π^n are $\mathscr{C} \cap \mathscr{C}^0$-valued predictable processes such that $\limsup_n g(\pi^n) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g$ $P \otimes A$-a.e. Lemma A.8 shows that π^n takes values in $\mathscr{C}^{0,*}$. $\square$

B Parametrization by Representative Portfolios

This appendix introduces an equivalent transformation of the model $(R, \mathscr{C})$ with specific properties (Theorem B.3); the main idea is to substitute the given assets by wealth processes that represent the investment opportunities of the model. While the result is of independent interest, the main conclusion in our context is that the approximation technique from Appendix A.1 for the case $p \in (0,1)$ can be avoided, at least under slightly stronger assumptions on $\mathscr{C}$: If the utility maximization problem is finite, the corresponding Lévy measure in the transformed model is p-suitable (cf. Definition A.5) and hence the corresponding function g is continuous. This is not only an alternative argument to prove Lemma 3.8. In applications, continuity can be useful to construct a maximizer for g (rather than a maximizing sequence) if one does not know *a priori* that there exists an optimal strategy. A static version of our construction was carried out for the case of Lévy processes in [25, §4].

In this appendix we use the following assumptions on the set-valued process $\mathscr{C}$ of constraints:

(C1) $\mathscr{C}$ is predictable.

(C2) $\mathscr{C}$ is closed.

(C4) $\mathscr{C}$ is star-shaped with respect to the origin: $\eta\mathscr{C} \subseteq \mathscr{C}$ for all $\eta \in [0,1]$.

Since we already obtained a proof of Lemma 3.8, we do not strive for minimal conditions here. Clearly (C4) implies condition (C3) from Section 2.4, but its main implication is that we can select a bounded (hence R-integrable) process in the subsequent lemma. The following result is the construction of the jth *representative portfolio*, a portfolio with the property that it invests in the jth asset *whenever* this is feasible.

Lemma B.1. *Fix $1 \leq j \leq d$ and let $H^j = \{x \in \mathbb{R}^d : x^j \neq 0\}$. There exists a bounded predictable $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process ϕ satisfying*

$$\{\phi^j = 0\} = \big\{\mathscr{C} \cap \mathscr{C}^{0,*} \cap H^j = \emptyset\big\}.$$

Proof. Let $B_1 = B_1(\mathbb{R}^d)$ be the closed unit ball and $H := H^j$. Condition (C4) implies $\big\{\mathscr{C} \cap \mathscr{C}^{0,*} \cap H = \emptyset\big\} = \big\{\mathscr{C} \cap B_1 \cap \mathscr{C}^{0,*} \cap H = \emptyset\big\}$, hence we may substitute $\mathscr{C}$ by $\mathscr{C} \cap B_1$. Define the closed sets $H_k = \{x \in \mathbb{R}^d : |x^j| \geq k^{-1}\}$ for $k \geq 1$, then $\bigcup_k H_k = H$. Moreover, let $\mathscr{D}_k = \mathscr{C} \cap \mathscr{C}^0 \cap H_k$. This is a compact-valued predictable process, so there exists a predictable process ϕ_k such that $\phi_k \in \mathscr{D}_k$ (hence $\phi_k^j \neq 0$) on the set $\Lambda_k := \{\mathscr{D}_k \neq \emptyset\}$ and $\phi_k = 0$ on the complement. Define $\Lambda^k := \Lambda_k \setminus (\Lambda_1 \cup \cdots \cup \Lambda_{k-1})$ and $\phi' := \sum_k \phi_k 1_{\Lambda^k}$. Then $|\phi'| \leq 1$ and $\{\phi'^j = 0\} = \big\{\mathscr{C} \cap \mathscr{C}^0 \cap H = \emptyset\big\} = \big\{\mathscr{C} \cap \mathscr{C}^{0,*} \cap H = \emptyset\big\}$; the second equality uses (C4) and Remark A.7. These two facts also show that $\phi := \frac{1}{2}\phi'$ has the same property while in addition being $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued. $\square$

Remark B.2. The previous proof also applies if instead of (C4), e.g., the diameter of $\mathscr{C}$ is uniformly bounded and $\mathscr{C}^0 = \mathscr{C}^{0,*}$.

