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OPTIMAL CONSUMPTION AND INVESTMENT WITH POWER UTILITY

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To Laura, whom I love.

Abstract

In this thesis we study the utility maximization problem for power utility random fields in a general semimartingale financial market, with and without intermediate consumption. The notion of an opportunity process is introduced as a reduced form of the value process for the resulting stochastic control problem. This process is shown to describe the key objects: the optimal strategy, the value function, and the convex-dual problem. We show that the existence of an optimal strategy implies that the opportunity process solves the so-called Bellman equation. The optimal strategy is described pointwise in terms of the opportunity process, which is also characterized as the minimal solution of the Bellman equation. Furthermore, we provide verification theorems for this equation. As an example, we consider exponential Lévy models, for which we construct an explicit solution in terms of the Lévy triplet. Finally, we study the asymptotic properties of the optimal strategy as the relative risk aversion tends to infinity or to one. The convergence of the optimal consumption is obtained for the general case, while the convergence of the optimal trading strategy is obtained for continuous semimartingale models.

Kurzfassung

Diese Dissertation beschäftigt sich mit dem Nutzenmaximierungs-Problem für Potenznutzen, mit oder ohne Konsum, in einem allgemeinen Semimartingalmodell eines Finanzmarktes. Der so genannte Opportunitätsprozess wird eingeführt als reduzierte Form des Wertprozesses des zugehörigen stochastischen Kontrollproblems. Dieser Prozess beschreibt die fundamentalen Grössen: die optimale Strategie, die Wertfunktion und das konvex-duale Problem. Der Opportunitätsprozess erfüllt die zugehörige Bellman-Gleichung, sobald eine optimale Strategie existiert. Umgekehrt wird dieser Prozess als die minimale Lösung dieser Gleichung charakterisiert, und die optimale Strategie wird punktweise mit Hilfe dieses Prozesses beschrieben. Desweiteren zeigen wir verschiedene Verifikationstheoreme für die Bellman-Gleichung. Für den Spezialfall eines exponentiellen Lévy-Modells leiten wir die explizite Lösung des Problems unter minimalen Annahmen her. Schliesslich untersuchen wir das asymptotische Verhalten der optimalen Strategien, wenn die relative Risikoaversion gegen unendlich oder gegen eins strebt. Wir zeigen die Konvergenz des optimalen Konsums im allgemeinen Fall und die Konvergenz der optimalen Handelsstrategie für stetige Semimartingalmodelle.

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Contents

Chapter I

Introduction

This chapter describes the optimization problem at hand, embeds the thesis in the literature and gives an overview of the main results.

I.1 Power Utility Maximization

Expected utility criteria are used to describe the preferences of a rational economic agent. We shall start with a given utility function U and refer to Föllmer and Schied [22] for a general introduction to the modeling of preferences and the relations to economic axioms for the latter.

We consider an agent who begins with an initial endowment $x_0 > 0$ and invests in a financial market in continuous time. Her preferences are modeled by an increasing and concave function U and her aim is to maximize a certain utility functional. We shall consider two problems of this type. The first one is the maximization of expected utility from terminal wealth, i.e, the agent chooses her trading strategy π such as to maximize the expectation

$$E\big[U\big(X_T(\pi)\big)\big],$$

where $X_T(\pi)$ denotes the wealth resulting from x_0 and π at a given time horizon T. In the second problem the agent is also allowed to consume during the interval $[0,T]$; i.e., she chooses a trading strategy π as well as a consumption rate c with the aim of maximizing

$$E\bigg[\int_0^T \widetilde{U}(c_t)\,dt + U\big(X_T(\pi,c)\big)\bigg],$$

where $\widetilde{U}$ is again some utility function. We shall treat both of these specifications in a unified notation, or more precisely, a slight extension involving a time-dependent (and possibly random) utility function $U_t(\cdot)$. For simplicity, we restrict our attention to classical utility functions in this introduction. As usual, the problem is simplified by assuming that the agent is "small" in the sense that her actions do not influence the financial market. Moreover, the market is supposed to be frictionless, i.e., free of transaction costs, liquidity effects, and so on.

The most frequently used explicit examples of utility functions $U(x)$ (defined for $x > 0$) are those for which the relative risk aversion $-xU''(x)/U'(x)$ is constant. This choice is a compromise between tractability and generality, and also due to the lack of other "canonical" examples. The logarithmic utility corresponds to unit relative risk aversion and yields the most explicit results as it leads to a "myopic" behavior[1]; but this also means that many interesting properties of general utility functions cannot be observed there. The functions with constant relative risk aversion in $(0, \infty) \setminus \{1\}$ are called *power utilities* and of the form $U(x) = \frac{1}{p}x^p$ with $p \in (-\infty, 0) \cup (0, 1)$. They entail non-myopic phenomena while their scaling properties still lead to a substantial simplification: in a suitable parametrization, the optimal strategies do not depend on the level of wealth. The frequent use of power utilities motivates their study in a general financial market model, and this is the content of the present thesis.

I.2 Classification of the Literature

There is a vast literature on the maximization of expected utility and we confine ourselves to a brief classification of the approaches. The book of Karatzas and Shreve [42, pp. 153] contains a survey of the literature up to its date of writing.

Existence. The existence of an optimal strategy in a frictionless semimartingale financial market has been established in satisfactory generality using martingale theory and convex duality. This works for general utility functions; see Kramkov and Schachermayer [49] for the case of terminal wealth (on $\mathbb{R}_+$) and Karatzas and Žitković [43] for the case with consumption (and random endowment).

In the light of these results, the focus of mathematical research in (single-agent) expected utility maximization has shifted to the study of the properties of the optimal strategy as well as to markets with friction. In the sequel, we focus on the first aspect and power utility.

Explicit Solutions. The most evident approach is to explicitly solve the utility maximization problem for suitable market models. In the case with consumption this turns out to be difficult: except for complete markets, an explicit solution can be expected only for Lévy models (see Chapter IV). In the case of terminal wealth, certain algebraic properties of a market

[1] "No doubt some will say: 'I'm not sure of my taste for risk. I lack a rule to act on. So I grasp at one that at least ends doubt: better to act to make the odds big that I win than to be left in doubt?' Not so. There is more than one rule to end doubt. Why pick on one odd one?" (from Samuelson's comment [66] on logarithmic utility—a paper whose most distinctive feature is to consist of words of one syllable.)

model can lead to an explicit solution also in the incomplete case. The exponentially-quadratic model of Kim and Omberg [47] was among the first of this kind; other examples include certain exponentially-affine specifications as in Kallsen and Muhle-Karbe [40] and Muhle-Karbe [58].

Markovian Dynamic Programming and PDEs. When the asset prices follow a Markov process, dynamic programming is often used to show that the value function satisfies the corresponding Hamilton-Jacobi-Bellman partial differential equation (PDE) in the viscosity sense (or integro-PDE in the case with jumps). In turn, the value function can be used to describe the optimal strategy—at least formally, in the sense that the expression at hand involves the derivatives of the value function while its regularity is known only under strong assumptions on the model. We refer to Fleming and Soner [21] for background and general references. Of course, the important early work of Merton [55, 56] falls in this category. Stoikov and Zariphopoulou [72] study optimal consumption in a diffusion model with constant correlation; we extend some of their results to semimartingale models in Chapter II.

Non-Markovian Dynamic Programming and BSDEs. In a suitable formulation, dynamic programming can be applied also when the asset prices are not Markovian, and this is the approach taken in this thesis. In contrast to the Markov case, the corresponding "local" equation is stochastic. In Chapter III we shall state it as a backward stochastic differential equation (BSDE) in the context of a general semimartingale model. For the case of terminal wealth, this was previously obtained for certain continuous models by Mania and Tevzadze [54] and Hu et al. [33]. It is worth noting that the BSDE formulation does not extend to general utility functions, whereas in the Markov case, the corresponding PDE (or integro-PDE) is standard.

I.3 Overview of the Thesis

In view of the general existence results mentioned above, the problem at the center of the thesis is *the description of the optimal strategy in a general model.* The results are divided into four chapters which correspond to the articles [61, 59, 60, 62]. The interdependencies are as follows:

$$\begin{tikzcd}
& \text{I Introduction} \arrow[d] & \\
& \text{II Opportunity Process} \arrow[d] & \\
& \text{III Bellman Equation} \arrow[dl] \arrow[dr] & \\
\text{IV Lévy Models} & & \text{V Risk Aversion Asymptotics}
\end{tikzcd}$$

However, each part is written in a self-contained way, i.e., necessary results from other chapters are always recalled. The following paragraphs give a synthesis of each part.

Opportunity Process. The concept of *dynamic programming* plays an important role in this thesis. Its fundamental quantity is the value function, i.e., the maximal expected utility that the agent can achieve, given certain initial conditions. In a non-Markovian context, this means that we freeze some strategy π (resp. (π, c)) to be used until some point t in time. Then we consider the maximal (conditional) expected utility $J_t(\pi)$ that can be achieved by optimizing the strategy on the remaining time interval $[t, T]$. The stochastic process $t \mapsto J_t(\pi)$ is called the *value process* corresponding to π.

The scaling property of our power utility functional leads to a factorization of the value process into one part which depends on the current wealth, and a process L. It is called *opportunity process* as L_t encodes the maximal conditional expected utility which can be attained from time t, starting from one unit of endowment. The factorization itself is very classical—for instance, it can already be found in Merton's work—and an opportunity process is present (in a more or less explicit form) in almost any paper dealing with so-called isoelastic utility functions. However, there was thus far no general study of L for power utility. We shall not indicate all the related literature but merely mention that the name "opportunity process" was introduced by Černý and Kallsen [11] for an analogous object in the context of mean-variance hedging.

In Chapter II we first introduce rigorously the opportunity process and then proceed to establish the connection to the "dual problem" in the sense of convex duality. On the one hand, the dual problem has a scaling property similar to the one of the (primal) utility maximization problem and this gives rise to a dual opportunity process denoted by L^*. It turns out that L^* is simply a power of L. This allows us to relate uniform bounds for L to so-called reverse Hölder inequalities for processes belonging to the dual domain. On the other hand, the optimal supermartingale solving the dual problem is expressed via L and the optimal wealth process. In the case with consumption, it is known that this supermartingale coincides with the marginal utility of the optimal consumption rate $\hat{c}$. Therefore, we obtain a feedback formula for $\hat{c}$ in terms of L. We exploit this connection to obtain model-independent bounds for the optimal consumption and to study how it is affected by certain changes in the model.

Bellman Equation. In contrast to the optimal consumption, nothing is said in Chapter II about the *trading strategy*, which forms the second part of the optimal strategy. Its description via the opportunity process requires a more involved stochastic analysis approach, which is the content of Chapter III. We present a local representation of the optimization problem that we call *Bellman equation* in analogy to classical Markovian control problems. Its main ingredient is a random function which is defined in terms of the semimartingale characteristics of the asset prices and the opportunity process. Optimal trading strategies are characterized as maximizers for this function. We also recover, with a quite different proof, the feedback formula for the optimal consumption.

The Bellman equation is stated in two forms: first as an equation of differential semimartingale characteristics and then as a backward stochastic differential equation (BSDE). The main result is that whenever an optimal strategy exists, the opportunity process (resp. the joint characteristics with the price process) solves the Bellman equation. Our construction relies purely on dynamic programming and necessitates neither additional no-arbitrage assumptions nor duality theory. This allows us to formulate the problem under portfolio constraints that need not be convex.

We can see the Bellman equation as a description for the opportunity process. In certain models the equation can be solved directly using existence results for BSDEs with quadratic growth. However, a verification result is needed to show that a solution of the equation corresponds to the solution of our optimization problem. We provide sufficient (and also necessary) conditions for this to hold. This is also a first answer to the question whether the opportunity process is fully described by the Bellman equation. While there are no general uniqueness results for BSDEs driven by semimartingales, we show that the opportunity process can be characterized as the minimal solution of the Bellman equation.

Lévy Models. In Chapter IV we consider the special case when the asset prices follow an exponential Lévy process. In the continuous case this corresponds to a drifted geometric Brownian motion, which is the specification in the original Merton problem. A classical observation in various models is that when the asset returns are i.i.d., the optimal portfolio and consumption are given by a constant and a deterministic function, respectively, in a suitable parametrization. The aim of Chapter IV is to establish this fact for convex-constrained Lévy models under minimal assumptions.

The optimal portfolio is characterized as the maximizer of a *deterministic* function $\mathfrak{g}$ defined in terms of the Lévy triplet; and the maximum value of $\mathfrak{g}$ yields the optimal consumption. The function $\mathfrak{g}$ is closely related to the random function mentioned above. While it is clear that the finiteness of the value function is a necessary requirement to study the utility maximization problem, it is in general impossible to describe this condition directly in terms of the model primitives. In the present special case, we succeed to state a description in terms of the Lévy triplet. We also consider the q-optimal equivalent martingale measures that are linked to utility maximization by convex duality ($q \in (-\infty, 1) \setminus \{0\}$); this results in an explicit existence characterization and a formula for the density process. Finally, we study some generalizations to non-convex constraints.

The approach in Chapter IV is classical and consists in solving the Bellman equation, which reduces to an ordinary differential equation in the Lévy setting. The main difficulty is to construct the maximizer for $\mathfrak{g}$; once this is achieved, we can apply the general verification results from Chapter III. The necessary compactness is obtained from a minimal no-free-lunch condition via scaling arguments which were developed by Kardaras [44] for log-utility. In our case, these arguments require certain additional integrability properties of the asset returns. Without compromising the generality, integrability is achieved by a transformation which replaces the given assets by certain portfolios. In fact, these portfolios are special cases of the "representative portfolios" that are introduced in the appendix of Chapter III to clarify certain technical issues (which we shall not detail in this overview).

Risk Aversion Asymptotics. Thus far, we have considered the power utility function $U^{(p)}(x) = \frac{1}{p}x^p$ for a fixed parameter $p \in (-\infty, 0) \cup (0, 1)$. In Chapter V we vary p and our main interest concerns the behavior of the optimal strategies *in the limits* $p \to -\infty$ and $p \to 0$.

The relative risk aversion of $U^{(p)}$ tends to infinity for $p \to -\infty$, hence we guess by the economic interpretation of this quantity that the optimal investment portfolio tends to zero. If there is no trading, optimizing the consumption becomes a deterministic problem that is readily solved. We prove (in a general semimartingale model) that the optimal consumption, expressed as a proportion of wealth, converges pointwise to a deterministic function. This function corresponds to the consumption which would be optimal in the case where trading is not allowed. In the continuous semimartingale case, we show that the optimal trading strategy tends to zero.

Our second result pertains to the same limit $p \to -\infty$ but concerns utility from terminal wealth only. It can be seen as a first-order asymptotic: in the continuous case, we show that the optimal trading strategy scaled by $1 - p$ converges to a strategy which is optimal for exponential utility.

For the limit $p \to 0$, we note that $p = 0$ formally corresponds to the logarithmic utility function. Again, we establish the convergence of the corresponding optimal consumption in the general case, and the convergence of the trading strategy in the continuous case.

In view of the feedback formula for the optimal consumption mentioned before, we study the dependence of the (primal and dual) opportunity processes on p and their convergence. This uses control-theoretic arguments and convex analysis. To obtain the convergence of the strategies, we study the asymptotics of the Bellman equation. The continuity assumption simplifies the equation and renders the optimal portfolio relatively explicit (in terms of the Kunita-Watanabe decomposition of the opportunity process). Despite this, we are not in a standard framework for quadratic BSDEs, and therefore we give the proofs by direct arguments.

Chapter II

Opportunity Process

In this chapter, which corresponds to the article [61], we lay the foundations for the rest of the thesis. We present the basic dynamic programming for the power utility maximization problem and we introduce the opportunity process. We also give applications to the study of the optimal consumption strategy.

II.1 Introduction

We consider the utility maximization problem in a semimartingale model for a financial market, with and without intermediate consumption. While the model is general, we focus on *power utilities.* If the maximization is seen as a stochastic control problem, the power form leads to a factorization of the value process into a part which depends on the current wealth and a process L around which our analysis is built. It is called *opportunity process* as L_t encodes the maximal conditional expected utility that can be attained from time t. This name was introduced by Černý and Kallsen [11] for an analogous object in the context of mean-variance hedging. Surprisingly, there exists no general study of L for the case of power utility, which is a gap we try to fill here.

The opportunity process is a suitable tool to derive *qualitative* results about the optimal consumption strategy. We present monotonicity properties and bounds which are quite explicit despite the generality of the model.

This chapter is organized as follows. After the introduction, we specify the optimization problem in detail. Section II.3 introduces the opportunity process L via dynamic programming and examines its basic properties. Section II.4 relates L to convex duality theory and reverse Hölder inequalities, which is useful to obtain bounds for the opportunity process. Section II.5 gives applications to the study of the optimal consumption. We establish a feedback formula in terms of L and use it to study how certain changes in the model affect the optimal consumption. These applications illustrate the usefulness of the opportunity process: they are general but have very simple proofs. Two appendices supply facts about dynamic programming and duality theory.

We refer to Jacod and Shiryaev [34] for unexplained notation.

II.2 The Optimization Problem

Financial Market. We fix the time horizon $T \in (0, \infty)$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual assumptions of right-continuity and completeness, as well as $\mathcal{F}_0 = \{\emptyset, \Omega\}$ P-a.s. We consider an $\mathbb{R}^d$-valued càdlàg semimartingale R with $R_0 = 0$. The (componentwise) stochastic exponential $S = \mathcal{E}(R)$ represents the discounted price processes of d risky assets, while R stands for their returns. Our agent also has a bank account paying zero interest at his disposal.

Trading Strategies and Consumption. The agent is endowed with a deterministic initial capital $x_0 > 0$. A *trading strategy* is a predictable R-integrable $\mathbb{R}^d$-valued process π, where the ith component is interpreted as the fraction of wealth (or the portfolio proportion) invested in the ith risky asset. A *consumption strategy* is a nonnegative optional process c such that $\int_0^T c_t\,dt < \infty$ P-a.s. We want to consider two cases. Either consumption occurs only at the terminal time T (utility from "terminal wealth" only); or there is intermediate consumption plus a bulk consumption at the time horizon. To unify the notation, define the measure μ on $[0,T]$ by

$$\mu(dt) := \begin{cases} 0 & \text{in the case without intermediate consumption,} \\ dt & \text{in the case with intermediate consumption.} \end{cases}$$

We also define $\mu^\circ := \mu + \delta_{\{T\}}$, where $\delta_{\{T\}}$ is the unit Dirac measure at T. The *wealth process* $X(\pi, c)$ corresponding to a pair (π, c) is described by the linear equation

$$X_t(\pi, c) = x_0 + \int_0^t X_{s-}(\pi, c)\pi_s\,dR_s - \int_0^t c_s\,\mu(ds), \quad 0 \le t \le T \qquad (2.1)$$

and the set of *admissible* trading and consumption pairs is

$$\mathcal{A}(x_0) = \big\{(\pi, c) : X(\pi, c) > 0,\ X_-(\pi, c) > 0 \text{ and } c_T = X_T(\pi, c)\big\}.$$

The convention $c_T = X_T(\pi, c)$ means that all the remaining wealth is consumed at time T; it is merely for notational convenience. Indeed, $X(\pi, c)$ does not depend on c_T, hence any given consumption strategy c can be redefined to satisfy $c_T = X_T(\pi, c)$. We fix the initial capital x_0 and usually write $\mathcal{A}$ for $\mathcal{A}(x_0)$. A consumption strategy c is called admissible if there exists π

such that $(\pi, c) \in \mathcal{A}$; we write $c \in \mathcal{A}$ for brevity. The meaning of $\pi \in \mathcal{A}$ is analogous.

Sometimes it is convenient to parametrize the consumption strategies as fractions of wealth. Let $(\pi, c) \in \mathcal{A}$ and let $X = X(\pi, c)$ be the corresponding wealth process. Then

$$\kappa := \frac{c}{X} \tag{2.2}$$

is called the *propensity to consume* corresponding to (π, c). Note that $\kappa_T = 1$ due to our convention that $c_T = X_T$.

Remark 2.1. (i) The parametrization (π, κ) allows to express wealth processes as stochastic exponentials: by (2.1),

$$X(\pi, \kappa) = x_0 \mathcal{E}\big(\pi \bullet R - \kappa \bullet \mu\big) \tag{2.3}$$

coincides with $X(\pi, c)$ for $\kappa := c/X(\pi, c)$, where we have used that $X(\pi, c) = X(\pi, c)_-$ μ-a.e. because it is càdlàg. The symbol $\bullet$ indicates an integral, e.g., $\pi \bullet R = \int \pi_s \, dR_s$.

(ii) Relation (2.2) induces a one-to-one correspondence between the pairs $(\pi, c) \in \mathcal{A}$ and the pairs (π, κ) such that $\pi \in \mathcal{A}$ and κ is a nonnegative optional process satisfying $\int_0^T \kappa_s \, ds < \infty$ P-a.s. and $\kappa_T = 1$. Indeed, given $(\pi, c) \in \mathcal{A}$, define κ by (2.2) with $X = X(\pi, c)$. As $X, X_- > 0$ and as X is càdlàg, almost every path of X is bounded away from zero and κ has the desired integrability. Conversely, given (π, κ), define X via (2.3) and $c := \kappa X$; then $X = X(\pi, c)$. From admissibility we deduce $\pi^\top \Delta R > -1$ up to evanescence, which in turn shows $X > 0$. Now $X_- > 0$ by a standard property of stochastic exponentials [34, II.8a], so $(\pi, c) \in \mathcal{A}$.

Preferences. Let D be a càdlàg adapted strictly positive process such that $E\big[\int_0^T D_s \, \mu^\circ(ds)\big] < \infty$ and fix $p \in (-\infty, 0) \cup (0, 1)$. We define the utility random field

$$U_t(x) := D_t \tfrac{1}{p} x^p, \quad x \in [0, \infty),\ t \in [0, T],$$

where $1/0 := \infty$. To wit, this is *any p-homogeneous* utility random field such that a constant consumption yields finite expected utility. The positive number $1 - p$ is called the relative risk aversion of U. Sometimes we shall assume that there are constants $0 < k_1 \leq k_2 < \infty$ such that

$$k_1 \leq D_t \leq k_2, \quad t \in [0, T]. \tag{2.4}$$

The *expected utility* corresponding to a consumption strategy $c \in \mathcal{A}$ is given by $E[\int_0^T U_t(c_t) \, \mu^\circ(dt)]$. We recall that this is either $E[U_T(c_T)]$ or $E[\int_0^T U_t(c_t) \, dt + U_T(c_T)]$. In the case without intermediate consumption, U_t is irrelevant for $t < T$. We remark that Zariphopoulou [74] and Tehranchi [73] have used utility functions modified by a multiplicative random variable, in the case where utility is obtained from terminal wealth.

Remark 2.2. The process D can be used for discounting utility and consumption, or to determine the weight of intermediate consumption compared to terminal wealth. Our utility functional can also be related to the usual power utility function $\frac{1}{p}x^p$ in the following ways. If we write

$$E\Big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\Big] = E\Big[\int_0^T \tfrac{1}{p}c_t^p\,dK_t\Big]$$

for $dK_t := D_t\,\mu^\circ(dt)$, we have the usual power utility, but with a *stochastic clock* K (cf. Goll and Kallsen [25]).

To model *taxation* of the consumption, let $\varrho > -1$ be the tax rate and $D := (1+\varrho)^{-p}$. If c represents the cashflow out of the portfolio, $c/(1+\varrho)$ is the effectively obtained amount of the consumption good, yielding the instantaneous utility $\frac{1}{p}(c_t/(1+\varrho_t))^p = U_t(c_t)$. Similarly, D_T can model a multiplicative *bonus payment*.

For yet another alternative, assume either that there is no intermediate consumption or that D is a martingale, and that $E[D_T]=1$. Then

$$E\Big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\Big] = E^{\widetilde{P}}\Big[\int_0^T \tfrac{1}{p}c_t^p\,\mu^\circ(dt)\Big]$$

with the equivalent probability $\widetilde{P}$ defined by $d\widetilde{P} = D_T\,dP$. This is the standard power utility problem for an agent with *subjective beliefs*, i.e., who uses $\widetilde{P}$ instead of the objective probability P.

Of course, these applications can be combined in a multiplicative way.

We assume that the value of the utility maximization problem is finite:

$$u(x_0) := \sup_{c\in\mathcal{A}(x_0)} E\Big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\Big] < \infty. \tag{2.5}$$

This is a **standing assumption** for the entire chapter. It is void if $p<0$ because then $U<0$. If $p>0$, it needs to be checked on a case-by-case basis (see also Remark 4.7). A strategy $(\hat{\pi},\hat{c})\in\mathcal{A}(x_0)$ is *optimal* if $E\big[\int_0^T U_t(c_t)\,\mu^\circ(dt)\big] = u(x_0)$. Of course, a no-arbitrage property is required to guarantee its existence. Let $\mathscr{M}^S$ be the set of equivalent σ-martingale measures for S. If

$$\mathscr{M}^S \neq \emptyset, \tag{2.6}$$

arbitrage is excluded in the sense of the NFLVR condition (see Delbaen and Schachermayer [17]). We can cite the following existence result of Karatzas and Žitković [43]; it was previously obtained by Kramkov and Schachermayer [49] for the case without intermediate consumption.

Proposition 2.3. *Under* (2.4) *and* (2.6)*, there exists an optimal strategy* $(\hat{\pi},\hat{c})\in\mathcal{A}$*. The corresponding wealth process* $\widehat{X} = X(\hat{\pi},\hat{c})$ *is unique. The consumption strategy* $\hat{c}$ *can be chosen to be càdlàg and is unique* $P\otimes\mu^\circ$*-a.e.*

In the sequel, $\hat{c}$ denotes a càdlàg version. We note that under (2.6), the requirement $X(\pi,c)_- > 0$ in the definition of $\mathcal{A}$ is automatically satisfied as soon as $X(\pi,c) > 0$, because $X(\pi,c)$ is then a positive supermartingale under an equivalent measure.

Remark 2.4. In Proposition 2.3, the assumption on D can be weakened by exploiting that (2.6) is invariant under equivalent changes of measure. Suppose that $D = D'D''$, where D' meets (2.4) and D'' is a martingale with unit expectation. As in Remark 2.2, we consider the problem under the probability $d\widetilde{P} = D''_T\,dP$, then Proposition 2.3 applies under $\widetilde{P}$ with D' instead of D, and we obtain the existence of a solution also under P.

II.3 The Opportunity Process

This section introduces the main object under discussion. We do not yet impose the existence of an optimal strategy, but recall the standing assumption (2.5). To apply dynamic programming, we introduce for each $(\pi,c) \in \mathcal{A}$ and $t \in [0,T]$ the set

$$\mathcal{A}(\pi,c,t) = \big\{(\tilde{\pi},\tilde{c}) \in \mathcal{A} :\, (\tilde{\pi},\tilde{c}) = (\pi,c) \text{ on } [0,t]\big\}. \tag{3.1}$$

These are the controls available on $(t,T]$ after having used (π,c) until t. The notation $\tilde{c} \in \mathcal{A}(\pi,c,t)$ means that there exists $\tilde{\pi}$ such that $(\tilde{\pi},\tilde{c}) \in \mathcal{A}(\pi,c,t)$. Given $(\pi,c) \in \mathcal{A}$, we consider the *value process*

$$J_t(\pi,c) := \operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]. \tag{3.2}$$

We choose the càdlàg version of this process (see Proposition 6.2 in the Appendix). The p-homogeneity of the utility functional leads to the following factorization of J.

Proposition 3.1. *There exists a unique càdlàg semimartingale L, called* opportunity process, *such that*

$$L_t\,\tfrac{1}{p}\big(X_t(\pi,c)\big)^p = J_t(\pi,c) = \operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \tag{3.3}$$

for any admissible strategy $(\pi,c) \in \mathcal{A}$. In particular, $L_T = D_T$.

Proof. Let $(\pi,c), (\check{\pi},\check{c}) \in \mathcal{A}$ and $X := X(\pi,c)$, $\check{X} := X(\check{\pi},\check{c})$. We claim that

$$\begin{aligned}\frac{1}{\check{X}_t^p} &\operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\check{\pi},\check{c},t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \\ &= \frac{1}{X_t^p}\operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big].\end{aligned} \tag{3.4}$$

Indeed, using the lattice property given in Fact 6.1, we can find a sequence (c^n) in $\mathcal{A}(\check{\pi}, \check{c}, t)$ such that, with a monotone increasing limit,

$$\begin{aligned}\frac{X_t^p}{\check{X}_t^p} \operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\check{\pi},\check{c},t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] &= \frac{X_t^p}{\check{X}_t^p} \lim_n E\Big[\int_t^T U_s(c_s^n)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]\\ = \lim_n E\Big[\int_t^T U_s\big(\tfrac{X_t}{\check{X}_t}c_s^n\big)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] &\leq \operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big],\end{aligned}$$

where we have used Fact 6.3 in the last step. The claim follows by symmetry. Thus, if we define $L_t := J_t(\pi,c)/\big[\frac{1}{p}\big(X_t(\pi,c)\big)^p\big]$, L does not depend on the choice of $(\pi,c)\in\mathcal{A}$ and inherits the properties of $J(\pi,c)$ and $X(\pi,c)>0$. $\square$

The opportunity process describes (p times) the maximal amount of conditional expected utility that can be accumulated on $[t,T]$ from one unit of wealth. Note that the value function (2.5) can be expressed as $u(x) = L_0\frac{1}{p}x^p$.

In a Markovian setting, the factorization of the value function (which then replaces the value process) is very classical; for instance, it can already be found in Merton [56]. Mania and Tevzadze [54] study power utility from terminal wealth in a continuous semimartingale model; that paper contains some of the basic notions used here as well.

Remark 3.2. Let D be a martingale with $D_0 = 1$ and $\widetilde{P}$ as in Remark 2.2. Bayes' rule and (3.3) show that $\widetilde{L} := L/D$ can be understood as "opportunity process under $\widetilde{P}$" for the standard power utility function.

Remark 3.3. We can now formalize the fact that the optimal strategies (in a suitable parametrization) do not depend on the current level of wealth, a special feature implied by the choice of power utility. If $(\hat{\pi},\hat{c})\in\mathcal{A}$ is optimal, $\widehat{X} = X(\hat{\pi},\hat{c})$, and $\hat{\kappa} = \hat{c}/\widehat{X}$ is the optimal propensity to consume, then $(\hat{\pi},\hat{\kappa})$ defines a conditionally optimal strategy for the problem

$$\operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big];\quad \text{for } any\ (\pi,c)\in\mathcal{A},\ t\in[0,T].$$

To see this, fix $(\pi,c)\in\mathcal{A}$ and $t\in[0,T]$. Define the pair $(\bar{\pi},\bar{c})$ by $\bar{\pi} = \pi 1_{[0,t]} + \hat{\pi}1_{(t,T]}$ and $\bar{c} = c1_{[0,t]} + \frac{X_t(\pi,c)}{\widehat{X}_t}\hat{c}1_{(t,T]}$ and let $\bar{X} := X(\bar{\pi},\bar{c})$. Note that $(\hat{\pi},\hat{c})$ is conditionally optimal in $\mathcal{A}(\hat{\pi},\hat{c},t)$, as otherwise Fact 6.1 yields a contradiction to the global optimality of $(\hat{\pi},\hat{c})$. Now (3.4) with $(\check{\pi},\check{c}) := (\hat{\pi},\hat{c})$ shows that $(\bar{\pi},\bar{c})$ is conditionally optimal in $\mathcal{A}(\pi,c,t)$. The result follows as $\bar{c}/\bar{X} = \hat{c}/\widehat{X} = \hat{\kappa}$ on $(t,T]$ by Fact 6.3.

The martingale optimality principle of dynamic programming takes the following form in our setting.

Proposition 3.4. *Let $(\pi, c) \in \mathcal{A}$ be an admissible strategy and assume that $E[\int_0^T U_s(c_s)\,\mu^\circ(ds)] > -\infty$. Then the process*

$$L_t \tfrac{1}{p}\big(X_t(\pi,c)\big)^p + \int_0^t U_s(c_s)\,\mu(ds), \quad t \in [0,T]$$

is a supermartingale; it is a martingale if and only if (π, c) is optimal.

Proof. Combine Proposition 3.1 and Proposition 6.2. □

The following lemma collects some elementary properties of L. The bounds are obtained by comparison with no-trade strategies, hence they are independent of the price process. If D is deterministic or if there are constants $k_1, k_2 > 0$ as in (2.4), we obtain bounds which are model-independent; they depend only on the utility function and the time to maturity.

Lemma 3.5. *The opportunity process L is a special semimartingale.*

(i) If $p \in (0,1)$, L is a supermartingale satisfying

$$L_t \geq \big(\mu^\circ[t,T]\big)^{-p} E\Big[\int_t^T D_s\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big], \quad 0 \leq t \leq T \tag{3.5}$$

and $L, L_- > 0$. In particular, $L \geq k_1$ if $D \geq k_1$.

(ii) If $p < 0$, L satisfies

$$0 \leq L_t \leq \big(\mu^\circ[t,T]\big)^{-p} E\Big[\int_t^T D_s\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big], \quad 0 \leq t \leq T \tag{3.6}$$

and in particular $L_t \leq k_2\big(\mu^\circ[t,T]\big)^{1-p}$ if $D \leq k_2$. In the case without intermediate consumption, L is a submartingale. Moreover, in both cases, $L, L_- > 0$ if there exists an optimal strategy $(\hat{\pi}, \hat{c})$.

Proof. Consider the cases where either $p > 0$, or $p < 0$ *and* there is no intermediate consumption. Then $\pi \equiv 0$, $c \equiv x_0 1_{\{T\}}$ is an admissible strategy and Proposition 3.4 shows that $L_t \frac{1}{p} x_0^p + \int_0^t U_s(0)\,\mu(ds) = L_t \frac{1}{p} x_0^p$ is a supermartingale, proving the super/submartingale properties in (i) and (ii).

Let p be arbitrary and assume there is no intermediate consumption. Applying (3.3) with $\pi \equiv 0$ and $c \equiv x_0 1_{\{T\}}$, we get $L_t \frac{1}{p} x_0^p \geq E[U_T(c_T)|\mathcal{F}_t] = E[D_T|\mathcal{F}_t]\frac{1}{p}x_0^p$. Hence $L_t \geq E[D_T|\mathcal{F}_t]$ if $p > 0$ and $L_t \leq E[D_T|\mathcal{F}_t]$ if $p < 0$, which corresponds to (3.5) and (3.6) for this case.

If there is intermediate consumption (and p is arbitrary), we consume at a constant rate after the fixed time t. That is, we use (3.3) with $\pi \equiv 0$ and $c = x_0(T-t+1)^{-1} 1_{[t,T]}$ to obtain $L_t \frac{1}{p} x_0^p \geq E\big[\int_t^T U_s(c_s)\,\mu^\circ(ds)\big|\mathcal{F}_t\big] = \frac{1}{p} x_0^p (1+T-t)^{-p} E\big[\int_t^T D_s\,\mu^\circ(ds)\big|\mathcal{F}_t\big]$. This ends the proof of (3.5) and (3.6).

In the case $p < 0$, (3.6) shows that L is dominated by a martingale, hence L is of class (D) and in particular a special semimartingale.

It remains to prove the positivity. If $p>0$, (3.5) shows $L>0$ and then $L_->0$ follows by the minimum principle for positive supermartingales. For $p<0$, let $\widehat{X}=X(\hat{\pi},\hat{c})$ be the optimal wealth process. Clearly $L>0$ follows from (3.3) with $(\hat{\pi},\hat{c})$. From Proposition 3.4 we have that $\frac{1}{p}\widehat{X}^pL+\int U_s(\hat{c}_s)\,\mu(ds)$ is a negative martingale, hence $\widehat{X}^pL$ is a positive supermartingale. Therefore $P[\inf_{0\leq t\leq T}\widehat{X}^p_tL_t>0]=1$ and it remains to note that the paths of $\widehat{X}^p$ are P-a.s. bounded because $\widehat{X},\widehat{X}_->0$. □

The following concerns the submartingale property in Lemma 3.5(ii).

Example 3.6. Consider the case *with* intermediate consumption and assume that $D\equiv 1$ and $S\equiv 1$. Then an optimal strategy is given by $(\hat{\pi},\hat{c})\equiv(0,x_0/(1+T))$ and $L_t=(1+T-t)^{1-p}$ is a decreasing function. In particular, L is not a submartingale.

Remark 3.7. We can also consider the utility maximization problem under *constraints* in the following sense. Suppose that for each $(\omega,t)\in\Omega\times[0,T]$ we are given a set $\mathscr{C}_t(\omega)\subseteq\mathbb{R}^d$. We assume that each of these sets contains the origin. A strategy $(\pi,c)\in\mathcal{A}$ is called *$\mathscr{C}$-admissible* if $\pi_t(\omega)\in\mathscr{C}_t(\omega)$ for all (ω,t), and the set of all these strategies is denoted by $\mathcal{A}^{\mathscr{C}}$. The example $(\pi,c)\equiv(0,x_0/\mu^\circ[0,T])$ shows that $\mathcal{A}^{\mathscr{C}}\neq\emptyset$.

We do not impose assumptions on the set-valued mapping $\mathscr{C}$ at this stage. For dynamic programming, the relevant point is that the constraints are specified as a pointwise condition in (ω,t), rather than as a set of processes π. We note that all arguments in this section remain valid if $\mathcal{A}$ is replaced by $\mathcal{A}^{\mathscr{C}}$ throughout. This generalization is not true for the subsequent section, and existence of an optimal strategy is not guaranteed for general $\mathscr{C}$.

II.4 Relation to the Dual Problem

We discuss how the problem dual to utility maximization relates to the opportunity process L. **We assume** (2.4) **and** (2.6) **in the entire Section II.4**, hence Proposition 2.3 applies. The dual problem will be defined on a domain $\mathscr{Y}$ introduced below. Since its definition is slightly cumbersome, we point out that to follow the results in the body of this chapter, only two facts about $\mathscr{Y}$ are needed. First, the density process of each martingale measure $Q\in\mathscr{M}^S$, scaled by a certain constant y_0, is contained in $\mathscr{Y}$. Second, each element of $\mathscr{Y}$ is a positive supermartingale.

Following [43], the *dual problem* is

$$\inf_{Y\in\mathscr{Y}(y_0)}E\Big[\int_0^T U_t^*(Y_t)\,\mu^\circ(dt)\Big],\tag{4.1}$$

where $y_0:=u'(x_0)=L_0x_0^{p-1}$ and U_t^* is the convex conjugate of $x\mapsto U_t(x)$,

$$U_t^*(y):=\sup_{x>0}\big\{U_t(x)-xy\big\}=-\tfrac{1}{q}y^qD_t^\beta.\tag{4.2}$$

We have denoted by

$$\beta := \frac{1}{1-p} > 0, \quad q := \frac{p}{p-1} \in (-\infty, 0) \cup (0,1) \tag{4.3}$$

the relative risk tolerance and the exponent conjugate to p, respectively. These constants will be used very often in the sequel and it is useful to note $\operatorname{sign}(p) = -\operatorname{sign}(q)$. It remains to define the domain $\mathscr{Y} = \mathscr{Y}(y_0)$. Let

$$\mathscr{X} = \{H \bullet S : H \in L(S),\ H \bullet S \text{ is bounded below}\}$$

be the set of gains processes from trading. The set of "supermartingale densities" is defined by

$$\mathscr{Y}^* = \{Y \geq 0 \text{ càdlàg} : Y_0 \leq y_0, \quad YG \text{ supermartingale for all } G \in \mathscr{X}\};$$

its subset corresponding to probability measures equivalent to P on $\mathcal{F}_T$ is

$$\mathscr{Y}^{\mathscr{M}} = \{Y \in \mathscr{Y}^* : Y > 0 \text{ is a martingale and } Y_0 = y_0\}.$$

We place ourselves in the setting of [43] by considering the same dual domain $\mathscr{Y}^{\mathscr{D}} \subseteq \mathscr{Y}^*$. It consists of density processes of (the regular parts of) the finitely additive measures in the $\sigma((L^\infty)^*, L^\infty)$-closure of $\{Y_T : Y \in \mathscr{Y}^{\mathscr{M}}\} \subset L^1 \subseteq (L^\infty)^*$. More precisely, we multiply each density with the constant y_0. We refer to [43] for details as the precise construction of $\mathscr{Y}^{\mathscr{D}}$ is not important here, it is relevant for us only that $\mathscr{Y}^{\mathscr{M}} \subseteq \mathscr{Y}^{\mathscr{D}} \subseteq \mathscr{Y}^*$. In particular, $y_0 \mathscr{M}^S \subseteq \mathscr{Y}^{\mathscr{D}}$ if we identify measures and their density processes. For notational reasons, we make the dual domain slightly smaller and let

$$\mathscr{Y} := \{Y \in \mathscr{Y}^{\mathscr{D}} : Y > 0\}.$$

By [43, Theorem 3.10] there exists a unique $\widehat{Y} = \widehat{Y}(y_0) \in \mathscr{Y}$ such that the infimum in (4.1) is attained, and it is related to the optimal consumption $\hat{c}$ via the marginal utility by

$$\widehat{Y}_t = \partial_x U_t(x)|_{x=\hat{c}_t} = D_t \hat{c}_t^{p-1} \tag{4.4}$$

on the support of μ°. In the case without intermediate consumption, an existence result was previously obtained in [49].

Remark 4.1. All the results stated below remain true if we replace $\mathscr{Y}$ by $\{Y \in \mathscr{Y}^* : Y > 0\}$; i.e., it is not important for our purposes whether we use the dual domain of [43] or the one of [49]. This is easily verified using the fact that $\mathscr{Y}^{\mathscr{D}}$ contains all maximal elements of $\mathscr{Y}^*$ (see [43, Theorem 2.10]). Here $Y \in \mathscr{Y}^*$ is called maximal if $Y = Y'B$, for some $Y' \in \mathscr{Y}^*$ and some càdlàg nonincreasing process $B \in [0,1]$, implies $B \equiv 1$.

Proposition 4.2. *Let* $(\hat{c}, \hat{\pi}) \in \mathcal{A}$ *be an optimal strategy and* $\widehat{X} = X(\hat{\pi}, \hat{c})$. *The solution to the dual problem is given by*

$$\widehat{Y} = L\widehat{X}^{p-1}.$$

Proof. As $L_T = D_T$ and $\hat{c}_T = \widehat{X}_T$, (4.4) already yields $\widehat{Y}_T = L_T\widehat{X}_T^{p-1}$. Moreover, by Lemma 7.1 in the Appendix, $\widehat{Y}$ has the property that

$$Z_t := \widehat{Y}_t\widehat{X}_t + \int_0^t \widehat{Y}_s\hat{c}_s\,\mu(ds) = \widehat{Y}_t\widehat{X}_t + p\int_0^t U_s(\hat{c}_s)\,\mu(ds)$$

is a martingale. By Proposition 3.4, $\widetilde{Z}_t := L_t\widehat{X}_t^p + p\int_0^t U_s(\hat{c}_s)\,\mu(ds)$ is also a martingale. The terminal values of these martingales coincide, hence $\widetilde{Z} = Z$. We deduce $\widehat{Y} = L\widehat{X}^{p-1}$ as $\widehat{X} > 0$. □

The formula $\widehat{Y} = L\widehat{X}^{p-1}$ could be used to *define* the opportunity process L. This is the approach taken in Muhle-Karbe [58] (see also Kallsen and Muhle-Karbe [40]), where utility from terminal wealth is considered and the opportunity process is used as a tool to verify the optimality of an explicit candidate solution. Our approach via the value process has the advantage that it immediately yields the properties in Lemma 3.5 and monotonicity results (see Section II.5).

II.4.1 The Dual Opportunity Process

We now introduce the analogue of L for the dual problem. Define for $Y \in \mathscr{Y}$ and $t \in [0,T]$ the set

$$\mathscr{Y}(Y,t) := \big\{\widetilde{Y} \in \mathscr{Y} :\, \widetilde{Y} = Y \text{ on } [0,t]\big\}.$$

We recall the constants (4.3) and the standing assumptions (2.4) and (2.6).

Proposition 4.3. *There exists a unique càdlàg process* L^*, *called* dual opportunity process, *such that for all* $Y \in \mathscr{Y}$ *and* $t \in [0,T]$,

$$-\tfrac{1}{q}Y_t^q L_t^* = \operatorname*{ess\,inf}_{\widetilde{Y}\in\mathscr{Y}(Y,t)} E\Big[\int_t^T U_s^*(\widetilde{Y}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big].$$

An alternative description is

$$L_t^* = \begin{cases} \operatorname{ess\,sup}_{Y\in\mathscr{Y}} E\Big[\int_t^T D_s^\beta (Y_s/Y_t)^q\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] & \text{if } q \in (0,1), \\ \operatorname{ess\,inf}_{Y\in\mathscr{Y}} E\Big[\int_t^T D_s^\beta (Y_s/Y_t)^q\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] & \text{if } q < 0 \end{cases}$$

and the extrema are attained at $Y = \widehat{Y}$.

Proof. The fork convexity of $\mathscr{Y}$ [43, Theorem 2.10] shows that if $Y, \check{Y} \in \mathscr{Y}$ and $\widetilde{Y} \in \mathscr{Y}(\check{Y}, t)$, then $Y 1_{[0,t)} + (Y_t/\check{Y}_t)\widetilde{Y} 1_{[t,T]}$ is in $\mathscr{Y}(Y,t)$. It also implies that if $A \in \mathcal{F}_t$ and $Y^1, Y^2 \in \mathscr{Y}(Y,t)$, then $Y^1 1_A + Y^2 1_{A^c} \in \mathscr{Y}(Y,t)$. The proof of the first claim is now analogous to that of Proposition 3.1. The second part follows by using that L^* does not depend on Y. □

The process L^* is related to L by a simple power transformation.

Proposition 4.4. *Let* $\beta = \frac{1}{1-p}$. *Then* $L^* = L^\beta$.