If Φ is a $d \times d$-matrix with columns $\phi_1, \dots, \phi_d \in L(R)$, the matrix stochastic integral $\widetilde{R} = \Phi \bullet R$ is the $\mathbb{R}^d$-valued process given by $\widetilde{R}^j = \phi_j \bullet R$. If $\psi \in L(\Phi \bullet R)$ is $\mathbb{R}^d$-valued, then $\Phi\psi \in L(R)$ and

$$\psi \bullet (\Phi \bullet R) = (\Phi\psi) \bullet R. \tag{B.1}$$

If $\mathscr{D}$ is a set-valued process which is predictable, closed and contains the origin, then the preimage $\Phi^{-1}\mathscr{D}$ shares these properties (cf. [28, 1Q]). Convexity and star-shape are also preserved.

We obtain the following model if we sequentially replace the given assets by representative portfolios; here e_j denotes the jth unit vector in $\mathbb{R}^d$ for $1 \leq j \leq d$ (i.e., $e_j^i = \delta_{ij}$).

Theorem B.3. *There exists a predictable $\mathbb{R}^{d\times d}$-valued uniformly bounded process Φ such that the financial market model with returns*

$$\widetilde{R} := \Phi \bullet R$$

and constraints $\widetilde{\mathscr{C}} := \Phi^{-1}\mathscr{C}$ has the following properties: for all $1 \leq j \leq d$,

(i) $\Delta \widetilde{R}^j > -1$ *(positive prices),*

(ii) $e_j \in \widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^{0,*}$*, where* $\widetilde{\mathscr{C}}^{0,*} = \Phi^{-1}\mathscr{C}^{0,*}$ *(entire wealth can be invested in each asset),*

(iii) the model $(\widetilde{R}, \widetilde{\mathscr{C}})$ *admits the same wealth processes as* $(R, \mathscr{C})$*.*

Proof. We treat the components one by one. Let $j = 1$ and let $\phi = \phi(1)$ be as in Lemma B.1. We replace the first asset R^1 by the process $\phi \bullet R$, or equivalently, we replace R by $\Phi \bullet R$, where $\Phi = \Phi(1)$ is the $d \times d$-matrix

$$\Phi = \begin{pmatrix} \phi^1 & & & \\ \phi^2 & 1 & & \\ \vdots & & \ddots & \\ \phi^d & & & 1 \end{pmatrix}.$$

The new natural constraints are $\Phi^{-1}\mathscr{C}^0$ and we replace $\mathscr{C}$ by $\Phi^{-1}\mathscr{C}$. Note that $e_1 \in \Phi^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$ because $\Phi e_1 = \phi \in \mathscr{C} \cap \mathscr{C}^{0,*}$ by construction.

We show that for every $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process $\pi \in L(R)$ there exists ψ predictable such that $\Phi\psi = \pi$. In view of (B.1), this will imply that the new model admits the same wealth processes as the old one. On the set $\{\phi^1 \neq 0\} = \{\Phi \text{ is invertible}\}$ we take $\psi = \Phi^{-1}\pi$ and on the complement we choose $\psi^1 \equiv 0$ and $\psi^j = \pi^j$ for $j \geq 2$; this is the same as inverting Φ on its image. Note that $\{\phi^1 = 0\} \subseteq \{\pi^1 = 0\}$ by the choice of ϕ.

We proceed with the second component of the new model in the same way, and then continue until the last one. We obtain matrices $\Phi(j)$ for $1 \leq j \leq d$ and set $\hat{\Phi} = \Phi(1) \cdots \Phi(d)$. Then $\hat{\Phi}$ has the required properties. Indeed, the construction and $\Phi(i)e_j = e_j$ for $i \neq j$ imply $e_j \in \hat{\Phi}^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$. This is (ii), and (i) is a consequence of (ii). $\square$