Proof. The martingale property of $Z_t := \widehat{X}_t\widehat{Y}_t + \int_0^t \hat{c}_s \widehat{Y}_s\,\mu(ds)$ from Lemma 7.1 implies that $\widehat{X}_t\widehat{Y}_t = E[Z_T|\mathcal{F}_t] - \int_0^t \hat{c}_s\widehat{Y}_s\,\mu(ds) = E\big[\int_t^T \hat{c}_s\widehat{Y}_s\,\mu^\circ(ds)\big|\mathcal{F}_t\big] = E\big[\int_t^T D_s^\beta \widehat{Y}_s^q\,\mu^\circ(ds)\big|\mathcal{F}_t\big]$, where the last equality is obtained by expressing $\hat{c}$ via (4.4). The right hand side equals $\widehat{Y}_t^q L_t^*$ by Proposition 4.3; so we have shown $\widehat{X}\widehat{Y} = \widehat{Y}^q L^*$. On the other hand, $(L\widehat{X}^{p-1})^q = \widehat{Y}^q$ by Proposition 4.2 and this can be written as $\widehat{X}\widehat{Y} = \widehat{Y}^q L^\beta$. We deduce $L^* = L^\beta$ as $\widehat{Y} > 0$. □

II.4.2 Reverse Hölder Inequality and Boundedness of L

Let $q = \frac{p}{p-1}$ be the exponent conjugate to p. Given a general positive process Y, we consider the following inequality of reverse Hölder type:

$$\begin{cases} \displaystyle\int_\tau^T E\big[(Y_s/Y_\tau)^q\big|\mathcal{F}_\tau\big]\,\mu^\circ(ds) \le C_q & \text{if } q < 0, \\ \displaystyle\int_\tau^T E\big[(Y_s/Y_\tau)^q\big|\mathcal{F}_\tau\big]\,\mu^\circ(ds) \ge C_q & \text{if } q \in (0,1), \end{cases} \tag{$\mathrm{R}_q(P)$}$$

for all stopping times $0 \le \tau \le T$ and some constant $C_q > 0$ independent of τ. It is useful to recall that $q < 0$ corresponds to $p \in (0,1)$ and vice versa.

Without intermediate consumption, $\mathrm{R}_q(P)$ reduces to $E[(Y_T/Y_\tau)^q|\mathcal{F}_\tau] \le C_q$ (resp. "$\ge$"). Inequalities of this type are well known. See, e.g., Doléans-Dade and Meyer [19] for an introduction or Delbaen et al. [16] and the references therein for some connections to finance. In most applications, the considered exponent q is greater than one; $\mathrm{R}_q(P)$ then takes the form as for $q < 0$. We recall once more the standing assumptions (2.4) and (2.6).

Proposition 4.5. *The following are equivalent:*

(i) The process L is uniformly bounded away from zero and infinity.

(ii) Inequality $\mathrm{R}_q(P)$ holds for the dual minimizer $\widehat{Y} \in \mathscr{Y}$.

(iii) Inequality $\mathrm{R}_q(P)$ holds for some *$Y \in \mathscr{Y}$.*

Proof. Under the standing assumption (2.4), a one-sided bound for L always holds by Lemma 3.5, namely $L \ge k_1$ if $p \in (0,1)$ and $L \le \mathit{const.}$ if $p < 0$.

(i) is equivalent to (ii): We use (2.4) and then Propositions 4.3 and 4.4 to obtain that $\int_\tau^T E\big[(\widehat{Y}_s/\widehat{Y}_\tau)^q\big|\mathcal{F}_\tau\big]\,\mu^\circ(ds) = E\big[\int_\tau^T (\widehat{Y}_s/\widehat{Y}_\tau)^q\,\mu^\circ(ds)\big|\mathcal{F}_\tau\big] \le$

$k_1^{-\beta} E\big[\int_\tau^T D_s^\beta (\widehat{Y}_s/\widehat{Y}_\tau)^q \,\mu^\circ(ds)\big|\mathcal{F}_\tau\big] = k_1^{-\beta} L_\tau^* = k_1^{-\beta} L_\tau^\beta$. Thus when $p \in (0,1)$ and hence $q < 0$, $\mathrm{R}_q(P)$ for $\widehat{Y}$ is equivalent to an upper bound for L. For $p < 0$, we replace k_1 by k_2.

(iii) implies (i): Assume $p \in (0,1)$. Using Propositions 4.4 and 4.3 and (4.2), $-\frac{1}{q} Y_t^q L_t^\beta \leq E\big[\int_t^T U_s^*(Y_s)\,\mu^\circ(ds)\big|\mathcal{F}_t\big] \leq -\frac{1}{q} k_2^\beta \int_t^T E[Y_s^q|\mathcal{F}_t]\,\mu^\circ(ds)$. Hence $L \leq k_2 C_q^{-\beta}$. If $p < 0$, we obtain $L \geq k_1 C_q^{-\beta}$ in the same way. □

If the equivalent conditions of Proposition 4.5 are satisfied, we say that "$\mathrm{R}_q(P)$ holds" for the given financial market model. Although quite frequent in the literature, this condition is rather restrictive in the sense that it often fails in explicit models that have stochastic dynamics. For instance, in the affine models of [40], L is an exponentially affine function of a typically unbounded factor process, in which case Proposition 4.5 implies that $\mathrm{R}_q(P)$ fails. Similarly, L is an exponentially quadratic function of an Ornstein-Uhlenbeck process in the model of Kim and Omberg [47]. On the other hand, exponential Lévy models have constant dynamics and here L turns out to be simply a smooth deterministic function.

In a given model, it may be hard to check whether $\mathrm{R}_q(P)$ holds. Recalling $y_0\mathscr{M}^S \subseteq \mathscr{Y}$, an obvious approach in view of Proposition 4.5(iii) is to choose for Y/y_0 the density process of some specific martingale measure. We illustrate this with an essentially classical example.

Example 4.6. Assume that R is a special semimartingale with decomposition

$$R = \alpha \bullet \langle R^c \rangle + M^R, \tag{4.5}$$

where R^c denotes the continuous local martingale part of R, $\alpha \in L^2_{loc}(R^c)$, and M^R is the local martingale part of R. Suppose that the process

$$\chi_t := \int_0^t \alpha_s^\top \, d\langle R^c \rangle_s \, \alpha_s\,, \quad t \in [0,T]$$

is *uniformly bounded*. Then $Z := \mathcal{E}(-\alpha \bullet R^c)$ is a martingale by Novikov's condition and the measure $Q \approx P$ with density $dQ/dP = Z_T$ is a local martingale measure for S as $Z\mathcal{E}(R) = \mathcal{E}(-\alpha \bullet R^c + M^R)$ by Yor's formula; hence $y_0 Z \in \mathscr{Y}$. Fix q. Using $Z^q = \mathcal{E}(-q\alpha \bullet R^c)\exp\big(\frac{1}{2}q(q-1)\chi\big)$, and that $\mathcal{E}(-q\alpha \bullet R^c)$ is a martingale by Novikov's condition, one readily checks that Z satisfies inequality $\mathrm{R}_q(P)$.

If R is continuous, (4.5) is the structure condition of Schweizer [71] and under (2.6) R is necessarily of this form. Then χ is called mean-variance tradeoff process and Q is the "minimal" martingale measure. In Itô process models, χ takes the form $\chi_t = \int_0^t \theta_s^\top \theta_s \, ds$, where θ is the market price of risk process. Thus χ will be bounded whenever θ is.

Remark 4.7. The example also gives a sufficient condition for (2.5). This is of interest only for $p \in (0,1)$ and we remark that for the case of Itô process models with bounded θ, the condition corresponds to Karatzas and Shreve [42, Remark 6.3.9].

Indeed, if there exists $Y \in \mathscr{Y}$ satisfying $\mathrm{R}_q(P)$, then with (4.2) and (2.4) it follows that the the value of the dual problem (4.1) is finite, and this suffices for (2.5), as in Kramkov and Schachermayer [50].

The rest of the section studies the dependence of $\mathrm{R}_q(P)$ on q.

Remark 4.8. Assume that Y satisfies $\mathrm{R}_q(P)$ with a constant C_q. If q_1 is such that $q < q_1 < 0$ or $0 < q < q_1 < 1$, then $\mathrm{R}_{q_1}(P)$ is satisfied with

$$C_{q_1} = \big(\mu^\circ[0,T]\big)^{1-q_1/q}(C_q)^{q_1/q}.$$

Similarly, if $q < 0 < q_1 < 1$, we can take $C_{q_1} = (C_q)^{q_1/q}$. This follows from Jensen's inequality.

There is also a partial converse.

Lemma 4.9. *Let $0 < q < q_1 < 1$ and let $Y > 0$ be a supermartingale. If Y satisfies $\mathrm{R}_{q_1}(P)$, it also satisfies $\mathrm{R}_q(P)$.*

In particular, the following dichotomy holds: Y satisfies either all or none of the inequalities $\big\{\mathrm{R}_q(P),\ q \in (0,1)\big\}$.

Proof. From Lemma 4.10 stated below we have $\int_t^T E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big]\,\mu^\circ(ds) \geq \int_t^T \big(E\big[(Y_s/Y_t)^{q_1}\big|\mathcal{F}_t\big]\big)^{\frac{1-q}{1-q_1}}\,\mu^\circ(ds)$. Noting that $\frac{1-q}{1-q_1} > 1$, we apply Jensen's inequality to the right-hand side and then use $\mathrm{R}_{q_1}(P)$ to deduce the claim with $C_q := \big(\mu^\circ[t,T]\big)^{\frac{q-q_1}{1-q_1}}\,(C_{q_1})^{\frac{1-q}{1-q_1}}$. The dichotomy follows by the previous remark. □

For future reference, we state separately the main step of the above proof.

Lemma 4.10. *Let $Y > 0$ be a supermartingale. For fixed $0 \leq t \leq s \leq T$,*

$$\phi:\ (0,1) \to \mathbb{R}_+, \quad q \mapsto \phi(q) := \Big(E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big]\Big)^{\frac{1}{1-q}}$$

is a monotone decreasing function P-a.s. If in addition Y is a martingale, then $\lim_{q\to 1-}\phi(q) = \exp\big(-E\big[(Y_s/Y_t)\log(Y_s/Y_t)\big|\mathcal{F}_t\big]\big)$ P-a.s., where the conditional expectation has values in $\mathbb{R}\cup\{+\infty\}$.

Proof. Suppose first that Y is a martingale; by scaling we may assume $E[Y_\cdot] = 1$. We define a probability $Q \approx P$ on $\mathcal{F}_s$ by $dQ/dP := Y_s$. With $r := (1-q) \in (0,1)$ and Bayes' formula,

$$\phi(q) = \Big(Y_t^{1-q}E^Q\big[Y_s^{q-1}\big|\mathcal{F}_t\big]\Big)^{\frac{1}{1-q}} = Y_t\Big(E^Q\big[(1/Y_s)^r\big|\mathcal{F}_t\big]\Big)^{\frac{1}{r}}.$$

This is increasing in r by Jensen's inequality, hence decreasing in q.

Now let Y be a supermartingale. We can decompose it as $Y_u = B_u M_u$, $u \in [0,s]$, where M is a martingale and $B_s = 1$. That is, $M_t = E[Y_s|\mathcal{F}_t]$ and $B_t = Y_t/E[Y_s|\mathcal{F}_t] \geq 1$, by the supermartingale property. Hence $B_t^{q/(q-1)}$ is decreasing in $q \in (0,1)$. Together with the first part, it follows that $\phi(q) = B_t^{q/(q-1)}\big(E\big[(M_s/M_t)^q\big|\mathcal{F}_t\big]\big)^{\frac{1}{1-q}}$ is decreasing.

Assume again that Y is a martingale. The limit $\lim_{q\to 1-}\log\big(\phi(q)\big)$ can be calculated as

$$\lim_{q\to 1-}\frac{\log\big(E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big]\big)}{1-q} = \lim_{q\to 1-} -\frac{E\big[(Y_s/Y_t)^q\log(Y_s/Y_t)\big|\mathcal{F}_t\big]}{E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big]} \quad P\text{-a.s.}$$

using l'Hôpital's rule and $E[(Y_s/Y_t)|\mathcal{F}_t] = 1$. The result follows using monotone and bounded convergence in the numerator and dominated convergence in the denominator. □

Remark 4.11. The limiting case $q = 1$ corresponds to the entropic inequality $\mathrm{R}_{L\log L}(P)$ which reads $\int_\tau^T E\big[(Y_s/Y_\tau)\log(Y_s/Y_\tau)\big|\mathcal{F}_\tau\big]\,\mu^\circ(ds) \leq C_1$. Lemma 4.10 shows that for a martingale $Y > 0$, $\mathrm{R}_{q_1}(P)$ with $q_1 \in (0,1)$ is weaker than $\mathrm{R}_{L\log L}(P)$, which, in turn, is obviously weaker than $\mathrm{R}_{q_0}(P)$ with $q_0 > 1$.

A much deeper argument [19, Proposition 5] shows that if Y is a martingale satisfying the "condition (S)" that $k^{-1}Y_- \leq Y \leq kY_-$ for some $k > 0$, then Y satisfies $\mathrm{R}_{q_0}(P)$ for some $q_0 > 1$ if and only if it satisfies $\mathrm{R}_q(P)$ for some $q < 0$, and then by Remark 4.8 also $\mathrm{R}_{q_1}(P)$ for all $q_1 \in (0,1)$.

Coming back to the utility maximization problem, we obtain the following dichotomy from Lemma 4.9 and the implication (iii) ⇒ (ii) in Proposition 4.5.

Corollary 4.12. *For the given market model, $\mathrm{R}_q(P)$ holds either for all or no values of $q \in (0,1)$.*

II.5 Applications

In this section we consider only the case *with* intermediate consumption. **We assume** (2.4) **and** (2.6). However, we remark that all results except for Proposition 5.4 and Remark 5.5 hold true as soon as there exists an optimal strategy $(\hat{\pi},\hat{c}) \in \mathcal{A}$.

We first show that given the opportunity process, the optimal propensity to consume $\hat{\kappa}$ can be expressed in feedback form, and therefore any result about L leads to a statement about $\hat{\kappa}$. This extends results known for special settings (e.g., Stoikov and Zariphopoulou [72]).

Theorem 5.1. *With $\beta = \frac{1}{1-p}$ we have*

$$\hat{c}_t = \Big(\frac{D_t}{L_t}\Big)^\beta \widehat{X}_t \qquad \textit{and hence} \qquad \hat{\kappa}_t = \Big(\frac{D_t}{L_t}\Big)^\beta. \tag{5.1}$$

Proof. This follows from Proposition 4.2 via (4.4) and (2.2). □

Remark 5.2. In Theorem III.3.2 (and Remark III.3.6) of Chapter III we establish the same formula for $\hat{\kappa}$ in the utility maximization problem under constraints as described in Remark 3.7, under the sole assumption that an optimal constrained strategy exists.

The special case where the constraints set $\mathscr{C} \subseteq \mathbb{R}^d$ is linear can be deduced from Theorem 5.1 by redefining the price process S. For instance, set $S^1 \equiv 1$ for $\mathscr{C} = \{(x^1, \dots, x^d) \in \mathbb{R}^d : x^1 = 0\}$.

In the remainder of the section we discuss how certain changes in the model and the discounting process D affect the optimal propensity to consume. This is based on (5.1) and the relation

$$\tfrac{1}{p} x_0^p L_t = \operatorname*{ess\,sup}_{c \in \mathcal{A}(0, x_0 1_{\{T\}}, t)} E\Big[\int_t^T D_s \tfrac{1}{p} c_s^p \, \mu^\circ(ds) \Big| \mathcal{F}_t\Big], \tag{5.2}$$

which is immediate from Proposition 3.1. In the present non-Markovian setting the parametrization by the propensity to consume is crucial as one cannot make statements for "fixed wealth". There is no immediate way to infer results about $\hat{c}$, except of course for the initial value $\hat{c}_0 = \hat{\kappa}_0 x_0$.

II.5.1 Variation of the Investment Opportunities

It is classical in economics to compare two "identical" agents with utility function U, where only one has access to a stock market. The opportunity to invest in risky assets gives rise to two contradictory effects. The presence of risk incites the agent to save cash for the uncertain future; this is the *precautionary savings effect* and its strength is related to the *absolute prudence* $\mathscr{P}(U) = -U'''/U''$. On the other hand, the agent may prefer to invest rather than to consume immediately. This *substitution effect* is related to the *absolute risk aversion* $\mathscr{A}(U) = -U''/U'$.

Classical economic theory (e.g., Gollier [27, Proposition 74]) states that in a one period model, the presence of a complete financial market makes the optimal consumption at time $t = 0$ smaller if $\mathscr{P}(U) \geq 2\mathscr{A}(U)$ holds everywhere on $(0, \infty)$, and larger if the converse inequality holds. For power utility, the former condition holds if $p < 0$ and the latter holds if $p \in (0, 1)$. We go a step further in the comparison by considering two different sets of constraints, instead of giving no access to the stock market at all (which is the constraint $\{0\}$).

Let $\mathscr{C}$ and $\mathscr{C}'$ be set-valued mappings of constraints as in Remark 3.7, and let $\mathscr{C}' \subseteq \mathscr{C}$ in the sense that $\mathscr{C}'_t(\omega) \subseteq \mathscr{C}_t(\omega)$ for all (t, ω). *Assume* that there exist corresponding optimal constrained strategies.

Proposition 5.3. *Let $\hat{\kappa}$ and $\hat{\kappa}'$ be the optimal propensities to consume for the constraints $\mathscr{C}$ and $\mathscr{C}'$, respectively. Then $\mathscr{C}' \subseteq \mathscr{C}$ implies $\hat{\kappa} \leq \hat{\kappa}'$ if $p > 0$ and $\hat{\kappa} \geq \hat{\kappa}'$ if $p < 0$. In particular, $\hat{c}_0 \leq \hat{c}'_0$ if $p > 0$ and $\hat{c}_0 \geq \hat{c}'_0$ if $p < 0$.*

Proof. Let L and L' be the corresponding opportunity processes; we make use of Remarks 3.7 and 5.2. Consider relation (5.2) with $\mathcal{A}^{\mathscr{C}}$ instead of $\mathcal{A}$ and the analogue for L' with $\mathcal{A}^{\mathscr{C}'}$. We see that $\mathcal{A}^{\mathscr{C}'} \subseteq \mathcal{A}^{\mathscr{C}}$ implies $\frac{1}{p}L' \leq \frac{1}{p}L$, as the supremum is taken over a larger set in the case of $\mathscr{C}$. By (5.1), $\hat{\kappa}$ is a decreasing function of L. □

Proposition 5.4. *The optimal propensity to consume satisfies*

$$\hat{\kappa}_t \leq \frac{(k_2/k_1)^\beta}{1+T-t} \quad \text{if } p \in (0,1) \quad \text{and} \quad \hat{\kappa}_t \geq \frac{(k_2/k_1)^\beta}{1+T-t} \quad \text{if } p < 0.$$

In particular, we have a model-independent deterministic threshold independent of p in the standard case $D \equiv 1$,

$$\hat{\kappa}_t \leq \frac{1}{1+T-t} \quad \text{if } p \in (0,1) \quad \text{and} \quad \hat{\kappa}_t \geq \frac{1}{1+T-t} \quad \text{if } p < 0.$$

Proof. This follows from Lemma 3.5 and (5.1). The second part can also be seen as special case of Proposition 5.3 with constraint set $\mathscr{C}' = \{0\}$ since then $\hat{\kappa}' = (1+T-t)^{-1}$ as in Example 3.6. □

The threshold $(1+T-t)^{-1}$ coincides with the optimal propensity to consume for the log-utility function (cf. [25]), which formally corresponds to $p = 0$. This suggests that the threshold is attained by $\hat{\kappa}(p)$ in the limit $p \to 0$, a result we prove in Chapter V.

Remark 5.5. Uniform bounds for $\hat{\kappa}$ *opposite* to the ones in Proposition 5.4 exist if and only if $\mathrm{R}_q(P)$ holds for the given financial market model. Quantitatively, if $C_q > 0$ is the constant for $\mathrm{R}_q(P)$, then

$$\hat{\kappa}_t \geq \Big(\frac{k_2}{k_1}\Big)^\beta \frac{1}{C_q} \quad \text{if } p \in (0,1) \quad \text{and} \quad \hat{\kappa}_t \leq \Big(\frac{k_1}{k_2}\Big)^\beta \frac{1}{C_q} \quad \text{if } p < 0.$$

This follows from (5.1) and (2.4) by (the proof of) Proposition 4.5. In view of Corollary 4.12 we have the following dichotomy: $\hat{\kappa} = \hat{\kappa}(p)$ has a uniform upper bound either for all values of $p < 0$, or for none of them.

II.5.2 Variation of D

We now study how $\hat{\kappa}$ is affected if we increase D on some time interval $[t_1, t_2)$. To this end, let $0 \leq t_1 < t_2 \leq T$ be two fixed points in time and ξ a bounded càdlàg adapted process which is strictly positive and nonincreasing on $[t_1, t_2)$. In addition to $U_t(x) = D_t \frac{1}{p} x^p$ we consider the utility random field

$$U'_t(x) := D'_t \tfrac{1}{p} x^p, \quad D' := \big(1 + \xi 1_{[t_1,t_2)}\big) D.$$

As an interpretation, recall the modeling of taxation by D from Remark 2.2. Then we want to find out how the agent reacts to a temporary change of the tax policy on $[t_1, t_2)$—in particular whether a reduction of the tax rate $\varrho := D^{-1/p} - 1$ stimulates consumption. For $p > 0$, the next result shows this to be true during $[t_1, t_2)$, while the contrary holds before the policy change and there is no effect after t_2. An agent with $p < 0$ reacts in the opposite way. Remark 2.2 also suggests other interpretations of the same result.

Proposition 5.6. *Let $\hat{\kappa}$ and $\hat{\kappa}'$ be the optimal propensities to consume for U and U', respectively. Then*

$$\begin{cases} \hat{\kappa}'_t < \hat{\kappa}_t & \text{if } t < t_1, \\ \hat{\kappa}'_t > \hat{\kappa}_t & \text{if } t \in [t_1, t_2), \\ \hat{\kappa}'_t = \hat{\kappa}_t & \text{if } t \geq t_2. \end{cases}$$

Proof. Let L and L' be the opportunity processes for U and U'. We consider (5.2) and compare it with its analogue for L', where D is replaced by D'. As $\xi > 0$, we then see that $L'_t > L_t$ for $t < t_1$; moreover, $L'_t = L_t$ for $t \geq t_2$. Since ξ is nonincreasing, we also see that $L'_t < (1+\xi_t)L_t$ for $t \in [t_1, t_2)$. It remains to apply (5.1). For $t < t_1$, $\hat{\kappa}' = (D'_t/L'_t)^\beta = (D_t/L'_t)^\beta < (D_t/L_t)^\beta = \hat{\kappa}$. For $t \in [t_1, t_2)$ we have

$$\hat{\kappa}' = (D'_t/L'_t)^\beta = \Big(\frac{(1+\xi_t)D_t}{L'_t}\Big)^\beta > \Big(\frac{(1+\xi_t)D_t}{(1+\xi_t)L_t}\Big)^\beta = \hat{\kappa},$$

while for $t \geq t_2$, $D'_t = D_t$ implies $\hat{\kappa}'_t = \hat{\kappa}_t$. □

Remark 5.7. (i) For $t_2 = T$, the statement of Proposition 5.6 remains true if the closed interval is chosen in the definition of $\widetilde{D}$.

(ii) One can see [72, Proposition 12] as a special case of Proposition 5.6. In our notation, the authors consider $D = 1_{[0,T)}K_1 + 1_{\{T\}}K_2$ for two constants $K_1, K_2 > 0$ and obtain monotonicity of the consumption with respect to the ratio K_2/K_1. This is proved in a Markovian setting by a comparison result for PDEs.

II.6 Appendix A: Dynamic Programming

This appendix collects the facts about dynamic programming which are used in this chapter. Recall the standing assumption (2.5), the set $\mathcal{A}(\pi, c, t)$ from (3.1) and the process J from (3.2). We begin with the lattice property.

Fact 6.1. Fix $(\pi, c) \in \mathcal{A}$ and let $\Gamma_t(\tilde{c}) := E[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)|\mathcal{F}_t]$. The set $\{\Gamma_t(\tilde{c}) : \tilde{c} \in \mathcal{A}(\pi, c, t)\}$ is upward filtering for each $t \in [0, T]$.

Indeed, if $(\pi^i, c^i) \in \mathcal{A}(\pi, c, t)$, $i = 1, 2$, we have $\Gamma_t(c^1) \vee \Gamma_t(c^2) = \Gamma_t(c^3)$ for $(\pi^3, c^3) := (\pi^1, c^1)1_A + (\pi^2, c^2)1_{A^c}$ with $A := \{\Gamma_t(c^1) > \Gamma_t(c^2)\}$. Clearly $(\pi^3, c^3) \in \mathcal{A}(\pi, c, t)$. Regarding Remark 3.7, we note that π^3 satisfies the constraints if π^1 and π^2 do.

Proposition 6.2. *Let* $(\pi, c) \in \mathcal{A}$ *and* $I_t(\pi, c) := J_t(\pi, c) + \int_0^t U_s(c_s)\,\mu(ds)$. *If* $E[\,|I_t(\pi, c)|\,] < \infty$ *for each* t, *then* $I(\pi, c)$ *is a supermartingale having a càdlàg version. It is a martingale if and only if* (π, c) *is optimal.*

Proof. The technique of proof is well known; see El Karoui and Quenez [46] or Laurent and Pham [52] for arguments in different contexts.

We fix $(\pi, c) \in \mathcal{A}$ as well as $0 \le t \le u \le T$ and prove the supermartingale property. Note that $I_t(\pi, c) = \operatorname{ess\,sup}_{\tilde{c} \in \mathcal{A}(\pi,c,t)} \Upsilon_t(\tilde{c})$ for the martingale $\Upsilon_t(\tilde{c}) := E\big[\int_0^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\big|\mathcal{F}_t\big]$. (More precisely, the expectation is well defined with values in $\mathbb{R} \cup \{-\infty\}$ by (2.5).)

As $\Upsilon_u(\tilde{c}) = \Gamma_u(\tilde{c}) + \int_0^u U_s(\tilde{c}_s)\,\mu(ds)$, Fact 6.1 implies that there exists a sequence (c^n) in $\mathcal{A}(\pi, c, u)$ such that $\lim_n \Upsilon_u(c^n) = I_u(\pi, c)$ P-a.s., where the limit is monotone increasing in n. We conclude that

$$\begin{aligned} E[I_u(\pi, c)|\mathcal{F}_t] &= E[\lim_n \Upsilon_u(c^n)|\mathcal{F}_t] = \lim_n E[\Upsilon_u(c^n)|\mathcal{F}_t] \\ &\le \operatorname{ess\,sup}_{\tilde{c} \in \mathcal{A}(\pi,c,u)} E[\Upsilon_u(\tilde{c})|\mathcal{F}_t] = \operatorname{ess\,sup}_{\tilde{c} \in \mathcal{A}(\pi,c,u)} \Upsilon_t(\tilde{c}) \\ &\le \operatorname{ess\,sup}_{\tilde{c} \in \mathcal{A}(\pi,c,t)} \Upsilon_t(\tilde{c}) = I_t(\pi, c). \end{aligned}$$

To construct the càdlàg version, denote by I' the process obtained by taking the right limits of $t \mapsto I_t(\pi, c) =: I_t$ through the rational numbers, with $I'_T := I_T$. Since I is a supermartingale and the filtration satisfies the "usual assumptions", these limits exist P-a.s., I' is a (càdlàg) supermartingale, and $I'_t \le I_t$ P-a.s. (see Dellacherie and Meyer [18, VI.1.2]). But in fact, equality holds here because for all $(\tilde{\pi}, \tilde{c}) \in \mathcal{A}(\pi, c, t)$ we have

$$\Upsilon_t(\tilde{c}) = E\Big[\int_0^T U_s(\tilde{c}_s)\,d\mu^\circ\Big|\mathcal{F}_t\Big] = E[I_T(\tilde{\pi}, \tilde{c})|\mathcal{F}_t] = E[I_T|\mathcal{F}_t] \le I'_t$$

due to $I_T = I'_T$, and hence also $I'_t \ge \operatorname{ess\,sup}_{\tilde{c} \in \mathcal{A}(\pi,c,t)} \Upsilon_t(\tilde{c}) = I_t$. Therefore I' is a càdlàg version of I.

Turning to the martingale property, let (π, c) be optimal. Then $I_0(\pi, c) = \Upsilon_0(\pi, c) = E[I_T(\pi, c)]$, so the supermartingale $I(\pi, c)$ is a martingale. Conversely, this relation states that (π, c) is optimal, by definition of $I(\pi, c)$. □

The following property was used in the body of the text.

Fact 6.3. Consider $(\pi, c), (\pi', c') \in \mathcal{A}$ with corresponding wealths X_t, X'_t at time $t \in [0, T]$ and $(\pi'', c'') \in \mathcal{A}(\pi', c', t)$. Then

$$c 1_{[0,t]} + \frac{X_t}{X'_t}\, c''\, 1_{(t,T]} \;\in \mathcal{A}(\pi, c, t).$$

Indeed, for the trading strategy $\pi 1_{[0,t]} + \pi'' 1_{(t,T]}$, the corresponding wealth process is $X 1_{[0,t]} + \frac{X_t}{X'_t} X'' 1_{(t,T]} > 0$ by (2.1).

II.7 Appendix B: Martingale Property of the Optimal Processes

The purpose of this appendix is to provide a statement which follows from [43] and is known to its authors, but which we could not find in the literature. For the case without intermediate consumption, the following assertion is contained in [49, Theorem 2.2].

Lemma 7.1. *Assume* (2.4) *and* (2.6). *Let* $(\pi, c) \in \mathcal{A}$, $X = X(\pi, c)$ *and* $Y \in \mathscr{Y}^{\mathscr{D}}$, *then*

$$Z_t := X_t Y_t + \int_0^t c_s Y_s \,\mu(ds), \quad t \in [0, T]$$

is a supermartingale. If $(X, c, Y) = (\widehat{X}, \hat{c}, \widehat{Y})$ *are the optimal processes solving the primal and the dual problem, respectively, then* Z *is a martingale.*

Proof. It follows from [43, Theorem 3.10(vi)] that $E[Z_T] = E[Z_0]$ for the optimal processes, so it suffices to prove the first part.

(i) Assume first that $Y \in \mathscr{Y}^{\mathscr{M}}$, i.e., Y/Y_0 is the density process of a measure $Q \approx P$. As $\mathscr{Y}^{\mathscr{M}} \subseteq \mathscr{Y}^*$, the process $X + \int c_u \,\mu(du) = x_0 + \int X_- \pi \,dR$ is a Q-supermartingale, that is, $E^Q[X_t + \int_0^t c_u \,\mu(du) | \mathcal{F}_s] \leq X_s + \int_0^s c_u \,\mu(du)$ for $s \leq t$. We obtain the claim by Bayes' rule,

$$E\Big[X_t Y_t + \int_s^t c_u Y_u \,\mu(du) \Big| \mathcal{F}_s\Big] \leq X_s Y_s.$$

(ii) Let $Y \in \mathscr{Y}^{\mathscr{D}}$ be arbitrary. By [43, Corollary 2.11], there is a sequence $Y^n \in \mathscr{Y}^{\mathscr{M}}$ which Fatou-converges to Y. Consider the supermartingale $Y' := \liminf_n Y^n$. By Žitković [75, Lemma 8], $Y'_t = Y_t$ P-a.s. for all t in a (dense) subset $\Lambda \subseteq [0, T]$ which contains T and whose complement is countable. It follows from Fatou's lemma and step (i) that Z is a supermartingale on Λ; indeed, for $s \leq t$ in Λ,

$$\begin{aligned} E\Big[X_t Y_t + \int_s^t c_u Y_u \,\mu(du) \Big| \mathcal{F}_s\Big] &= E\Big[X_t Y'_t + \int_s^t c_u Y'_u \,\mu(du) \Big| \mathcal{F}_s\Big] \\ &\leq \liminf_n E\Big[X_t Y^n_t + \int_s^t c_u Y^n_u \,\mu(du) \Big| \mathcal{F}_s\Big] \\ &\leq \liminf_n X_s Y^n_s = X_s Y_s \quad P\text{-a.s.} \end{aligned}$$

We can extend $Z|_\Lambda$ to $[0, T]$ by taking right limits in Λ and obtain a right-continuous supermartingale Z' on $[0, T]$, by right-continuity of the filtration. But Z' is indistinguishable from Z because Z is also right-continuous. Hence Z is a supermartingale as claimed. □

Chapter III

Bellman Equation

In this chapter, which corresponds to the article [59], we consider a general semimartingale model with closed portfolio constraints. We focus on the local representation of the optimization problem, which is formalized by the Bellman equation.

III.1 Introduction

This chapter presents the local dynamic programming for power utility maximization in a general constrained semimartingale framework. We have seen that the homogeneity of these utility functions leads to a factorization of the value process into a power of the current wealth and the opportunity process L. In our setting, the Bellman equation describes the drift rate of L and clarifies the local structure of our problem. Finding an optimal strategy boils down to maximizing a random function $y \mapsto g(\omega, t, y)$ on $\mathbb{R}^d$ for every state ω and date t. This function is given in terms of the semimartingale characteristics of L as well as the asset returns, and its maximum yields the drift rate of L. The role of the opportunity process is to augment the information contained in the return characteristics in order to have a local sufficient statistic for the global optimization problem.

We present three main results. First, we show that if there exists an optimal strategy for the utility maximization problem, *the opportunity process L solves the Bellman equation* and we provide a local description of the optimal strategies. We state the Bellman equation in two forms, as an identity for the drift rate of L and as a backward stochastic differential equation (BSDE) for L. Second, we characterize the opportunity process as the *minimal solution* of this equation. Finally, given some solution and an associated strategy, one can ask whether the strategy is optimal and the solution is the opportunity process. We present two different approaches which lead to two *verification theorems* not comparable in strength unless the constraints are convex.

The present dynamic programming approach should be seen as complementary to convex duality, which remains the only method to obtain *existence* of optimal strategies in general models; see Kramkov and Schachermayer [49], Karatzas and Žitković [43], Karatzas and Kardaras [41]. In some cases the Bellman equation can be solved directly, e.g., in the setting of Example 5.8 with continuous asset prices or in the Lévy process setting of Chapter IV. In addition to existence, one then typically obtains additional properties of the optimal strategies.

This chapter is organized as follows. The next section specifies the optimization problem in detail, recalls the opportunity process and the martingale optimality principle, and fixes the notation for the characteristics. We also introduce set-valued processes describing the budget condition and state the assumptions on the portfolio constraints. Section III.3 derives the Bellman equation, first as a drift condition and then as a BSDE. It becomes more explicit as we specialize to the case of continuous asset prices. The definition of a solution of the Bellman equation is given in Section III.4, where we show the minimality of the opportunity process. Section III.5 deals with the verification problem, which is converse to the derivation of the Bellman equation since it requires the passage from the local maximization to the global optimization problem. We present an approach via the value process and a second approach via a deflator, which corresponds to the dual problem in a suitable setting. Appendix A is linked to Section III.3 and contains the measurable selections for the construction of the Bellman equation. It is complemented by Appendix B, where we construct an alternative parametrization of the market model by representative portfolios.

III.2 Preliminaries

The following notation is used. If $x, y \in \mathbb{R}$ are reals, $x^+ = \max\{x, 0\}$ and $x \wedge y = \min\{x, y\}$. We set $1/0 := \infty$ where necessary. If $z \in \mathbb{R}^d$ is a d-dimensional vector, z^i is its ith coordinate, $z^\top$ its transpose, and $|z| = (z^\top z)^{1/2}$ the Euclidean norm. If X is an $\mathbb{R}^d$-valued semimartingale and π is an $\mathbb{R}^d$-valued predictable integrand, the vector stochastic integral is a scalar semimartingale with initial value zero and denoted by $\int \pi \, dX$ or by $\pi \bullet X$. The quadratic variation of X is the $d \times d$-matrix $[X] := [X, X]$ and if Y is a scalar semimartingale, $[X, Y]$ is the d-vector which is given by $[X, Y]^i := [X^i, Y]$. Relations between measurable functions hold almost everywhere unless otherwise mentioned. Our reference for any unexplained notion from stochastic calculus is Jacod and Shiryaev [34].

III.2.1 The Optimization Problem

We fix the time horizon $T \in (0, \infty)$ and a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ satisfies the usual assumptions of right-continuity and completeness as well as $\mathcal{F}_0 = \{\emptyset, \Omega\}$ P-a.s. We consider an

$\mathbb{R}^d$-valued càdlàg semimartingale R with $R_0 = 0$ representing the returns of d risky assets. Their discounted prices are given by the stochastic exponential $S = \mathcal{E}(R) = (\mathcal{E}(R^1), \dots, \mathcal{E}(R^d))$. Our agent also has a bank account at his disposal; it does not pay interest.

The agent is endowed with a deterministic initial capital $x_0 > 0$. A *trading strategy* is a predictable R-integrable $\mathbb{R}^d$-valued process π, where π^i indicates the fraction of wealth (or the portfolio proportion) invested in the ith risky asset. A *consumption strategy* is a nonnegative optional process c such that $\int_0^T c_t\,dt < \infty$ P-a.s. We want to consider two cases. Either consumption occurs only at the terminal time T (utility from "terminal wealth" only); or there is intermediate consumption plus a bulk consumption at the time horizon. To unify the notation, we introduce the measure μ on $[0,T]$ by

$$\mu(dt) := \begin{cases} 0 & \text{in the case without intermediate consumption,} \\ dt & \text{in the case with intermediate consumption.} \end{cases}$$

Let also $\mu^\circ := \mu + \delta_{\{T\}}$, where $\delta_{\{T\}}$ is the unit Dirac measure at T. The *wealth process* $X(\pi, c)$ corresponding to a pair (π, c) is defined by the equation

$$X_t(\pi, c) = x_0 + \int_0^t X_{s-}(\pi, c)\pi_s\,dR_s - \int_0^t c_s\,\mu(ds), \quad 0 \le t \le T.$$

We define the set of trading and consumption pairs

$$\mathcal{A}^0(x_0) := \big\{(\pi, c) : X(\pi, c) > 0,\ X_-(\pi, c) > 0 \text{ and } c_T = X_T(\pi, c)\big\}.$$

These are the strategies that satisfy the budget constraint. The convention $c_T = X_T(\pi, c)$ means that all the remaining wealth is consumed at time T. We consider also exogenous constraints imposed on the agent. For each $(\omega, t) \in \Omega \times [0, T]$ we are given a set $\mathscr{C}_t(\omega) \subseteq \mathbb{R}^d$ which contains the origin. The set of (constrained) *admissible* strategies is

$$\mathcal{A}(x_0) := \big\{(\pi, c) \in \mathcal{A}^0(x_0) : \pi_t(\omega) \in \mathscr{C}_t(\omega) \text{ for all } (\omega, t)\big\};$$

it is nonempty as $0 \in \mathscr{C}_t(\omega)$. Further assumptions on the set-valued mapping $\mathscr{C}$ will be introduced in Section III.2.4. We fix the initial capital x_0 and usually write $\mathcal{A}$ for $\mathcal{A}(x_0)$. We write $c \in \mathcal{A}$ and call c admissible if there exists π such that $(\pi, c) \in \mathcal{A}$; an analogous convention is used for similar expressions.

We will often parametrize the consumption strategies as a fraction of wealth. Let $(\pi, c) \in \mathcal{A}$ and $X = X(\pi, c)$. Then

$$\kappa := \frac{c}{X}$$

is called the *propensity to consume* corresponding to (π, c). This yields a one-to-one correspondence between the pairs $(\pi, c) \in \mathcal{A}$ and the pairs (π, κ) such that $\pi \in \mathcal{A}$ and κ is a nonnegative optional process satisfying $\int_0^T \kappa_s \, ds < \infty$ P-a.s. and $\kappa_T = 1$ (see Remark II.2.1 for details). We shall abuse the notation and identify a consumption strategy with the corresponding propensity to consume, e.g., we write $(\pi, \kappa) \in \mathcal{A}$. Note that

$$X(\pi, \kappa) = x_0 \mathcal{E}\big(\pi \bullet R - \kappa \bullet \mu\big).$$

This simplifies verifying that some pair (π, κ) is admissible as $X(\pi, \kappa) > 0$ implies $X_-(\pi, \kappa) > 0$ (cf. [34, II.8a]).

The preferences of the agent are modeled by a time-additive random utility function as follows. Let D be a càdlàg, adapted, strictly positive process such that $E\big[\int_0^T D_s \, \mu^\circ(ds)\big] < \infty$ and fix $p \in (-\infty, 0) \cup (0, 1)$. We define the power utility random field

$$U_t(x) := D_t \tfrac{1}{p} x^p, \quad x \in (0, \infty), \ t \in [0, T].$$

This is the general form of a *p-homogeneous* utility random field such that a constant consumption yields finite expected utility. Interpretations and applications for the process D are discussed in Chapter II. We denote by U^* the convex conjugate of $x \mapsto U_t(x)$,

$$U_t^*(y) = \sup_{x>0} \big\{U_t(x) - xy\big\} = -\tfrac{1}{q} y^q D_t^\beta; \tag{2.1}$$

here $q := \frac{p}{p-1} \in (-\infty, 0) \cup (0, 1)$ is the exponent conjugate to p and the constant $\beta := \frac{1}{1-p} > 0$ is the relative risk tolerance of U. Note that we exclude the well-studied logarithmic utility (e.g., Goll and Kallsen [25]) which corresponds to $p = 0$.

The *expected utility* corresponding to a consumption strategy $c \in \mathcal{A}$ is $E\big[\int_0^T U_t(c_t) \, \mu^\circ(dt)\big]$, i.e., either $E[U_T(c_T)]$ or $E[\int_0^T U_t(c_t) \, dt + U_T(c_T)]$. The utility maximization problem is said to be *finite* if

$$u(x_0) := \sup_{c \in \mathcal{A}(x_0)} E\Big[\int_0^T U_t(c_t) \, \mu^\circ(dt)\Big] < \infty. \tag{2.2}$$

Note that this condition is void if $p < 0$ as then $U < 0$. If (2.2) holds, a strategy $(\pi, c) \in \mathcal{A}(x_0)$ is called *optimal* if $E\big[\int_0^T U_t(c_t) \, \mu^\circ(dt)\big] = u(x_0)$.

Finally, we introduce the following sets; they are of minor importance and used only in the case $p < 0$:

$$\mathcal{A}^f := \big\{(\pi, c) \in \mathcal{A} : \ \textstyle\int_0^T U_t(c_t) \, \mu^\circ(dt) > -\infty\big\},$$
$$\mathcal{A}^{fE} := \big\{(\pi, c) \in \mathcal{A} : \ E\big[\textstyle\int_0^T U_t(c_t) \, \mu^\circ(dt)\big] > -\infty\big\}.$$

Anticipating that (2.2) will be in force, the indices stand for “finite” and “finite expectation”. Clearly $\mathcal{A}^{fE} \subseteq \mathcal{A}^f \subseteq \mathcal{A}$, and equality holds if $p \in (0, 1)$.

III.2.2 Opportunity Process

We recall the opportunity process. We assume (2.2) in this section, which ensures that the following process is finite. By Proposition II.3.1 and Remark II.3.7 there exists a unique càdlàg semimartingale L, called *opportunity process*, such that

$$L_t \,\tfrac{1}{p}\big(X_t(\pi,c)\big)^p = \operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \tag{2.3}$$

for any $(\pi,c)\in\mathcal{A}$, where $\mathcal{A}(\pi,c,t):=\big\{(\tilde{\pi},\tilde{c})\in\mathcal{A}:\,(\tilde{\pi},\tilde{c})=(\pi,c)\text{ on }[0,t]\big\}$. We note that $L_T = D_T$ and that $u(x_0) = L_0\frac{1}{p}x_0^p$ is the value function from (2.2). The following is contained in Lemma II.3.5.

Lemma 2.1. *L is a special semimartingale for all p. If $p\in(0,1)$, then $L, L_- > 0$. If $p<0$, the same holds provided that an optimal strategy exists.*

Proposition 2.2 (Proposition II.3.4)**.** *Let $(\pi,c)\in\mathcal{A}^{fE}$. Then the process*

$$L_t\tfrac{1}{p}\big(X_t(\pi,c)\big)^p + \int_0^t U_s(c_s)\,\mu(ds),\quad t\in[0,T]$$

is a supermartingale; it is a martingale if and only if (π,c) is optimal.

This is the "martingale optimality principle". The expected terminal value of this process equals $E[\int_0^T U_t(c_t)\,\mu^\circ(dt)]$, hence the assertion fails for $(\pi,c)\in\mathcal{A}\setminus\mathcal{A}^{fE}$.

III.2.3 Semimartingale Characteristics

In the remainder of this section we introduce tools which are necessary to describe the optimization problem locally. The use of semimartingale characteristics and set-valued processes follows [25] and [41], which consider logarithmic utility and convex constraints. That problem differs from ours in that it is "myopic", i.e., the characteristics of R are sufficient to describe the local problem and so there is no opportunity process.

We refer to [34] for background regarding semimartingale characteristics and random measures. Let μ^R be the integer-valued random measure associated with the jumps of R and let $h:\mathbb{R}^d\to\mathbb{R}^d$ be a cut-off function, i.e., h is bounded and $h(x)=x$ in a neighborhood of $x=0$. Let (B^R, C^R, ν^R) be the predictable characteristics of R relative to h. The *canonical representation* of R (cf. [34, II.2.35]) is

$$R = B^R + R^c + h(x)*(\mu^R-\nu^R) + (x-h(x))*\mu^R. \tag{2.4}$$

The finite variation process $(x-h(x))*\mu^R$ contains essentially the "large" jumps of R. The rest is the canonical decomposition of the semimartingale

$\bar{R} = R - (x - h(x)) * \mu^R$, which has bounded jumps: $B^R = B^R(h)$ is predictable of finite variation, R^c is a continuous local martingale, and finally $h(x) * (\mu^R - \nu^R)$ is a purely discontinuous local martingale.