Coming back to the utility maximization problem, note that property (iii) implies that the value functions and the opportunity processes for the models $(R, \mathscr{C})$ and $(\widetilde{R}, \widetilde{\mathscr{C}})$ coincide up to evanescence; we identify them in the sequel. Furthermore, if $\tilde{g}$ denotes the analogue of g in the model $(\widetilde{R}, \widetilde{\mathscr{C}})$, cf. (A.1), we have the relation

$$\tilde{g}(y) = g(\Phi y), \quad y \in \widetilde{\mathscr{C}}^0.$$

Finding a maximizer for $\tilde{g}$ is equivalent to finding one for g and if $(\tilde{\pi}, \kappa)$ is an optimal strategy for $(\widetilde{R}, \widetilde{\mathscr{C}})$ then $(\Phi\tilde{\pi}, \kappa)$ is optimal for $(R, \mathscr{C})$. In fact, most properties of interest carry over from $(R, \mathscr{C})$ to $(\widetilde{R}, \widetilde{\mathscr{C}})$, in particular any no-arbitrage property that is defined via the set of admissible (positive) wealth processes.

Remark B.4. A classical no-arbitrage condition defined in a slightly different way is that there exist a probability measure $Q \approx P$ under which $\mathcal{E}(R)$ is a σ-martingale; cf. Delbaen and Schachermayer [9]. In this case, $\mathcal{E}(\widetilde{R})$ is even a local martingale under Q, as it is a σ-martingale with positive components.

Property (ii) from Theorem B.3 is useful to apply the following result.

Lemma B.5. *Let $p \in (0,1)$ and assume $e_j \in \mathscr{C} \cap \mathscr{C}^{0,*}$ for $1 \leq j \leq d$. Then $u(x_0) < \infty$ implies that $F^{R,L}$ is p-suitable. If, in addition, there exists a constant k_1 such that $D \geq k_1 > 0$, it follows that $\int_{\{|x|>1\}} |x|^p \, F^R(dx) < \infty$.*

Proof. As $p > 0$ and $u(x_0) < \infty$, L is well defined and $L, L_- > 0$ by Section 2.2. No further properties were used to establish Lemma 3.4, whose formula shows that $g(\pi)$ is finite $P \otimes A$-a.e. for all $\pi \in \mathcal{A} = \mathcal{A}^{fE}$. In particular, from the definition of g, it follows that $\int (L_- + x')\big\{p^{-1}(1+\pi^\top x)^p - p^{-1} - \pi^\top h(x)\big\} \, F^{R,L}(d(x,x'))$ is finite. If $D \geq k_1$, [26, Lemma 3.5] shows that $L \geq k_1$, hence $L_- + x' \geq k_1$ $F^L(dx')$-a.e. and $\int \big\{p^{-1}(1+\pi^\top x)^p - p^{-1} - \pi^\top h(x)\big\} \, F^R(dx) < \infty$. We choose $\pi = e_j$ (and κ arbitrary) for $1 \leq j \leq d$ to deduce the result. $\square$

In general, the condition $u(x_0) < \infty$ does not imply any properties of R; for instance, in the trivial cases $\mathscr{C} = \{0\}$ or $\mathscr{C}^{0,*} = \{0\}$. The transformation changes the geometry of $\mathscr{C}$ and $\mathscr{C}^{0,*}$ such that Theorem B.3(ii) holds, and then the situation is different.

Corollary B.6. *Let $p \in (0,1)$ and $u(x_0) < \infty$. In the model $(\widetilde{R}, \widetilde{\mathscr{C}})$ of Theorem B.3, $F^{\widetilde{R},L}$ is p-suitable and hence $\tilde{g}$ is continuous.*

Therefore, to prove Lemma 3.8 under (C4), we may substitute $(R, \mathscr{C})$ by $(\widetilde{R}, \widetilde{\mathscr{C}})$ and avoid the use of p-suitable approximating sequences. In some cases, Lemma B.5 applies directly in $(R, \mathscr{C})$. In particular, if the asset prices are strictly positive ($\Delta R^j > -1$ for $1 \leq j \leq d$), then the positive orthant of $\mathbb{R}^d$ is contained in $\mathscr{C}^{0,*}$ and the condition of Lemma B.5 is satisfied as soon as $e_j \in \mathscr{C}$ for $1 \leq j \leq d$.

C Omitted Calculation

This appendix contains a calculation which was omitted in the proof of Proposition 5.12.