As L is a special semimartingale (Lemma 2.1), it has a canonical decomposition $L = L_0 + A^L + M^L$. Here L_0 is constant, A^L is predictable of finite variation and also called the *drift* of L, M^L is a local martingale, and $A^L_0 = M^L_0 = 0$. Analogous notation will be used for other special semimartingales. It is then possible to consider the characteristics (A^L, C^L, ν^L) of L with respect to the identity instead of a cut-off function. Writing x' for the identity on $\mathbb{R}$, the canonical representation is

$$L = L_0 + A^L + L^c + x' * (\mu^L - \nu^L);$$

see [34, II.2.38]. It will be convenient to use the joint characteristics of the $\mathbb{R}^d \times \mathbb{R}$-valued process (R, L). We denote a generic point in $\mathbb{R}^d \times \mathbb{R}$ by (x, x') and let $(B^{R,L}, C^{R,L}, \nu^{R,L})$ be the characteristics of (R, L) with respect to the function $(x, x') \mapsto (h(x), x')$. More precisely, we choose "good" versions of the characteristics so that they satisfy the properties given in [34, II.2.9]. For the $(d+1)$-dimensional process (R, L) we have the canonical representation

$$\begin{pmatrix} R \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ L_0 \end{pmatrix} + \begin{pmatrix} B^R \\ A^L \end{pmatrix} + \begin{pmatrix} R^c \\ L^c \end{pmatrix} + \begin{pmatrix} h(x) \\ x' \end{pmatrix} * (\mu^{R,L} - \nu^{R,L}) + \begin{pmatrix} x - h(x) \\ 0 \end{pmatrix} * \mu^{R,L}.$$

We denote by $(b^{R,L}, c^{R,L}, F^{R,L}; A)$ the *differential* characteristics with respect to a predictable locally integrable increasing process A, e.g.,

$$A_t := t + \sum_i \mathrm{Var}(B^{RL,i})_t + \sum_{i,j} \mathrm{Var}(C^{RL,ij})_t + \big(|(x, x')|^2 \wedge 1\big) * \nu^{R,L}_t.$$

Then $b^{R,L} \bullet A = B^{R,L}$, $c^{R,L} \bullet A = C^{R,L}$, and $F^{R,L} \bullet A = \nu^{R,L}$. We write $b^{R,L} = (b^R, a^L)^\top$ and $c^{R,L} = \left(\begin{smallmatrix} c^R & c^{RL} \\ (c^{RL})^\top & c^L \end{smallmatrix}\right)$, i.e., c^{RL} is a d-vector satisfying $(c^{RL}) \bullet A = \langle R^c, L^c \rangle$. We will often use that

$$\int_{\mathbb{R}^d \times \mathbb{R}} (|x|^2 + |x'|^2) \wedge (1 + |x'|) \; F^{R,L}(d(x, x')) < \infty \tag{2.5}$$

because L is a special semimartingale (cf. [34, II.2.29]). Let Y be any scalar semimartingale with differential characteristics (b^Y, c^Y, F^Y) relative to A and a cut-off function $\bar{h}$. We call

$$a^Y := b^Y + \int \big(x - \bar{h}(x)\big) \, F^Y(dx)$$

the *drift rate* of Y whenever the integral is well defined with values in $[-\infty, \infty]$, even if it is not finite. Note that a^Y does not depend on the choice of $\bar{h}$. If Y is special, the drift rate is finite and even A-integrable (and vice versa). As an example, a^L is the drift rate of L and $a^L \bullet A = A^L$ yields the drift.

Remark 2.3. Assume Y is a *nonpositive* scalar semimartingale. Then its drift rate a^Y is well defined with values in $[-\infty, \infty)$. Indeed, the fact that $Y = Y_- + \Delta Y \leq 0$ implies that $x \leq -Y_-$ $F^Y(dx)$-a.e.

If Y is a scalar semimartingale with drift rate $a^Y \in [-\infty, 0]$, we call Y a *semimartingale with nonpositive drift rate.* Here a^Y need not be finite, as in the case of a compound Poisson process with negative, non-integrable jumps. We refer to Kallsen [39] for the concept of σ-localization. Recalling that $\mathcal{F}_0$ is trivial, we conclude the following, e.g., from [41, Appendix 3].

Lemma 2.4. *Let Y be a semimartingale with nonpositive drift rate.*

(i) Y is a σ-supermartingale $\Leftrightarrow$ a^Y is finite $\Leftrightarrow$ Y is σ-locally of class (D).

(ii) Y is a local supermartingale $\Leftrightarrow$ $a^Y \in L(A)$ $\Leftrightarrow$ Y is locally of class (D).

(iii) If Y is uniformly bounded from below, it is a supermartingale.

III.2.4 Constraints and Degeneracies

We introduce some set-valued processes that will be used in the sequel, that is, for each (ω, t) they describe a subset of $\mathbb{R}^d$. We refer to Rockafellar [64] and Aliprantis and Border [1, §18] for background.

We start by expressing the budget constraint in this fashion. The process

$$\mathscr{C}_t^0(\omega) := \Big\{ y \in \mathbb{R}^d : \, F_t^R(\omega)\big\{x \in \mathbb{R}^d : \, y^\top x < -1\big\} = 0 \Big\}$$

was called the *natural constraints* in [41]. Clearly $\mathscr{C}^0$ is closed, convex, and contains the origin. Moreover, one can check (see [41, §3.3]) that it is *predictable* in the sense that for each closed $G \subseteq \mathbb{R}^d$, the lower inverse image $(\mathscr{C}^0)^{-1}(G) = \{(\omega, t) : \mathscr{C}_t(\omega) \cap G \neq \emptyset\}$ is predictable. (Here one can replace closed by compact or by open; see [64, 1A].) A statement such as "$\mathscr{C}^0$ is closed" means that $\mathscr{C}_t^0(\omega)$ is closed for all (ω, t); moreover, we will often omit the arguments (ω, t). We also consider the slightly smaller set-valued process

$$\mathscr{C}^{0,*} := \Big\{ y \in \mathbb{R}^d : \, F^R\big\{x \in \mathbb{R}^d : \, y^\top x \leq -1\big\} = 0 \Big\}.$$

These processes relate to the budget constraint as follows.

Lemma 2.5. *A process $\pi \in L(R)$ satisfies $\mathcal{E}(\pi \bullet R) \geq 0 \, (> 0)$ up to evanescence if and only if $\pi \in \mathscr{C}^0 \, (\mathscr{C}^{0,*})$ $P \otimes A$-a.e.*

Proof. Recall that $\mathcal{E}(\pi \bullet R) > 0$ if and only if $1 + \pi^\top \Delta R > 0$ ([34, II.8a]). Writing $V(x) = 1_{\{x:\, 1 + \pi^\top x \leq 0\}}(x)$, we have that $(P \otimes A)\{\pi \notin \mathscr{C}^{0,*}\} = E[V(x) * \nu_T^R] = E[V(x) * \mu_T^R] = E\big[\sum_{s \leq T} 1_{\{x:\, 1 + \pi_s^\top \Delta R_s \leq 0\}}\big]$. For the equivalence with $\mathscr{C}^0$, interchange strict and non-strict inequality signs. $\square$

The process $\mathscr{C}^{0,*}$ is not closed in general (nor relatively open). Clearly $\mathscr{C}^{0,*} \subseteq \mathscr{C}^0$, and in fact $\mathscr{C}^0$ is the closure of $\mathscr{C}^{0,*}$: for $y \in \mathscr{C}^0_t(\omega)$, the sequence $\{(1+n^{-1})y\}_{n\geq 1}$ is in $\mathscr{C}^{0,*}_t(\omega)$ and converges to y. This implies that $\mathscr{C}^{0,*}$ is predictable; cf. [1, 18.3]. We will not be able to work directly with $\mathscr{C}^{0,*}$ because closedness is essential for the measurable selection arguments that will be used.

We turn to the exogenous portfolio constraints, i.e., the set-valued process $\mathscr{C}$ containing the origin. We consider the following conditions:

(C1) $\mathscr{C}$ is predictable.

(C2) $\mathscr{C}$ is closed.

(C3) *If* $p \in (0,1)$: There exists a $(0,1)$-valued process $\underline{\eta}$ such that
$y \in (\mathscr{C} \cap \mathscr{C}^0) \setminus \mathscr{C}^{0,*} \implies \eta y \in \mathscr{C}$ for all $\eta \in (\underline{\eta}, 1)$, $P \otimes A$-a.e.

Condition (C3) is clearly satisfied if $\mathscr{C} \cap \mathscr{C}^0 \subseteq \mathscr{C}^{0,*}$, which includes the case of a continuous process R, and it is always satisfied if $\mathscr{C}$ is star-shaped with respect to the origin or even convex. If $p < 0$, (C3) should be read as always being satisfied. We motivate (C3) by

Example 2.6. We assume that there is no intermediate consumption and $x_0 = 1$. Consider the one-period binomial model of a financial market, i.e., $S = \mathcal{E}(R)$ is a scalar process which is constant up to time T, where it has a single jump, say, $P[\Delta R_T = -1] = p_0$ and $P[\Delta R_T = K] = 1 - p_0$, where $K > 0$ is a constant and $p_0 \in (0,1)$. The filtration is generated by R and we consider $\mathscr{C} \equiv \{0\} \cup \{1\}$. Then $E[U(X_T(\pi))] = U(1)$ if $\pi_T = 0$ and $E[U(X_T(\pi))] = p_0 U(0) + (1-p_0)U(1+K)$ if $\pi_T = 1$. If $U(0) > -\infty$, and if K is large enough, $\pi_T = 1$ performs better and its *terminal wealth vanishes* with probability $p_0 > 0$. Of course, this cannot happen if $U(0) = -\infty$, i.e., $p < 0$. The constants can also be chosen such that both strategies are optimal, so there is *no uniqueness*.

We have included only positive wealth processes in our definition of $\mathcal{A}$; only these match our multiplicative setting. Under (C3), the Inada condition $U'(0) = \infty$ ensures that vanishing wealth is not optimal.

The final set-valued process is related to linear dependencies of the assets. As in [41], the predictable process of *null-investments* is

$$\mathscr{N} := \big\{ y \in \mathbb{R}^d : y^\top b^R = 0,\, y^\top c^R = 0,\, F^R\{x : y^\top x \neq 0\} = 0 \big\}.$$

Its values are linear subspaces of $\mathbb{R}^d$, hence closed, and provide the pointwise description of the null-space of $H \mapsto H \bullet R$. That is, $H \in L(R)$ satisfies $H \bullet R \equiv 0$ if and only if $H \in \mathscr{N}$ $P \otimes A$-a.e. An investment with values in $\mathscr{N}$ has no effect on the wealth process.

III.3 The Bellman Equation

We have now introduced the necessary notation to formulate our first main result. Two special cases of our Bellman equation can be found in the pioneering work of Mania and Tevzadze [54] and Hu *et al.* [33]. These articles consider models with continuous asset prices and we shall indicate the connections as we specialize to that case in Section III.3.3. A related equation also arises in the study of mean-variance hedging by Černý and Kallsen [11] in the context of locally square-integrable semimartingales, although they do not use dynamic programming explicitly. Due to the quadratic setting, that equation is more explicit than ours and the mathematical treatment is quite different. Czichowsky and Schweizer [13] study a cone-constrained version of the related Markowitz problem and there the equation is no longer explicit.

The Bellman equation highlights the local structure of our utility maximization problem. In addition, it has two main benefits. First, it can be used as an abstract tool to derive properties of the optimal strategies and the opportunity process. Second, one can try to solve the equation directly in a given model and to deduce the optimal strategies. This is the point of view taken in Section III.5 and obviously requires the precise form of the equation.

The following assumptions are in force for the entire Section III.3.

Assumptions 3.1. The utility maximization problem is finite, there exists an optimal strategy $(\hat{\pi}, \hat{c}) \in \mathcal{A}$, and $\mathscr{C}$ satisfies (C1)-(C3).

III.3.1 Bellman Equation in Joint Characteristics

Our first main result is the Bellman equation stated as a description of the drift rate of the opportunity process. We recall the conjugate function $U_t^*(y) = -\frac{1}{q} y^q D_t^\beta$.

Theorem 3.2. *The drift rate a^L of the opportunity process satisfies*

$$-p^{-1} a^L = U^*(L_-)\, \tfrac{d\mu}{dA} + \max_{y \in \mathscr{C} \cap \mathscr{C}^0} g(y), \tag{3.1}$$

where g is the predictable random function

$$\begin{aligned} g(y) := L_- y^\top \Big(b^R + \tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2} c^R y \Big) + \int_{\mathbb{R}^d \times \mathbb{R}} x' y^\top h(x)\, F^{R,L}(d(x,x')) \\ + \int_{\mathbb{R}^d \times \mathbb{R}} (L_- + x') \big\{ p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x) \big\}\, F^{R,L}(d(x,x')). \end{aligned} \tag{3.2}$$

The unique ($P \otimes \mu^\circ$-a.e.) optimal propensity to consume is

$$\hat{\kappa} = \Big(\frac{D}{L} \Big)^{\frac{1}{1-p}}. \tag{3.3}$$

Any optimal trading strategy π^* *satisfies*

$$\pi^* \in \arg\max_{\mathscr{C}\cap\mathscr{C}^0} g \tag{3.4}$$

and the corresponding optimal wealth process and consumption are given by

$$X^* = x_0\mathcal{E}\big(\pi^* \bullet R - \hat{\kappa} \bullet \mu\big); \quad c^* = X^*\hat{\kappa}.$$

Note that the maximization in (3.1) can be understood as a local version of the optimization problem. Indeed, recalling (2.1), the right hand side of (3.1) is the maximum of a single function over certain points $(k, y) \in \mathbb{R}_+ \times \mathbb{R}^d$ that correspond to the admissible controls (κ, π). Moreover, optimal controls are related to maximizers of this function, a characteristic feature of any dynamic programming equation. The maximum of g is not explicit due to the jumps of R; this simplifies in the continuous case considered in Section III.3.3 below. Some mathematical comments are also in order.

Remark 3.3. (i) The random function g is well defined on $\mathscr{C}^0$ in the extended sense (see Lemma 6.2) and it does not depend on the choice of the cut-off function h by [34, II.2.25].

(ii) For $p < 0$ we have a more precise statement: Given $\pi^* \in L(R)$ and $\hat{\kappa}$ as in (3.3), $(\pi^*, \hat{\kappa})$ is optimal *if and only if* π^* takes values in $\mathscr{C} \cap \mathscr{C}^0$ and maximizes g. This will follow from Corollary 5.4 applied to the triplet $(L, \pi^*, \hat{\kappa})$.

(iii) For $p \in (0, 1)$, partial results in this direction follow from Section III.5. The question is trivial for convex $\mathscr{C}$ by the next item.

(iv) If $\mathscr{C}$ is convex, $\arg\max_{\mathscr{C}\cap\mathscr{C}^0} g$ is unique in the sense that the difference of any two elements lies in $\mathscr{N}$ (see Lemma 6.3).

We split the proof of Theorem 3.2 into several steps; the plan is as follows. Let $(\pi, \kappa) \in \mathcal{A}^{fE}$ and denote $X = X(\pi, \kappa)$. We recall from Proposition 2.2 that

$$Z(\pi, \kappa) := L\tfrac{1}{p}X^p + \int U_s(\kappa_s X_s)\, \mu(ds)$$

is a supermartingale, and a martingale if and only if (π, κ) is optimal. Hence we shall calculate its drift rate and then maximize over (π, κ); the maximum will be attained at any optimal strategy. This is fairly straightforward and essentially the content of Lemma 3.7 below. In the Bellman equation, we maximize over a subset of $\mathbb{R}^d$ for each (ω, t) and not over a set of strategies. This final step is a measurable selection problem and its solution will be the second part of the proof.

Lemma 3.4. *Let* $(\pi, \kappa) \in \mathcal{A}^f$. *The drift rate of* $Z(\pi, \kappa)$ *is*

$$a^{Z(\pi,\kappa)} = X(\pi, \kappa)^p_- \big(p^{-1}a^L + f(\kappa)\tfrac{d\mu}{dA} + g(\pi)\big) \in [-\infty, \infty),$$

where $f_t(k) := U_t(k) - L_{t-}k$ *and* g *is given by* (3.2). *Moreover,* $a^{Z(\hat{\pi},\hat{\kappa})} = 0$, *and* $a^{Z(\pi,\kappa)} \in (-\infty, 0]$ *for* $(\pi, \kappa) \in \mathcal{A}^{fE}$.

Proof. We can assume that the initial capital is $x_0 = 1$. Let $(\pi,\kappa) \in \mathcal{A}^f$, then in particular $Z := Z(\pi,\kappa)$ is finite. We also set $X := X(\pi,\kappa)$. By Itô's formula, we have $X^p = \mathcal{E}(\pi \bullet R - \kappa \bullet \mu)^p = \mathcal{E}(Y)$ with

$$Y = p(\pi \bullet R - \kappa \bullet \mu) + \tfrac{p(p-1)}{2}\pi^\top c^R \pi \bullet A + \big\{(1+\pi^\top x)^p - 1 - p\pi^\top x\big\} * \mu^R.$$

Integrating by parts in the definition of Z and using $X_s = X_{s-}$ $\mu(ds)$-a.e. (path-by-path), we have $X_-^{-p} \bullet Z = p^{-1}(L - L_0 + L_- \bullet Y + [L,Y]) + U(\kappa) \bullet \mu$. Here

$$\begin{aligned}[L,Y] &= [L^c, Y^c] + \sum \Delta L \Delta Y \\ &= p\pi^\top c^{RL} \bullet A + p x' \pi^\top x * \mu^{R,L} + x'\big\{(1+\pi^\top x)^p - 1 - p\pi^\top x\big\} * \mu^{R,L}.\end{aligned}$$

Thus $X_-^{-p} \bullet Z$ equals

$$\begin{aligned}&p^{-1}(L - L_0) + L_-\pi \bullet R + f(\kappa) \bullet \mu + L_- \tfrac{(p-1)}{2}\pi^\top c^R \pi \bullet A + \pi^\top c^{RL} \bullet A \\ &+ x'\pi^\top x * \mu^{R,L} + (L_- + x')\big\{p^{-1}(1+\pi^\top x)^p - p^{-1} - \pi^\top x\big\} * \mu^{R,L}.\end{aligned}$$

Writing $x = h(x) + x - h(x)$ and $\bar{R} = R - (x - h(x)) * \mu^R$ as in (2.4),

$$\begin{aligned}&X_-^{-p} \bullet Z = \qquad (3.5)\\ &p^{-1}(L - L_0) + L_-\pi \bullet \bar{R} + f(\kappa) \bullet \mu + L_-\pi^\top\big(\tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2}c^R\pi\big) \bullet A \\ &+ x'\pi^\top h(x) * \mu^{R,L} + (L_- + x')\big\{p^{-1}(1+\pi^\top x)^p - p^{-1} - \pi^\top h(x)\big\} * \mu^{R,L}.\end{aligned}$$

Since π need not be locally bounded, we use from now on a predictable cut-off function h such that $\pi^\top h(x)$ is bounded, e.g., $h(x) = x 1_{\{|x|\leq 1\}\cap\{|\pi^\top x|\leq 1\}}$. Then the compensator of $x'\pi^\top h(x) * \mu^{R,L}$ exists, since L is special.

Let $(\pi,\kappa) \in \mathcal{A}^{fE}$. Then the compensator of the last integral in the right hand side of (3.5) also exists; indeed, all other terms in that equality are special, since Z is a supermartingale. The drift rate can now be read from (3.5) and (2.4), and it is nonpositive by the supermartingale property. The drift rate vanishes for the optimal $(\hat{\pi},\hat{\kappa})$ by the martingale condition from Proposition 2.2.

Now consider $(\pi,\kappa) \in \mathcal{A}^f \setminus \mathcal{A}^{fE}$. Note that necessarily $p < 0$ (otherwise $\mathcal{A}^f = \mathcal{A}^{fE}$). Thus $Z \leq 0$, so by Remark 2.3 the drift rate a^Z is well defined with values in $[-\infty,\infty)$—alternatively, this can also be read from the integrals in (3.5) via (2.5). Using directly the definition of a^Z, we find the same formula for a^Z is as above. □

We do not have the supermartingale property for $(\pi,\kappa) \in \mathcal{A}^f \setminus \mathcal{A}^{fE}$, so it is not evident that $a^{Z(\pi,\kappa)} \leq 0$ in that case. However, we have the following

Lemma 3.5. *Let $(\pi,\kappa) \in \mathcal{A}^f$. Then $a^Z(\pi,\kappa) \in [0,\infty]$ implies $a^Z(\pi,\kappa) = 0$.*

Proof. Denote $Z = Z(\pi,\kappa)$. For $p>0$ we have $\mathcal{A}^f = \mathcal{A}^{fE}$ and the claim is immediate from Lemma 3.4. Let $p<0$. Then $Z\leq 0$ and by Lemma 2.4(iii), $a^Z \in [0,\infty]$ implies that Z is a submartingale . Hence $E[Z_T] = E\big[\int_0^T U_t(\kappa_t X_t(\pi,\kappa))\,\mu^\circ(dt)\big] > -\infty$, that is, $(\pi,\kappa)\in\mathcal{A}^{fE}$. Now Lemma 3.4 yields $a^Z(\pi,\kappa)\leq 0$. □

We observe in Lemma 3.4 that the drift rate splits into separate functions involving κ and π, respectively. For this reason, we can single out the

Proof of the consumption formula (3.3). Let $(\pi,\kappa)\in\mathcal{A}$. Note the following feature of our parametrization: we have $(\pi,\kappa^*)\in\mathcal{A}$ for *any* nonnegative optional process κ^* such that $\int_0^T \kappa^*_s\,\mu(ds)<\infty$ and $\kappa^*_T = 1$. Indeed, $X(\pi,\kappa) = x_0\mathcal{E}(\pi\bullet R - \kappa\bullet\mu)$ is positive by assumption. As μ is continuous, $X(\pi,\kappa^*) = x_0\mathcal{E}(\pi\bullet R-\kappa^*\bullet\mu)$ is also positive.

In particular, let $(\hat\pi,\hat\kappa)$ be optimal, $\beta=(1-p)^{-1}$ and $\kappa^* = (D/L)^\beta$; then $(\hat\pi,\kappa^*)\in\mathcal{A}$. In fact the paths of $U(\kappa^* X(\hat\pi,\kappa^*)) = p^{-1}D^{\beta p+1}X(\hat\pi,\kappa^*)^p L^{-\beta p}$ are bounded P-a.s. (because the processes are càdlàg; $L, L_- > 0$ and $\beta p+1 = \beta>0$) so that $(\hat\pi,\kappa^*)\in\mathcal{A}^f$.

Note that $P\otimes\mu$-a.e., we have $\kappa^* = (D/L_-)^\beta = \arg\max_{k\geq 0} f(k)$, hence $f(\kappa^*)\geq f(\hat\kappa)$. Suppose $(P\otimes\mu)\{f(\kappa^*)>f(\hat\kappa)\}>0$, then the formula from Lemma 3.4 and $a^{Z(\hat\pi,\hat\kappa)}=0$ imply $a^{Z(\hat\pi,\kappa^*)}\geq 0$ and $(P\otimes A)\{a^{Z(\hat\pi,\kappa^*)}>0\}>0$, a contradiction to Lemma 3.5. It follows that $\hat\kappa = \kappa^*$ $P\otimes\mu$-a.e. since f has a unique maximum. □

Remark 3.6. The previous proof does not use the assumptions (C1)-(C3).

Lemma 3.7. *Let π be a predictable process with values in $\mathscr{C}\cap\mathscr{C}^{0,*}$. Then*

$$(P\otimes A)\big\{g(\hat\pi)<g(\pi)\big\} = 0.$$

Proof. We argue by contradiction and assume $(P\otimes A)\{g(\hat\pi)<g(\pi)\}>0$. By redefining π, we may assume that $\pi=\hat\pi$ on the complement of this predictable set. Then

$$g(\hat\pi)\leq g(\pi) \quad\text{and}\quad (P\otimes A)\{g(\hat\pi)<g(\pi)\}>0. \tag{3.6}$$

As π is σ-bounded, we can find a constant $C>0$ such that the process $\tilde\pi := \pi 1_{|\pi|\leq C} + \hat\pi 1_{|\pi|>C}$ again satisfies (3.6); that is, we may assume that π is R-integrable. Since $\pi\in\mathscr{C}\cap\mathscr{C}^{0,*}$, this implies $(\pi,\hat\kappa)\in\mathcal{A}$ (as observed above, the consumption $\hat\kappa$ plays no role here). The contradiction follows as in the previous proof. □

In view of Lemma 3.7, the main task will be to construct a *measurable* maximizing sequence for g.

Lemma 3.8. *Under Assumptions 3.1, there exists a sequence* (π^n) *of predictable* $\mathscr{C} \cap \mathscr{C}^{0,*}$*-valued processes such that*

$$\limsup_n g(\pi^n) = \sup_{\mathscr{C}\cap\mathscr{C}^0} g \quad P \otimes A\text{-a.e.}$$

We defer the proof of this lemma to Appendix III.6, together with the study of the properties of g. The theorem can then be proved as follows.

Proof of Theorem 3.2. Let π^n be as in Lemma 3.8. Then Lemma 3.7 with $\pi = \pi^n$ yields $g(\hat{\pi}) = \sup_{\mathscr{C}\cap\mathscr{C}^0} g$, which is (3.4). By Lemma 3.4 we have $0 = a^{Z(\hat{\pi},\hat{\kappa})} = p^{-1}a^L + f(\hat{\kappa})\frac{d\mu}{dA} + g(\hat{\pi})$. This is (3.1) as $f(\hat{\kappa}) = U^*(L_-)$ $P \otimes \mu$-a.e. due to (3.3). $\square$

III.3.2 Bellman Equation as BSDE

In this section we express the Bellman equation as a BSDE. The unique orthogonal decomposition of the local martingale M^L with respect to R (cf. [34, III.4.24]) leads to the representation

$$L = L_0 + A^L + \varphi^L \bullet R^c + W^L * (\mu^R - \nu^R) + N^L, \tag{3.7}$$

where, using the notation of [34], $\varphi^L \in L^2_{loc}(R^c)$, $W^L \in G_{loc}(\mu^R)$, and N^L is a local martingale such that $\langle (N^L)^c, R^c\rangle = 0$ and $M^P_{\mu^R}(\Delta N^L|\widetilde{\mathcal{P}}) = 0$. The last statement means that $E[(V\Delta N^L)*\mu^R_T] = 0$ for any sufficiently integrable predictable function $V = V(\omega, t, x)$. We also introduce

$$\widehat{W}^L_t := \int_{\mathbb{R}^d} W^L(t,x)\, \nu^R(\{t\} \times dx),$$

then $\Delta\big(W^L * (\mu^R - \nu^R)\big) = W^L(\Delta R)1_{\{\Delta R \neq 0\}} - \widehat{W}^L$ by definition of the purely discontinuous local martingale $W^L * (\mu^R - \nu^R)$ and we can write

$$\Delta L = \Delta A^L + W^L(\Delta R)1_{\{\Delta R\neq 0\}} - \widehat{W}^L + \Delta N^L.$$

We recall that Assumptions 3.1 are in force. Now (3.1) can be restated as follows, the random function g being the same as before but in new notation.

Corollary 3.9. *The opportunity process L and the processes defined by* (3.7) *satisfy the BSDE*

$$L = L_0 - pU^*(L_-) \bullet \mu - p \max_{y\in\mathscr{C}\cap\mathscr{C}^0} g(y) \bullet A + \varphi^L \bullet R^c + W^L * (\mu^R - \nu^R) + N^L \tag{3.8}$$

with terminal condition $L_T = D_T$, where g is given by

$$g(y) :=$$

$$L_- y^\top \Big(b^R + c^R\big(\tfrac{\varphi^L}{L_-} + \tfrac{(p-1)}{2}y\big)\Big) + \int_{\mathbb{R}^d} \big(\Delta A^L + W^L(x) - \widehat{W}^L\big) y^\top h(x)\, F^R(dx)$$

$$+ \int_{\mathbb{R}^d} \big(L_- + \Delta A^L + W^L(x) - \widehat{W}^L\big)\big\{p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx).$$

We observe that the orthogonal part N^L plays a minor role here. In a suitable setting, it is linked to the "dual problem"; see Remark 5.18.

It is possible (but notationally more cumbersome) to prove a version of Lemma 3.4 using g as in Corollary 3.9 and the decomposition (3.7), thus involving only the characteristics of R instead of the joint characteristics of (R,L). Using this approach, we see that the increasing process A in the BSDE can be chosen based on R and without reference to L. This is desirable if we want to consider other solutions of the equation, as in Section III.4. One consequence is that A can be chosen to be continuous if and only if R is quasi left continuous (cf. [34, II.2.9]). Since $p^{-1}A^L = -f(\hat{\kappa}) \bullet \mu - g(\hat{\pi}) \bullet A$, $\mathrm{Var}(A^L)$ is absolutely continuous with respect to $A + \mu$, and we conclude:

Remark 3.10. If R is quasi left continuous, A^L is continuous.

If R is quasi left continuous, $\nu^R(\{t\} \times \mathbb{R}^d) = 0$ for all t by [34, II.1.19], hence $\widehat{W}^L = 0$ and we have the simpler formula

$$\begin{aligned} g(y) = L_- y^\top \Big(b^R + c^R\big(\tfrac{\varphi^L}{L_-} + \tfrac{(p-1)}{2} y\big)\Big) + \int_{\mathbb{R}^d} W^L(x) y^\top h(x)\, F^R(dx) \\ + \int_{\mathbb{R}^d} \big(L_- + W^L(x)\big)\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx). \end{aligned}$$

III.3.3 The Case of Continuous Prices

In this section we specialize the previous results to the case where R is a continuous semimartingale and mild additional conditions are satisfied. As usual in this setting, the martingale part of R will be denoted by M rather than R^c. In addition to Assumptions 3.1, the following conditions are in force for the present Section III.3.3.

Assumptions 3.11.

(i) R is continuous,

(ii) $R = M + \int d\langle M\rangle \lambda$ for some $\lambda \in L^2_{loc}(M)$ (*structure condition*),

(iii) the orthogonal projection of $\mathscr{C}$ onto $\mathscr{N}^\perp$ is closed.

Note that $\mathscr{C}^{0,*} = \mathbb{R}^d$ due to (i), in particular (C3) is void. When R is continuous, it necessarily satisfies (ii) when a no-arbitrage property holds; see Schweizer [71]. By (i) and (ii) we can write the differential characteristics of R with respect to, e.g., $A_t := t + \sum_{i=1}^d \langle M^i\rangle_t$. It will be convenient to factorize $c^R = \sigma\sigma^\top$, where σ is a predictable matrix-valued process; hence $\sigma\sigma^\top\, dA = d\langle M\rangle$. Then (ii) implies $\mathscr{N} = \ker \sigma^\top$ because $\sigma\sigma^\top y = 0$ implies $(\sigma^\top y)^\top(\sigma^\top y) = 0$. Since $\sigma^\top : \ker(\sigma^\top)^\perp \to \sigma^\top\mathbb{R}^d$ is a homeomorphism, we see that (iii) is equivalent to

$$\sigma^\top \mathscr{C} \quad \text{is closed.}$$

This condition depends on the semimartingale R. It is equivalent to the closedness of $\mathscr{C}$ itself if σ has full rank. For certain constraint sets (e.g., closed polyhedral or compact) the condition is satisfied for *all* matrices σ, but not so, e.g., for non-polyhedral cone constraints. We mention that violation of (iii) leads to nonexistence of optimal strategies in simple examples (cf. Example IV.3.5) and we refer to Czichowsky and Schweizer [14] for background.

Under (i), (3.7) is the more usual Kunita-Watanabe decomposition

$$L = L_0 + A^L + \varphi^L \bullet M + N^L,$$

where $\varphi^L \in L^2_{loc}(M)$ and N^L is a local martingale such that $[M, N^L] = 0$; see Ansel and Stricker [2, *cas* 3]. If $\emptyset \neq K \subseteq \mathbb{R}^d$ is a closed set, we denote the Euclidean distance to K by $d_K(x) = \min\{|x-y| : y \in K\}$, and d_K^2 is the squared distance. We also define the (set-valued) projection Π^K which maps $x \in \mathbb{R}^d$ to the points in K with minimal distance to x,

$$\Pi^K(x) = \big\{y \in K : |x-y| = d_K(x)\big\} \neq \emptyset.$$

If K is convex, Π^K is the usual (single-valued) Euclidean projection. In the present continuous setting, the random function g simplifies considerably:

$$g(y) = L_- y^\top \sigma\sigma^\top \Big(\lambda + \frac{\varphi^L}{L_-} + \tfrac{p-1}{2} y\Big) \tag{3.9}$$

and so the Bellman BSDE becomes more explicit.

Corollary 3.12. *Any optimal trading strategy* π^* *satisfies*

$$\sigma^\top \pi^* \in \Pi^{\sigma^\top \mathscr{C}} \Big\{\sigma^\top (1-p)^{-1}\Big(\lambda + \frac{\varphi^L}{L_-}\Big)\Big\}.$$

The opportunity process satisfies the BSDE

$$L = L_0 - pU^*(L_-) \bullet \mu + F(L_-, \varphi^L) \bullet A + \varphi^L \bullet M + N^L; \quad L_T = D_T,$$

where

$$F(L_-, \varphi^L) = \\ \frac{1}{2} L_- \Big\{p(1-p) d^2_{\sigma^\top \mathscr{C}}\Big(\sigma^\top (1-p)^{-1}\Big(\lambda + \frac{\varphi^L}{L_-}\Big)\Big) + \tfrac{p}{p-1}\Big|\sigma^\top\Big(\lambda + \frac{\varphi^L}{L_-}\Big)\Big|^2\Big\}.$$

If $\mathscr{C}$ *is a convex cone,* $F(L_-, \varphi^L) = \frac{p}{2(p-1)} L_- \big|\Pi^{\sigma^\top \mathscr{C}}\big\{\sigma^\top\big(\lambda + \frac{\varphi^L}{L_-}\big)\big\}\big|^2$. *If* $\mathscr{C} = \mathbb{R}^d$, *then* $F(L_-, \varphi^L) \bullet A = \frac{p}{2(p-1)} \int L_- \big(\lambda + \frac{\varphi^L}{L_-}\big)^\top d\langle M\rangle \big(\lambda + \frac{\varphi^L}{L_-}\big)$ *and the unique (mod.* $\mathscr{N}$*) optimal trading strategy is* $\pi^* = (1-p)^{-1}\big(\lambda + \frac{\varphi^L}{L_-}\big)$.

Proof. Let $\beta = (1-p)^{-1}$. We obtain $\sigma^\top(\arg\max_{\mathscr{C}} g) = \Pi^{\sigma^\top\mathscr{C}}\{\sigma^\top\beta(\lambda+\frac{\varphi^L}{L_-})\}$ by completing the square in (3.9), moreover, for any $\pi^* \in \arg\max_{\mathscr{C}} g$,

$$g(\pi^*) = \tfrac{1}{2}L_-\Big\{\beta\Big(\lambda+\frac{\varphi^L}{L_-}\Big)^\top\sigma\sigma^\top\Big(\lambda+\frac{\varphi^L}{L_-}\Big) - \beta^{-1}d^2_{\sigma^\top\mathscr{C}}\Big(\sigma^\top\beta\Big(\lambda+\frac{\varphi^L}{L_-}\Big)\Big)\Big\}.$$

In the case where $\mathscr{C}$, and hence $\sigma^\top\mathscr{C}$, is a convex cone, $\Pi := \Pi^{\sigma^\top\mathscr{C}}$ is single-valued, positively homogeneous, and Πx is orthogonal to $x - \Pi x$ for any $x \in \mathbb{R}^d$. Writing $\Psi := \sigma^\top(\lambda + \frac{\varphi^L}{L_-})$ we get $g(\pi^*) = L_-\beta(\Pi\Psi)^\top(\Psi - \frac{1}{2}\Pi\Psi) = L_-\frac{1}{2}\beta(\Pi\Psi)^\top(\Pi\Psi)$. Finally, $\Pi\Psi = \Psi$ if $\mathscr{C} = \mathbb{R}^d$. The result follows from Corollary 3.9. □

Of course the consumption formula (3.3) and Remark 3.3 still apply. We remark that the BSDE for the unconstrained case $\mathscr{C} = \mathbb{R}^d$ (and with $\mu = 0$, $D = 1$) was previously obtained in [54] in a similar spirit. A variant of the constrained BSDE for an Itô process model (and $\mu = 0$, $D = 1$) appears in [33], where a converse approach is taken: the equation is derived only formally and then existence results for BSDEs are employed together with a verification argument. We shall extend that result in Section III.5 (Example 5.8) when we study verification.

If L is continuous, the BSDE of Corollary 3.12 simplifies if it is stated for $\log(L)$ rather than L, but in general the given form is more convenient as the jumps are "hidden" in N^L.

Remark 3.13. (i) Continuity of R does not imply that L is continuous. For instance, in the Itô process model of Barndorff-Nielsen and Shephard [3] with Lévy driven coefficients, the opportunity process is not continuous. See, e.g., Theorem 3.3 and the subsequent remark in Kallsen and Muhle-Karbe [40]. If R satisfies the structure condition and the filtration $\mathbb{F}$ is continuous, it clearly follows that L is continuous. Here $\mathbb{F}$ is called continuous if all $\mathbb{F}$-martingales are continuous, as, e.g., for the Brownian filtration. In general, L is related to the predictable characteristics of the asset returns rather than their levels. As an example, Lévy models have jumps but constant characteristics; here L turns out to be a smooth function (see Chapter IV).

(ii) In the present setting we see that F has quadratic growth in φ^L, so that the Bellman equation is a "quadratic BSDE" (see also Example 5.8). In general, F does not satisfy the bounds which are usually assumed in the theory of such BSDEs. Together with existence results for the utility maximization problem (see the citations from the introduction), the Bellman equation yields various examples of BSDEs with the opportunity process as a solution. This includes terminal conditions D_T which are integrable and unbounded (see also Remark II.2.4).

III.4 Minimality of the Opportunity Process

This section considers the Bellman equation as such, having possibly many solutions, and we characterize the opportunity process as the minimal solution. As mentioned above, it seems more natural to use the BSDE formulation for this purpose (but see Remark 4.4). We first have to clarify what we mean by a solution of the BSDE. We consider R and A as given. Since the finite variation part in the BSDE is predictable, a solution will certainly be a special semimartingale. If ℓ is any special semimartingale, there exists a unique orthogonal decomposition [34, III.4.24]

$$\ell = \ell_0 + A^\ell + \varphi^\ell \bullet R^c + W^\ell * (\mu^R - \nu^R) + N^\ell, \tag{4.1}$$

using the same notation as in (3.7). These processes are unique in the sense that the integrals are uniquely determined, and so it suffices to consider the left hand side of the BSDE for the notion of a solution. (In BSDE theory, a solution would be, at least, a quadruple.) We define the random function g^ℓ as in Corollary 3.9, with L replaced by ℓ. Since ℓ is special, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}} (|x|^2 + |x'|^2) \wedge (1 + |x'|)\ F^{R,\ell}(d(x,x')) < \infty \tag{4.2}$$

and the arguments from Lemma 6.2 show that g^ℓ is well defined on $\mathscr{C}^0$ with values in $\mathbb{R} \cup \{\operatorname{sign}(p)\infty\}$. Hence we can consider (formally at first) the BSDE (3.8) with L replaced by ℓ, i.e.,

$$\ell = \ell_0 - pU^*(\ell_-) \bullet \mu - p \max_{y \in \mathscr{C} \cap \mathscr{C}^0} g^\ell(y) \bullet A + \varphi^\ell \bullet R^c + W^\ell * (\mu^R - \nu^R) + N^\ell \tag{4.3}$$

with terminal condition $\ell_T = D_T$.

Definition 4.1. A càdlàg special semimartingale ℓ is called a *solution of the Bellman equation* (4.3) if

- $\ell, \ell_- > 0$,
- there exists a $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process $\check{\pi} \in L(R)$ such that $g^\ell(\check{\pi}) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g^\ell < \infty$,
- ℓ and the processes from (4.1) satisfy (4.3) with $\ell_T = D_T$.

Moreover, we define $\check{\kappa} := (D/\ell)^\beta$, where $\beta = (1-p)^{-1}$. We call $(\check{\pi}, \check{\kappa})$ the strategy associated with ℓ, and for brevity, we also call $(\ell, \check{\pi}, \check{\kappa})$ a solution.

If the process $\check{\pi}$ is not unique, we choose and fix one. The assumption $\ell > 0$ excludes pathological cases where ℓ jumps to zero and becomes positive immediately afterwards and thereby ensures that $\check{\kappa}$ is admissible. More precisely, the following holds.

Remark 4.2. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation.

(i) $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^{fE}$.

(ii) $\sup_{\mathscr{C}\cap\mathscr{C}^0} g^\ell$ is a predictable, A-integrable process.

(iii) If $p \in (0,1)$, g^ℓ is finite on $\mathscr{C}\cap\mathscr{C}^0$.

(iv) The condition $\ell > 0$ is automatically satisfied if (a) $p \in (0,1)$ or if (b) $p < 0$ and there is no intermediate consumption and Assumptions 3.1 are satisfied.

Proof. (i) We have $\int_0^T \check{\kappa}_s\,\mu(ds) < \infty$ P-a.s. since the paths of ℓ are bounded away from zero. Moreover, $\int_0^T U_t(\check{\kappa}_t X_t(\check{\pi}, \check{\kappa}))\,\mu(dt) < \infty$ as in the proof of (3.3) (stated after Lemma 3.5). This shows $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^f$. The fact that $(\check{\pi}, \check{\kappa}) \in \mathcal{A}^{fE}$ is contained in the proof of Lemma 4.9 below.

(ii) We have $0 = g^\ell(0) \leq \sup_{\mathscr{C}\cap\mathscr{C}^0} g^\ell = g^\ell(\check{\pi})$. Hence $\sup_{\mathscr{C}\cap\mathscr{C}^0} g^\ell \bullet A$ is well defined, and it is finite because otherwise (4.3) could not hold.

(iii) Note that $p > 0$ implies $g^\ell > -\infty$ by its definition and (4.2), while $g^\ell < \infty$ by assumption.

(iv) If $p > 0$, (4.3) states that A^ℓ is decreasing. As $\ell_- > 0$ implies $\ell \geq 0$, ℓ is a supermartingale by Lemma 2.4. Since $\ell_T = D_T > 0$, the minimum principle for nonnegative supermartingales shows $\ell > 0$. Under (b) the assertion is a consequence of Theorem 4.5 below (which shows $\ell \geq L > 0$) upon noting that the condition $\ell > 0$ is not used in its proof when there is no intermediate consumption. □

It may seem debatable to make existence of the maximizer $\check{\pi}$ part of the definition of a solution. However, associating a control with the solution is crucial for the following theory. Some justification is given by the following result for the continuous case (where $\mathscr{C}^{0,*} = \mathbb{R}^d$).

Proposition 4.3. *Let ℓ be any càdlàg special semimartingale such that $\ell, \ell_- > 0$. Under Assumptions 3.11, (C1) and (C2), there exists a $\mathscr{C}\cap\mathscr{C}^{0,*}$-valued predictable process $\check{\pi}$ such that $g^\ell(\check{\pi}) = \sup_{\mathscr{C}\cap\mathscr{C}^0} g^\ell < \infty$, and any such process is R-integrable.*

Proof. As g^ℓ is analogous to (3.9), it is continuous and its supremum over $\mathbb{R}^d$ is finite. By continuity of R and the structure condition, $\pi \in L(R)$ if and only if $\int_0^T \pi^\top\,d\langle M\rangle\pi = \int_0^T |\sigma^\top\pi|^2\,dA < \infty$ P-a.s.

Assume first that $\mathscr{C}$ is compact, then Lemma 6.4 yields a measurable selector π for $\arg\max_{\mathscr{C}} g$. As in the proof of Corollary 3.12, $\sigma^\top\pi \in \Pi^{\sigma^\top\mathscr{C}}\sigma^\top\psi$ for $\psi := \beta\big(\lambda + \frac{\varphi^\ell}{\ell_-}\big)$, which satisfies $\int_0^T |\sigma^\top\psi|^2\,dA < \infty$ by definition of λ and φ^ℓ. We note that $|\sigma^\top\pi| \leq |\sigma^\top\psi| + |\sigma^\top\pi - \sigma^\top\psi| \leq 2|\sigma^\top\psi|$ due to the definition of the projection and $0 \in \mathscr{C}$.

In the general case we approximate $\mathscr{C}$ by a sequence of compact constraints $\mathscr{C}^n := \mathscr{C} \cap \{x \in \mathbb{R}^d : |x| \leq n\}$, each of which yields a selector π^n for $\arg\max_{\mathscr{C}^n} g$. By the above, $|\sigma^\top\pi^n| \leq 2|\sigma^\top\psi|$, so the sequence $(\sigma^\top\pi^n)_n$

is bounded for fixed (ω, t). A random index argument as in the proof of Lemma 6.4 yields a selector ϑ for a cluster point of this sequence. We have $\vartheta \in \sigma^\top \mathscr{C}$ by closedness of this set and we find a selector $\check{\pi}$ for the preimage $((\sigma^\top)^{-1}\vartheta) \cap \mathscr{C}$ using [64, 1Q]. We have $\check{\pi} \in \arg\max_{\mathscr{C}} g$ as the sets $\mathscr{C}^n$ increase to $\mathscr{C}$, and $\int_0^T |\sigma^\top \check{\pi}|^2 \, dA \leq 2 \int_0^T |\sigma^\top \psi|^2 \, dA < \infty$ shows $\check{\pi} \in L(R)$. $\square$

Another example for the construction of $\check{\pi}$ is given in Chapter IV. In general, two ingredients are needed: Existence of a maximizer for fixed (ω, t) will typically require a compactness condition in the form of a no-arbitrage assumption (in the previous proof, this is the structure condition). Moreover, a measurable selection is required; here the techniques from the appendices may be useful.

Remark 4.4. The BSDE formulation of the Bellman equation has the advantage that we can choose A based on R and speak about the class of all solutions. However, we do not want to write proofs in this cumbersome notation. Once we fix a solution ℓ (and maybe L, and finitely many other semimartingales), we can choose a new reference process $\tilde{A} = A + A'$ (where A' is increasing), with respect to which our semimartingales admit differential characteristics; in particular we can use the joint characteristics $(b^{R,\ell}, c^{R,\ell}, F^{R,\ell}; \tilde{A})$. As we change A, all drift rates change in that they are multiplied by $d\tilde{A}/dA$, so any (in)equalities between them are preserved. With this in mind, we shall use the joint characteristics of (R, ℓ) in the sequel without further comment and treat the two formulations of the Bellman equation as equivalent.

Our definition of a solution of the Bellman equation is loose in terms of integrability assumptions. Even in the continuous case, it is unclear "how many" solutions exist. The next result shows that we can always identify L by taking the smallest one, i.e., $L \leq \ell$ for any solution ℓ.