Lemma C.1. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation, $(\pi, \kappa) \in \mathcal{A}$, $X := X(\pi, \kappa)$ and $\check{X} := X(\check{\pi}, \check{\kappa})$. Define $\bar{R} = R - (x - h(x)) * \mu^R$ as well as $\bar{\pi} := (p-1)\check{\pi} + \pi$ and $\bar{\kappa} := (p-1)\check{\kappa} + \kappa$. Then $\xi := \ell \check{X}^{p-1} X$ satisfies*

$$\big(\check{X}_-^{p-1} X_-\big)^{-1} \bullet \xi =$$

$$\ell - \ell_0 + \ell_- \bar{\pi} \bullet \bar{R} - \ell_- \bar{\kappa} \bullet \mu + \ell_-(p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A + \bar{\pi}^\top c^{R\ell} \bullet A$$
$$+ \bar{\pi}^\top x' h(x) * \mu^{R,\ell} + (\ell_- + x')\big\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top h(x)\big\} * \mu^{R,\ell}.$$

Proof. We may assume $x_0 = 1$. This calculation is similar to the one in the proof of Lemma 3.4 and therefore we shall be brief. By Itô's formula we have $\check{X}^{p-1} = \mathcal{E}(\zeta)$ for

$$\begin{aligned}\zeta = (p-1)(\check{\pi} \bullet R - \check{\kappa} \bullet \mu) + \tfrac{(p-1)(p-2)}{2} \check{\pi}^\top c^R \check{\pi} \bullet A \\ + \big\{(1+\check{\pi}^\top x)^{p-1} - 1 - (p-1)\check{\pi}^\top x\big\} * \mu^R.\end{aligned}$$

Thus $\check{X}^{p-1} X = \mathcal{E}\big(\zeta + \pi \bullet R - \kappa \bullet \mu + [\zeta, \pi \bullet R]\big) =: \mathcal{E}(\Psi)$ with

$$\begin{aligned}[R,\zeta] &= [R^c, \zeta^c] + \sum \Delta R \Delta \zeta \\ &= (p-1) c^R \check{\pi} \bullet A + (p-1)\check{\pi}^\top x x * \mu^R \\ &\quad + x\big\{(1+\check{\pi}^\top x)^{p-1} - 1 - \check{\pi}^\top x\big\} * \mu^R\end{aligned}$$

and recombining the terms yields

$$\begin{aligned}\Psi = \bar{\pi} \bullet R - \bar{\kappa} \bullet \mu + (p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A \\ + \big\{(1+\check{\pi}^\top x)^{p-1}(1+\pi^\top x) - 1 - \bar{\pi}^\top x\big\} * \mu^R.\end{aligned}$$

Then $\big(\check{X}_-^{p-1} X_-\big)^{-1} \bullet \xi = \ell - \ell_0 + \ell_- \bullet \Psi + [\ell, \Psi]$, where

$$\begin{aligned}[\ell, \Psi] &= [\ell^c, \Psi^c] + \sum \Delta \ell \Delta \Psi \\ &= \bar{\pi}^\top c^{R\ell} \bullet A + \bar{\pi}^\top x' x * \mu^{R,\ell} \\ &\quad + x'\big\{(1+\check{\pi}^\top x)^{p-1}(1+\pi^\top x) - 1 - \bar{\pi}^\top x\big\} * \mu^{R,\ell}.\end{aligned}$$

We arrive at

$$\begin{aligned}&\big(\check{X}_-^{p-1} X_-\big)^{-1} \bullet \xi = \\ &\ell - \ell_0 + \ell_- \bar{\pi} \bullet R - \ell_- \bar{\kappa} \bullet \mu + \ell_-(p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A + \bar{\pi}^\top c^{R\ell} \bullet A \\ &+ \bar{\pi}^\top x' x * \mu^{R,\ell} + (\ell_- + x')\big\{(1+\check{\pi}^\top x)^{p-1}(1+\pi^\top x) - 1 - \bar{\pi}^\top x\big\} * \mu^{R,\ell}.\end{aligned}$$

The result follows by writing $x = h(x) + x - h(x)$. □

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