Theorem 4.5. *Under Assumptions 3.1, the opportunity process L is characterized as the minimal solution of the Bellman equation.*

Remark 4.6. As a consequence, the Bellman equation has a bounded solution if and only if the opportunity process is bounded (and similarly for other integrability properties). In conjunction with Section II.4.2 this yields examples of quadratic BSDEs which have bounded terminal value (for D_T bounded), but no bounded solution.

The proof of Theorem 4.5 is based on the following result; it is the fundamental property of any Bellman equation.

Proposition 4.7. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. For any $(\pi, \kappa) \in \mathcal{A}^f$,*

$$Z(\pi, \kappa) := \ell \tfrac{1}{p} \big(X(\pi, \kappa)\big)^p + \int U_s\big(\kappa_s X_s(\pi, \kappa)\big) \, \mu(ds) \tag{4.4}$$

is a semimartingale with nonpositive drift rate. Moreover, $Z(\check{\pi},\check{\kappa})$ is a local martingale.

Proof. Let $(\pi,\kappa)\in\mathcal{A}^f$. Note that $Z:=Z(\pi,\kappa)$ satisfies $\operatorname{sign}(p)Z\geq 0$, hence has a well defined drift rate a^Z by Remark 2.3. The drift rate can be calculated as in Lemma 3.4: If f^ℓ is defined similarly to the function f in that lemma but with L replaced by ℓ, then

$$\begin{aligned}a^Z &= X(\pi,\kappa)^p_-\big\{p^{-1}a^\ell + f^\ell(\kappa)\,\tfrac{d\mu}{dA} + g^\ell(\pi)\big\}\\ &= X(\pi,\kappa)^p_-\big\{\big(f^\ell(\kappa)-f^\ell(\check{\kappa})\big)\,\tfrac{d\mu}{dA} + g^\ell(\pi) - g^\ell(\check{\pi})\big\}.\end{aligned}$$

This is nonpositive because $\check{\kappa}$ and $\check{\pi}$ maximize f^ℓ and g^ℓ. For the special case $(\pi,\kappa):=(\check{\pi},\check{\kappa})$ we have $a^Z=0$ and so Z is a σ-martingale, thus a local martingale as $\operatorname{sign}(p)Z\geq 0$. □

Remark 4.8. In Proposition 4.7, "semimartingale with nonpositive drift rate" can be replaced by "σ-supermartingale" if g^ℓ is finite on $\mathscr{C}\cap\mathscr{C}^0$.

Theorem 4.5 follows from the next lemma (which is actually stronger). We recall that for $p<0$ the opportunity process L can be defined without further assumptions.

Lemma 4.9. *Let ℓ be a solution of the Bellman equation. If $p<0$, then $L\leq\ell$. For $p\in(0,1)$, the same holds if* (2.2) *is satisfied and there exists an optimal strategy.*

Proof. Let $(\ell,\check{\pi},\check{\kappa})$ be a solution and define $Z(\pi,\kappa)$ as in (4.4).

Case $p<0$: We choose $(\pi,\kappa):=(\check{\pi},\check{\kappa})$. As $Z(\check{\pi},\check{\kappa})$ is a negative local martingale by Proposition 4.7, it is a submartingale. In particular, $E[Z_T(\check{\pi},\check{\kappa})]>-\infty$, and using $L_T=D_T$, this is the statement that the expected utility is finite, i.e., $(\check{\pi},\check{\kappa})\in\mathcal{A}^{fE}$—this completes the proof of Remark 4.2(i). Recall that $\mu^\circ=\mu+\delta_{\{T\}}$. With $\check{X}:=X(\check{\pi},\check{\kappa})$ and $\check{c}:=\check{\kappa}\check{X}$, and using $\ell_T=D_T=L_T$, we deduce

$$\begin{aligned}\ell_t\tfrac{1}{p}\check{X}_t^p + \int_0^t U_s(\check{c}_s)\,\mu(ds) &= Z_t(\check{\pi},\check{\kappa}) \leq E\big[Z_T(\check{\pi},\check{\kappa})\big|\mathcal{F}_t\big]\\ &\leq \operatorname{ess\,sup}_{\tilde{c}\in\mathcal{A}(\check{\pi},\check{c},t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] + \int_0^t U_s(\check{c}_s)\,\mu(ds)\\ &= L_t\tfrac{1}{p}\check{X}_t^p + \int_0^t U_s(\check{c}_s)\,\mu(ds),\end{aligned}$$

where the last equality holds by (2.3). As $\frac{1}{p}\check{X}_t^p<0$, we have $\ell_t\geq L_t$.

Case $p\in(0,1)$: We choose $(\pi,\kappa):=(\hat{\pi},\hat{\kappa})$ to be an optimal strategy. Then $Z(\hat{\pi},\hat{\kappa})\geq 0$ is a supermartingale by Proposition 4.7 and Lemma 2.4(iii), and we obtain

$$\ell_t \tfrac{1}{p} \widehat{X}_t^p + \int_0^t U_s(\hat{c}_s)\,\mu(ds) = Z_t(\hat{\pi},\hat{\kappa}) \geq E\big[Z_T(\hat{\pi},\hat{\kappa})\big|\mathcal{F}_t\big]$$
$$= E\Big[\int_0^T U_s(\hat{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] = L_t \tfrac{1}{p} \widehat{X}_t^p + \int_0^t U_s(\hat{c}_s)\,\mu(ds)$$

by the optimality of $(\hat{\pi},\hat{\kappa})$ and (2.3). More precisely, we have used the fact that $(\hat{\pi},\hat{\kappa})$ is also conditionally optimal (see Remark II.3.3). As $\frac{1}{p}\widehat{X}_t^p > 0$, we conclude $\ell_t \geq L_t$. □

III.5 Verification

Suppose that we have found a solution of the Bellman equation; then we want to know whether it is the opportunity process and whether the associated strategy is optimal. In applications, it might not be clear *a priori* that an optimal strategy exists or even that the utility maximization problem is finite. Therefore, we stress that in this section these properties are not assumed. Also, we do not need the assumptions on $\mathscr{C}$ made in Section III.2.4—they are not necessary because we start with a given solution.

Generally speaking, verification involves the candidate for an optimal control, $(\check{\pi},\check{\kappa})$ in our case, and all the competing ones. It is often very difficult to check a condition involving all these controls, so it is desirable to have a verification theorem whose assumptions involve only $(\check{\pi},\check{\kappa})$.

We present two verification approaches. The first one is via the value process and is classical for general dynamic programming: it uses little structure of the given problem. For $p \in (0,1)$, it yields the desired result. However, in a general setting, this is not the case for $p < 0$. The second approach uses the concavity of the utility function. To fully exploit this and make the verification conditions necessary, we will assume that $\mathscr{C}$ is convex. In this case, we shall obtain the desired verification theorem for all values of p.

III.5.1 Verification via the Value Process

The basis of this approach is the following simple result; we state it separately for better comparison with Lemma 5.10 below. In the entire section, $Z(\pi,\kappa)$ is defined by (4.4) whenever ℓ is given.

Lemma 5.1. *Let ℓ be any positive càdlàg semimartingale with $\ell_T = D_T$ and let $(\check{\pi},\check{\kappa}) \in \mathcal{A}$. Assume that for all $(\pi,\kappa) \in \mathcal{A}^{fE}$, the process $Z(\pi,\kappa)$ is a supermartingale. Then $Z(\check{\pi},\check{\kappa})$ is a martingale if and only if* (2.2) *holds* and $(\check{\pi},\check{\kappa})$ *is optimal* and $\ell = L$.

Proof. "⇒": Recall that $Z_0(\pi,\kappa) = \ell_0 \frac{1}{p} x_0^p$ does not depend on (π,κ) and that $E[Z_T(\pi,\kappa)] = E[\int_0^T U_t(\kappa_t(X_t(\pi,\kappa)))\,\mu^\circ(dt)]$ is the expected utility corresponding to (π,κ). With $\check{X} := X(\check{\pi},\check{\kappa})$, the (super)martingale condition implies that $E[\int_0^T U_t(\check{\kappa}_t\check{X}_t)\,\mu^\circ(dt)] \geq E[\int_0^T U_t(\kappa_t X_t(\pi,\kappa))\,\mu^\circ(dt)]$ for all $(\pi,\kappa)\in\mathcal{A}^{fE}$. Since for $(\pi,\kappa)\in\mathcal{A}\setminus\mathcal{A}^{fE}$ the expected utility is $-\infty$, this shows that $(\check{\pi},\check{\kappa})$ is optimal with $E[Z_T(\check{\pi},\check{\kappa})] = Z_0(\check{\pi},\check{\kappa}) = \ell_0\frac{1}{p}x_0^p < \infty$. In particular, the opportunity process L is well defined. By Proposition 2.2, $L\frac{1}{p}\check{X}^p + \int U_s(\check{c}_s)\,\mu(ds)$ is a martingale, and as its terminal value equals $Z_T(\check{\pi},\check{\kappa})$, we deduce $\ell = L$ by comparison with (4.4), using $\check{X} > 0$.

The converse is contained in Proposition 2.2. $\square$

We can now state our first verification theorem.

Theorem 5.2. *Let $(\ell,\check{\pi},\check{\kappa})$ be a solution of the Bellman equation.*

(i) If $p\in(0,1)$, the following are equivalent:
   - *(a) $Z(\check{\pi},\check{\kappa})$ is of class (D),*
   - *(b) $Z(\check{\pi},\check{\kappa})$ is a martingale,*
   - *(c) (2.2) holds and $(\check{\pi},\check{\kappa})$ is optimal and $\ell = L$.*

(ii) If $p<0$, the following are equivalent:
   - *(a) $Z(\pi,\kappa)$ is of class (D) for all $(\pi,\kappa)\in\mathcal{A}^{fE}$,*
   - *(b) $Z(\pi,\kappa)$ is a supermartingale for all $(\pi,\kappa)\in\mathcal{A}^{fE}$,*
   - *(c) $(\check{\pi},\check{\kappa})$ is optimal and $\ell = L$.*

Proof. When $p>0$ and $(\pi,\kappa)\in\mathcal{A}^f$, $Z(\pi,\kappa)$ is positive and $a^{Z(\pi,\kappa)}\leq 0$ by Proposition 4.7, hence $Z(\pi,\kappa)$ is a supermartingale according to Lemma 2.4. By Proposition 4.7, $Z(\check{\pi},\check{\kappa})$ is a local martingale, so it is a martingale if and only if it is of class (D). Lemma 5.1 implies the result.

If $p<0$, $Z(\pi,\kappa)$ is negative. Thus the local martingale $Z(\check{\pi},\check{\kappa})$ is a submartingale, and a martingale if and only if it is also a supermartingale. Note that a class (D) semimartingale with nonpositive drift rate is a supermartingale. Conversely, any negative supermartingale Z is of class (D) due to the bounds $0\geq Z\geq E[Z_T|\mathbb{F}]$. Lemma 5.1 implies the result after noting that if $\ell = L$, then Proposition 2.2 yields (b). $\square$

Theorem 5.2 is "as good as it gets" for $p>0$, but as announced, the result for $p<0$ is not satisfactory. In particular settings, this can be improved.

Remark 5.3 ($p<0$)**.** (i) Assume we know *a priori* that *if* there is an optimal strategy $(\hat{\pi},\hat{\kappa})\in\mathcal{A}$, then

$$(\hat{\pi},\hat{\kappa})\in\mathcal{A}^{(D)} := \big\{(\pi,\kappa)\in\mathcal{A}:\ X(\pi,\kappa)^p \text{ is of class (D)}\big\}.$$

In this case we can reduce our optimization problem to the class $\mathcal{A}^{(D)}$. If furthermore ℓ is bounded (which is not a strong assumption when $p<0$), the class (D) condition in Theorem 5.2(ii) is automatically satisfied for any $(\pi,\kappa)\in\mathcal{A}^{(D)}$. The verification then reduces to checking that $(\check{\pi},\check{\kappa})\in\mathcal{A}^{(D)}$.

(ii) How can we establish the condition needed for (i)? One possibility is to show that L is uniformly bounded away from zero; then the condition follows (see the argument in the next proof). Of course, L is not known when we try to apply this. However, Section II.4.2 gives verifiable conditions for L to be (bounded and) bounded away from zero. They are stated for the unconstrained case $\mathscr{C} = \mathbb{R}^d$, but can be used nevertheless: if $L^{\mathbb{R}^d}$ is the opportunity process corresponding to $\mathscr{C} = \mathbb{R}^d$, the actual L satisfies $L \geq L^{\mathbb{R}^d}$ because the supremum in (2.3) is taken over a smaller set in the constrained case.

In the situation where ℓ and L^{-1} are bounded, we can also use the following result. Note also its use in Remark 3.3(ii) and recall that $1/0 := \infty$.

Corollary 5.4. *Let $p < 0$ and let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. Let L be the opportunity process and assume that ℓ/L is uniformly bounded. Then $(\check{\pi}, \check{\kappa})$ is optimal and $\ell = L$.*

Proof. Fix arbitrary $(\pi, \kappa) \in \mathcal{A}^{fE}$ and let $X = X(\pi, \kappa)$. The process $L\frac{1}{p}\big(X(\pi,\kappa)\big)^p + \int U_s(\kappa_s X_s)\,\mu(ds)$ is a negative supermartingale by Proposition 2.2, hence of class (D). Since $\int U_s(\kappa_s X_s)\,\mu(ds)$ is decreasing and its terminal value is integrable (definition of $\mathcal{A}^{fE}$), $L\frac{1}{p}X^p$ is also of class (D). The assumption yields that $\ell\frac{1}{p}X^p$ is of class (D), and then so is $Z(\pi, \kappa)$. □

As bounded solutions are of special interest in BSDE theory, let us note the following consequence.

Corollary 5.5. *Let $p < 0$. Under Assumptions 3.1 the following are equivalent:*

(i) L is bounded and bounded away from zero.

(ii) There exists a unique bounded solution of the Bellman equation, and this solution is bounded away from zero.

One can note that in the setting of Section II.4.2, these conditions are further equivalent to a reverse Hölder inequality for the market model.

We give an illustration of Theorem 5.2 also for the case $p \in (0,1)$. Thus far, we have considered only the given exponent p and assumed (2.2). In many situations, there will exist some $p_0 \in (p, 1)$ such that, if we consider the exponent p_0 instead of p, the utility maximization problem is still finite. Note that by Jensen's inequality this is a stronger assumption. We define for $q_0 \geq 1$ the class of semimartingales ℓ bounded in $L^{q_0}(P)$,

$$\mathbf{B}(q_0) := \{\ell :\, \sup_\tau \|\ell_\tau\|_{L^{q_0}(P)} < \infty\},$$

where the supremum ranges over all stopping times τ.

Corollary 5.6. *Let $p \in (0,1)$ and let there be a constant $k_1 > 0$ such that $D \geq k_1$. Assume that the utility maximization problem is finite for some $p_0 \in (p,1)$ and let $q_0 \geq 1$ be such that $q_0 > p_0/(p_0 - p)$. If $(\ell, \check{\pi}, \check{\kappa})$ is a solution of the Bellman equation (for p) with $\ell \in \mathbf{B}(q_0)$, then $\ell = L$ and $(\check{\pi}, \check{\kappa})$ is optimal.*

Proof. Let $\ell \in \mathbf{B}(q_0)$ be a solution, $(\check{\pi}, \check{\kappa})$ the associated strategy, and $\check{X} = X(\check{\pi}, \check{\kappa})$. By Theorem 5.2 and an argument as in the previous proof, it suffices to show that $\ell \check{X}^p$ is of class (D). Let $\delta > 1$ be such that $\delta/q_0 + \delta p/p_0 = 1$. For every stopping time τ, Hölder's inequality yields

$$E[(\ell_\tau \check{X}_\tau^p)^\delta] = E[(\ell_\tau^{q_0})^{\delta/q_0}(\check{X}_\tau^{p_0})^{\delta p/p_0}] \leq E[\ell_\tau^{q_0}]^{\delta/q_0} E[\check{X}_\tau^{p_0}]^{\delta p/p_0}.$$

We show that this is bounded uniformly in τ; then $\{\ell_\tau \check{X}_\tau^p : \tau \text{ stopping time}\}$ is bounded in $L^\delta(P)$ and hence uniformly integrable. Indeed, $E[\ell_\tau^{q_0}]$ is bounded by assumption. The set of wealth processes corresponding to admissible strategies is stable under stopping. Therefore $E[D_T \frac{1}{p_0} \check{X}_\tau^{p_0}] \leq u^{(p_0)}(x_0)$, the value function for the utility maximization problem with exponent p_0. The result follows as $D_T \geq k_1$. □

Remark 5.7. In Example II.4.6 we give a condition which implies that the utility maximization problem is finite for *all* $p_0 \in (0,1)$. Conversely, given such a $p_0 \in (p,1)$, one can show that $L \in \mathbf{B}(p_0/p)$ if D is uniformly bounded from above (see Corollary V.4.2).

Example 5.8. We apply our results in an Itô model with bounded mean variance tradeoff process together with an existence result for BSDEs. For the case of utility from terminal wealth only, we retrieve (a minor generalization of) the pioneering result of [33, §3]; the case with intermediate consumption is new. Let W be an m-dimensional standard Brownian motion ($m \geq d$) and assume that $\mathbb{F}$ is generated by W. We consider

$$dR_t = b_t \, dt + \sigma_t \, dW_t,$$

where b is predictable $\mathbb{R}^d$-valued and σ is predictable $\mathbb{R}^{d \times m}$-valued with everywhere full rank; moreover, we consider constraints $\mathscr{C}$ satisfying (C1) and (C2). We are in the situation of Assumptions III.3.3 with $dM = \sigma \, dW$ and $\lambda = (\sigma\sigma^\top)^{-1} b$. The process $\theta := \sigma^\top \lambda$ is called *market price of risk*. We assume that there are constants $k_i > 0$ such that

$$0 < k_1 \leq D \leq k_2 \quad \text{and} \quad \int_0^T |\theta_s|^2 \, ds \leq k_3.$$

The latter condition is called *bounded mean-variance tradeoff*. Note that $dQ/dP = \mathcal{E}(-\lambda \bullet M)_T = \mathcal{E}(-\theta \bullet W)_T$ defines a local martingale measure for $\mathcal{E}(R)$. By Section II.4.2 the utility maximization problem is finite for all

p and the opportunity process L is bounded and bounded away from zero. It is continuous due to Remark 3.13(i).

As suggested above, we write the Bellman BSDE for $Y := \log(L)$ rather than L in this setting. If $Y = A^Y + \varphi^Y \bullet M + N^Y$ is the Kunita-Watanabe decomposition, we write $Z := \sigma^\top \varphi^Y$ and choose $Z^\perp$ such that $Z^\perp \bullet W = N^Y$ by Brownian representation. The orthogonality of the decomposition implies $\sigma^\top Z^\perp = 0$ and that $Z^\top Z^\perp = 0$. We write $\delta = 1$ if there is intermediate consumption and $\delta = 0$ otherwise. Then Itô's formula and Corollary 3.12 (with $A_t := t$) yield the BSDE

$$dY = f(Y, Z, Z^\perp)\,dt + (Z + Z^\perp)\,dW; \quad Y_T = \log(D_T) \tag{5.1}$$

with

$$\begin{aligned} f(Y, Z, Z^\perp) = \tfrac{1}{2}p(1-p)\, d^2_{\sigma^\top \mathscr{C}}\big(\beta(\theta + Z)\big) + \tfrac{q}{2}|\theta + Z|^2 \\ + \delta(p-1)D^\beta \exp\big((q-1)Y\big) - \tfrac{1}{2}(|Z|^2 + |Z^\perp|^2). \end{aligned}$$

Here $\beta = (1-p)^{-1}$ and $q = p/(p-1)$; the dependence on (ω, t) is suppressed in the notation. Using the orthogonality relations and $p(1-p)\beta^2 = -q$, one can check that $f(Y, Z, Z^\perp) = f(Y, Z+Z^\perp, 0) =: f(Y, \widetilde{Z})$, where $\widetilde{Z} := Z+Z^\perp$. As $0 \in \mathscr{C}$, we have $d^2_{\sigma^\top \mathscr{C}}(x) \leq |x|^2$. Hence there exist a constant $C > 0$ and an increasing continuous function ϕ such that

$$|f(y, \tilde{z})| \leq C\big(|\theta|^2 + \phi(y) + |\tilde{z}|^2\big).$$

The following monotonicity property handles the exponential nonlinearity caused by the consumption: As $p - 1 < 0$ and $q - 1 < 0$,

$$-y\big[f(y, \tilde{z}) - f(0, \tilde{z})\big] \leq 0.$$

Thus we have Briand and Hu's [9, Condition (A.1)] after noting that they call $-f$ what we call f, and [9, Lemma 2] states the existence of a bounded solution Y to the BSDE (5.1). Let us check that $\ell := \exp(Y)$ is the opportunity process. We define an associated strategy $(\check{\pi}, \check{\kappa})$ by $\check{\kappa} := (D/\ell)^\beta$ and Proposition 4.3; then we have a solution $(\ell, \check{\pi}, \check{\kappa})$ of the Bellman equation in the sense of Definition 4.1. For $p < 0$ ($p \in (0,1)$), Corollary 5.4 (Corollary 5.6) yields $\ell = L$ and the optimality of $(\check{\pi}, \check{\kappa})$. In fact, the same verification argument applies if we replace $\check{\pi}$ by any other predictable $\mathscr{C}$-valued π^* such that $\sigma^\top \pi^* \in \Pi^{\sigma^\top \mathscr{C}}\{\beta(\theta + Z)\}$; recall from Proposition 4.3 that $\pi^* \in L(R)$ automatically. To conclude: we have that

$$L = \exp(Y) \text{ is the opportunity process}$$

and the set of optimal strategies equals the set of all $(\pi^*, \hat{\kappa})$ such that

- $\hat{\kappa} = (D/L)^\beta$ μ°-a.e.
- π^* is predictable, $\mathscr{C}$-valued and $\sigma^\top \pi^* \in \Pi^{\sigma^\top \mathscr{C}}\{\beta(\theta + Z)\}$ $P \otimes dt$-a.e.

One can remark that the previous arguments show $Y' = \log(L)$ whenever Y' is a solution of the BSDE (5.1) which is uniformly bounded from above. Hence we have proved uniqueness for (5.1) in this class of solutions, which is not immediate from BSDE theory. One can also note that, in contrast to [33], we did not use the theory of BMO martingales in this example.

We close this section with a formula intended for future applications.

Remark 5.9. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. Sometimes exponential formulas can be used to verify that $Z(\check{\pi}, \check{\kappa})$ is of class (D).

Let h be a predictable cut-off function such that $\check{\pi}^\top h(x)$ is bounded, e.g., $h(x) = x1_{\{|x|\leq 1\}\cap\{|\check{\pi}^\top x|\leq 1\}}$, and define Ψ to be the local martingale

$$\ell_-^{-1} \bullet M^\ell + p\check{\pi} \bullet R^c + p\check{\pi}^\top h(x) * (\mu^R - \nu^R) + p(x'/\ell_-)\check{\pi}^\top h(x) * (\mu^{R,\ell} - \nu^{R,\ell})$$
$$+ (1 + x'/\ell_-)\big\{(1 + \check{\pi}^\top x)^p - 1 - p\check{\pi}^\top h(x)\big\} * (\mu^{R,\ell} - \nu^{R,\ell}).$$

Then $\mathcal{E}(\Psi) > 0$, and if $\mathcal{E}(\Psi)$ is of class (D), then $Z(\check{\pi}, \check{\kappa})$ is also of class (D).

Proof. Let $Z = Z(\check{\pi}, \check{\kappa})$. By a calculation as in the proof of Lemma 3.4 and the local martingale condition from Proposition 4.7, $(\frac{1}{p}\check{X}_-^p)^{-1} \bullet Z = \ell_- \bullet \Psi$. Hence $Z = Z_0\mathcal{E}(\Psi)$ in the case without intermediate consumption. For the general case, we have seen in the proof of Corollary 5.4 that Z is of class (D) whenever $\ell\frac{1}{p}\check{X}^p$ is. Writing the definition of $\check{\kappa}$ as $\check{\kappa}^{p-1} = \ell_-/D$ μ-a.e., we have $\ell\frac{1}{p}\check{X}^p = Z - \int \check{\kappa}\ell_- \frac{1}{p}\check{X}^p \, d\mu = (\ell_- \frac{1}{p}\check{X}_-^p) \bullet (\Psi - \check{\kappa} \bullet \mu)$, hence $\ell\frac{1}{p}\check{X}^p = Z_0\mathcal{E}(\Psi - \check{\kappa} \bullet \mu) = Z_0\mathcal{E}(\Psi)\exp(-\check{\kappa} \bullet \mu)$. It remains to note that $\exp(-\check{\kappa} \bullet \mu) \leq 1$. $\square$

III.5.2 Verification via Deflator

The goal of this section is a verification theorem which involves only the candidate for the optimal strategy and holds for general semimartingale models. Our plan is as follows. Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation and assume for the moment that $\mathscr{C}$ is convex. As the concave function g^ℓ has a maximum at $\check{\pi}$, the directional derivatives at $\check{\pi}$ in all directions should be nonpositive (if they can be defined). A calculation will show that, at the level of processes, this yields a supermartingale property which is well known from duality theory and allows for verification. In the case of non-convex constraints, the directional derivatives need not be defined in any sense. Nevertheless, the formally corresponding quantities yield the expected result. To make the first order conditions necessary, we later specialize to convex $\mathscr{C}$. As in the previous section, we first state a basic result; it is essentially classical.

Lemma 5.10. *Let ℓ be any positive càdlàg semimartingale with $\ell_T = D_T$. Suppose there exists $(\check{\pi}, \check{\kappa}) \in \mathcal{A}$ with $\check{\kappa} = (D/\ell)^\beta$ and let $\check{X} := X(\check{\pi}, \check{\kappa})$. Assume $Y := \ell\check{X}^{p-1}$ has the property that for all $(\pi, \kappa) \in \mathcal{A}$,*

$$\Gamma(\pi, \kappa) := X(\pi, \kappa)Y + \int \kappa_s X_s(\pi, \kappa) Y_s \, \mu(ds)$$

is a supermartingale. Then $\Gamma(\check\pi,\check\kappa)$ *is a martingale if and only if* (2.2) *holds and* $(\check\pi,\check\kappa)$ *is optimal and* $\ell = L$.

Proof. "$\Rightarrow$": Let $(\pi,\kappa)\in\mathcal{A}$ and denote $c=\kappa X(\pi,\kappa)$ and $\check c=\check\kappa\check X$. Note the partial derivative $\partial U(\check c)=D\check\kappa^{p-1}\check X^{p-1}=\ell\check X^{p-1}=Y$. Concavity of U implies $U(c)-U(\check c)\le\partial U(\check c)(c-\check c)=Y(c-\check c)$, hence

$$E\Big[\int_0^T U_s(c_s)\,\mu^\circ(ds)\Big]-E\Big[\int_0^T U_s(\check c_s)\,\mu^\circ(ds)\Big]\le E\Big[\int_0^T Y_s(c_s-\check c_s)\,\mu^\circ(ds)\Big]$$
$$=E[\Gamma_T(\pi,\kappa)]-E[\Gamma_T(\check\pi,\check\kappa)].$$

Let $\Gamma(\check\pi,\check\kappa)$ be a martingale; then $\Gamma_0(\pi,\kappa)=\Gamma_0(\check\pi,\check\kappa)$ and the supermartingale property imply that the last line is nonpositive. As (π,κ) was arbitrary, $(\check\pi,\check\kappa)$ is optimal with expected utility $E\big[\int_0^T U_s(\check c_s)\,\mu^\circ(ds)\big]=E[\frac{1}{p}\Gamma_T(\check\pi,\check\kappa)]=\frac{1}{p}\Gamma_0(\check\pi,\check\kappa)=\frac{1}{p}x_0^p\ell_0<\infty$. The rest is as in the proof of Lemma 5.1. □

The process Y is a supermartingale deflator in the language of [41]. We refer to Chapter II for the connection of the opportunity process with convex duality, which in fact suggests Lemma 5.10. Note that unlike $Z(\pi,\kappa)$ from the previous section, $\Gamma(\pi,\kappa)$ is positive for all values of p.

Our next goal is to link the supermartingale property to local first order conditions. Let $y,\check y\in\mathscr{C}\cap\mathscr{C}^0$ (we will plug in $\check\pi$ for $\check y$). The formal directional derivative of g^ℓ at $\check y$ in the direction of y is $(y-\check y)^\top\nabla g^\ell(\check y)=G^\ell(y,\check y)$, where, by formal differentiation under the integral sign (cf. (3.2)),

$$G^\ell(y,\check y):= \tag{5.2}$$
$$\ell_-(y-\check y)^\top\Big(b^R+\tfrac{c^{R\ell}}{\ell_-}+(p-1)c^R\check y\Big)+\int_{\mathbb{R}^d\times\mathbb{R}}(y-\check y)^\top x'h(x)\,F^{R,\ell}(d(x,x'))$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}}(\ell_-+x')\big\{(1+\check y^\top x)^{p-1}(y-\check y)^\top x-(y-\check y)^\top h(x)\big\}\,F^{R,\ell}(d(x,x')).$$

We take this expression as the definition of $G^\ell(y,\check y)$ whenever the last integral is well defined (the first one is finite by (4.2)). The differentiation cannot be justified in general, but see the subsequent section.

Lemma 5.11. *Let* $y\in\mathscr{C}^0$ *and* $\check y\in\mathscr{C}^{0,*}\cap\{g^\ell>-\infty\}$. *Then* $G^\ell(y,\check y)$ *is well defined with values in* $(-\infty,\infty]$ *and* $G^\ell(\cdot,\check y)$ *is lower semicontinuous on* $\mathscr{C}^0$.

Proof. Writing $(y-\check y)^\top x=1+y^\top x-(1+\check y^\top x)$, we can express $G^\ell(y,\check y)$ as

$$\ell_-(y-\check y)^\top\Big(b^R+\tfrac{c^{R\ell}}{\ell_-}+(p-1)c^R\check y\Big)+\int_{\mathbb{R}^d\times\mathbb{R}}(y-\check y)^\top x'h(x)\,F^{R,\ell}(d(x,x'))$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}}(\ell_-+x')\Big\{\frac{1+y^\top x}{(1+\check y^\top x)^{1-p}}-1-(y+(p-1)\check y)^\top h(x)\Big\}\,F^{R,\ell}(d(x,x'))$$
$$-\int_{\mathbb{R}^d\times\mathbb{R}}(\ell_-+x')\big\{(1+\check y^\top x)^p-1-p\check y^\top h(x)\big\}\,F^{R,\ell}(d(x,x')).$$

The first integral is finite and continuous in y by (4.2). The last integral above occurs in the definition of $g^\ell(\check{y})$, cf. (3.2), and it is finite if $g^\ell(\check{y}) > -\infty$ and equals $+\infty$ otherwise. Finally, consider the second integral above and call its integrand $\psi = \psi(y, \check{y}, x, x')$. The Taylor expansion $\frac{1+y^\top x}{(1+\check{y}^\top x)^{1-p}} = 1 + (y + (p-1)\check{y})^\top x + \frac{(p-1)}{2}\big(2y + (p-2)\check{y}\big)^\top x\; x^\top \check{y} + o(|x|^3)$ shows that $\int_{\{|x|+|x'|\leq 1\}} \psi\, dF^{R,\ell}$ is well defined and finite. It also shows that given a compact $K \subset \mathbb{R}^d$, there is $\varepsilon > 0$ such that $\int_{\{|x|+|x'|\leq \varepsilon\}} \psi\, dF^{R,\ell}$ is continuous in $y \in K$ (and also in $\check{y} \in K$). The details are as in Lemma 6.2. Moreover, for $y \in \mathscr{C}^0$ we have the lower bound $\psi \geq (\ell_- + x')\{-1 - (y + (p-1)\check{y})^\top h(x)\}$, which is $F^{R,\ell}$-integrable on $\{|x| + |x'| > \varepsilon\}$ for any $\varepsilon > 0$, again by (4.2). The result now follows by Fatou's lemma. $\square$

We can now connect the local first order conditions for g^ℓ and the global supermartingale property: it turns out that the formal derivative G^ℓ determines the sign of the drift rate of Γ (cf. (5.3) below), which leads to the following proposition. Here and in the sequel, we denote $\check{X} = X(\check{\pi}, \check{\kappa})$.

Proposition 5.12. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation and $(\pi, \kappa) \in \mathcal{A}$. Then $\Gamma(\pi,\kappa) := \ell \check{X}^{p-1} X(\pi,\kappa) + \int \kappa_s \ell_s \check{X}_s^{p-1} X_s(\pi,\kappa)\,\mu(ds)$ is a supermartingale (local martingale) if and only if $G^\ell(\pi, \check{\pi}) \leq 0$ $(= 0)$.*

Proof. Define $\bar{R} = R - (x - h(x)) * \mu^R$ as in (2.4). We abbreviate $\bar{\pi} := (p-1)\check{\pi} + \pi$ and similarly $\bar{\kappa} := (p-1)\check{\kappa} + \kappa$. We defer to Lemma 8.1 a calculation showing that $\big(\check{X}_-^{p-1} X_-(\pi,\kappa)\big)^{-1} \bullet \big(\ell \check{X}^{p-1} X(\pi,\kappa)\big)$ equals

$$\ell - \ell_0 + \ell_- \bar{\pi} \bullet \bar{R} - \ell_- \bar{\kappa} \bullet \mu + \ell_-(p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A + \bar{\pi}^\top c^{R\ell} \bullet A$$
$$+\, \bar{\pi}^\top x' h(x) * \mu^{R,\ell} + (\ell_- + x')\big\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top h(x)\big\} * \mu^{R,\ell}.$$

Here we use a predictable cut-off function h such that $\bar{\pi}^\top h(x)$ is bounded, e.g., $h(x) = x 1_{\{|x|\leq 1\}\cap\{|\bar{\pi}^\top x|\leq 1\}}$. Since $(\ell, \check{\pi}, \check{\kappa})$ is a solution, the drift of ℓ is

$$A^\ell = -pU^*(\ell_-) \bullet \mu - p g^\ell(\check{\pi}) \bullet A = (p-1)\ell_- \check{\kappa} \bullet \mu - p g^\ell(\check{\pi}) \bullet A.$$

By Remark 2.3, $\Gamma := \Gamma(\pi,\kappa)$ has a well defined drift rate a^Γ with values in $(-\infty, \infty]$. From the two formulas above and (2.4) we deduce

$$a^\Gamma = \check{X}_-^{p-1} X(\pi,\kappa)_- G^\ell(\pi, \check{\pi}). \tag{5.3}$$

Here $\check{X}_-^{p-1} X(\pi,\kappa)_- > 0$ by admissibility. If Γ is a supermartingale, then $a^\Gamma \leq 0$, and the converse holds by Lemma 2.4 in view of $\Gamma \geq 0$. $\square$

We obtain our second verification theorem from Proposition 5.12 and Lemma 5.10.

Theorem 5.13. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation. Assume that $P \otimes A$-a.e., $G^\ell(y, \check{\pi}) \in [-\infty, 0]$ for all $y \in \mathscr{C} \cap \mathscr{C}^{0,*}$. Then*

$$\Gamma(\check{\pi}, \check{\kappa}) := \ell \check{X}^p + \int \check{\kappa}_s \ell_s \check{X}^p_s \, \mu(ds)$$

is a local martingale. It is a martingale if and only if (2.2) *holds and $(\check{\pi}, \check{\kappa})$ is optimal and $\ell = L$ is the opportunity process.*

If $\mathscr{C}$ is not convex, one can imagine situations where the directional derivative of g^ℓ at the maximum is positive—i.e., the assumption on $G^\ell(y, \check{\pi})$ is sufficient but not necessary. This changes in the subsequent section.

The Convex-Constrained Case

We assume in this section that $\mathscr{C}$ is convex; then $\mathscr{C} \cap \mathscr{C}^0$ is also convex. Our aim is to show that the nonnegativity condition on G^ℓ in Theorem 5.13 is automatically satisfied in this case. We start with an elementary but crucial observation about "differentiation under the integral sign".

Lemma 5.14. *Consider two distinct points y_0 and $\check{y}$ in $\mathbb{R}^d$ and let $C = \{\eta y_0 + (1-\eta)\check{y} : 0 \leq \eta \leq 1\}$. Let ρ be a function on $\Sigma \times C$, where Σ is some Borel space with measure ν, such that $x \mapsto \rho(x, y)$ is ν-measurable, $\int \rho^+(x, \cdot)\, \nu(dx) < \infty$ on C, and $y \mapsto \rho(x, y)$ is concave. In particular, the directional derivative*

$$D_{\check{y},y}\rho(x, \cdot) := \lim_{\varepsilon \to 0+} \frac{\rho\big(x, \check{y} + \varepsilon(y - \check{y})\big) - \rho(x, \check{y})}{\varepsilon}$$

exists in $(-\infty, \infty]$ for all $y \in C$. Let α be another concave function on C.

Define $\gamma(y) := \alpha(y) + \int \rho(x, y)\, \nu(dx)$ and assume that $\gamma(y_0) > -\infty$ and that $\gamma(\check{y}) = \max_C \gamma < \infty$. Then for all $y \in C$,

$$D_{\check{y},y}\gamma = D_{\check{y},y}\alpha + \int D_{\check{y},y}\rho(x, \cdot)\, \nu(dx) \; \in (-\infty, 0] \tag{5.4}$$

and in particular $D_{\check{y},y}\rho(x, \cdot) < \infty$ $\nu(dx)$-a.e.

Proof. Note that γ is concave, hence we also have $\gamma > -\infty$ on C. Let $v = (y - \check{y})$ and $\varepsilon > 0$, then $\frac{\gamma(\check{y}+\varepsilon v) - \gamma(\check{y})}{\varepsilon} = \frac{\alpha(\check{y}+\varepsilon v) - \alpha(\check{y})}{\varepsilon} + \int \frac{\rho(x,\check{y}+\varepsilon v) - \rho(x,\check{y})}{\varepsilon}\, \nu(dx)$. By concavity, these quotients increase monotonically as $\varepsilon \downarrow 0$, in particular their limits exist. The left hand side is nonpositive as $\check{y}$ is a maximum and monotone convergence yields (5.4). □

For completeness, let us mention that if $\gamma(y_0) = -\infty$, there are examples where the left hand side of (5.4) is $-\infty$ but the right hand side is finite; we shall deal with this case separately. We deduce the following version of Theorem 5.13; as discussed, it involves only the control $(\check{\pi}, \check{\kappa})$.

Theorem 5.15. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation and assume that $\mathscr{C}$ is convex. Then $\Gamma(\check{\pi}, \check{\kappa}) := \ell \check{X}^p + \int \check{\kappa}_s \ell_s \check{X}_s^p \,\mu(ds)$ is a local martingale. It is a martingale if and only if* (2.2) *holds and $(\check{\pi}, \check{\kappa})$ is optimal and $\ell = L$.*

Proof. To apply Theorem 5.13, we have to check that $G^\ell(y, \check{\pi}) \in [-\infty, 0]$ for $y \in \mathscr{C} \cap \mathscr{C}^{0,*}$. Recall that $\check{\pi}$ is a maximizer for g^ℓ and that G^ℓ was defined by differentiation under the integral sign. Lemma 5.14 yields $G^\ell(y, \check{\pi}) \leq 0$ whenever $y \in \{g^\ell > -\infty\}$. This ends the proof for $p \in (0, 1)$ as g^ℓ is then finite. If $p < 0$, the definition of g^ℓ and Remark 6.7 show that the set $\{g^\ell > -\infty\}$ contains the set $\bigcup_{\eta \in [0,1)} \eta(\mathscr{C} \cap \mathscr{C}^0)$ which, in turn, is clearly dense in $\mathscr{C} \cap \mathscr{C}^{0,*}$. Hence $\{g^\ell > -\infty\}$ is dense in $\mathscr{C} \cap \mathscr{C}^{0,*}$ and we obtain $G^\ell(y, \check{\pi}) \in [-\infty, 0]$ for all $y \in \mathscr{C} \cap \mathscr{C}^{0,*}$ using the lower semicontinuity from Lemma 5.11. □

Remark 5.16. (i) We note that $\Gamma(\check{\pi}, \check{\kappa}) = pZ(\check{\pi}, \check{\kappa})$ if Z is defined as in (4.4). In particular, Remark 5.9 can be used also for $\Gamma(\check{\pi}, \check{\kappa})$.

(ii) Muhle-Karbe [58] considers certain one-dimensional (unconstrained) affine models and introduces a sufficient optimality condition in the form of an algebraic inequality (see [58, Theorem 4.20(3)]). This condition can be seen as a special case of the statement that $G^L(y, \check{\pi}) \in [-\infty, 0]$ for $y \in \mathscr{C}^{0,*}$; in particular, we have shown its necessity.

Of course, all our verification results can be seen as a uniqueness result for the Bellman equation. As an example, Theorem 5.15 yields:

Corollary 5.17. *If $\mathscr{C}$ is convex, there is at most one solution of the Bellman equation in the class of solutions $(\ell, \check{\pi}, \check{\kappa})$ such that $\Gamma(\check{\pi}, \check{\kappa})$ is of class (D).*

Similarly, one can give corollaries for the other results. We close with a comment concerning convex duality.

Remark 5.18. (i) A major insight in [49] was that the "dual domain" for utility maximization (here with $\mathscr{C} = \mathbb{R}^d$) should be a set of supermartingales rather than (local) martingales when the price process has jumps. A one-period example for log-utility [49, Example 5.1'] showed that the supermartingale solving the dual problem can indeed have nonvanishing drift. In that example it is clear that this arises when the budget constraint becomes binding. For general models and log-utility, [25] comments on this phenomenon. The calculations of this section yield an instructive "local" picture also for power utility.

Under Assumptions 3.1, the opportunity process L and the optimal strategy $(\hat{\pi}, \hat{\kappa})$ solve the Bellman equation. Assume that $\mathscr{C}$ is convex and let $\widehat{X} = X(\hat{\pi}, \hat{\kappa})$. Consider $\widehat{Y} = L\widehat{X}^{p-1}$, which was the solution to the dual problem in Chapter II. We have shown that $\widehat{Y}\mathcal{E}(\pi \bullet R)$ is a supermartingale for every $\pi \in \mathcal{A}$, i.e., $\widehat{Y}$ is a supermartingale deflator. Choosing $\pi = 0$, we see that $\widehat{Y}$ is itself a supermartingale, and by (5.3) its drift rate satisfies

$$a^{\widehat{Y}} = \widehat{X}_-^{p-1} G^L(0, \hat{\pi}) = -\widehat{X}_-^{p-1} \hat{\pi}^\top \nabla g(\hat{\pi}).$$

Hence $\widehat{Y}$ is a local martingale if and only if $\hat{\pi}^\top \nabla g(\hat{\pi}) = 0$. One can say that $-\hat{\pi}^\top \nabla g(\hat{\pi}) < 0$ means that the constraints are binding, whereas in an "unconstrained" case the gradient of g would vanish; i.e., $\widehat{Y}$ has nonvanishing drift rate at a given (ω, t) whenever the constraints are binding. Even if $\mathscr{C} = \mathbb{R}^d$, we still have the budget constraint $\mathscr{C}^0$ in the maximization of g. If in addition R is continuous, $\mathscr{C}^0 = \mathbb{R}^d$ and we are truly in an unconstrained situation. Then $\widehat{Y}$ is a local martingale; indeed, in the setting of Corollary 3.12 we calculate

$$\widehat{Y} = y_0 \mathcal{E}\Big(-\lambda \bullet M + \frac{1}{L_-} \bullet N^L \Big), \quad y_0 := L_0 x_0^{p-1}.$$

Note how N^L, the martingale part of L orthogonal to R, yields the solution to the dual problem.

(ii) From the proof of Proposition 5.12 we have that the general formula for the local martingale part of $\widehat{Y}$ is

$$M^{\widehat{Y}} = \widehat{X}_-^{p-1} \bullet \bigg(M^L + L_-(p-1)\hat{\pi} \bullet M^{\bar{R}} + (p-1)\hat{\pi}^\top x' h(x) * (\mu^{R,L} - \nu^{R,L})$$
$$+ (L_- + x')\big\{(1+\hat{\pi}^\top x)^{p-1} - 1 - (p-1)\hat{\pi}^\top h(x)\big\} * (\mu^{R,L} - \nu^{R,L}) \bigg).$$

This is relevant in the problem of *q-optimal equivalent martingale measures*; cf. Goll and Rüschendorf [26] for a general perspective. Let $u(x_0) < \infty$, $D \equiv 1$, $\mu = 0$, $\mathscr{C} = \mathbb{R}^d$, and assume that the set $\mathscr{M}$ of equivalent local martingale measures for $S = \mathcal{E}(R)$ is nonempty. Given $q = p/(p-1) \in (-\infty, 0) \cup (0, 1)$ conjugate to p, $Q \in \mathscr{M}$ is called *q-optimal* if $E[-q^{-1}(dQ/dP)^q]$ is finite and minimal over $\mathscr{M}$. If $q < 0$, i.e., $p \in (0,1)$, then $u(x_0) < \infty$ is equivalent to the existence of some $Q \in \mathscr{M}$ such that $E[-q^{-1}(dQ/dP)^q] < \infty$; moreover, Assumptions 3.1 are satisfied (see Kramkov and Schachermayer [49, 50]). Using [49, Theorem 2.2(iv)] we conclude that

(a) the q-optimal martingale measure exists if and only if $a^{\widehat{Y}} \equiv 0$ and $M^{\widehat{Y}}$ is a true martingale;

(b) in that case, $1 + y_0^{-1} M^{\widehat{Y}}$ is its P-density process.

This generalizes earlier results of [26] as well as of Grandits [28], Jeanblanc *et al.* [35] and Choulli and Stricker [12].

III.6 Appendix A: Proof of Lemma 3.8

This main goal of this appendix is to construct a measurable maximizing sequence for the random function g (cf. Lemma 3.8). The entire section is under Assumptions 3.1. Before beginning the proof, we discuss the properties of g; recall that

$$g(y) := L_- y^\top \Big(b^R + \tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2} c^R y\Big) + \int_{\mathbb{R}^d \times \mathbb{R}} x' y^\top h(x)\, F^{R,L}(d(x,x')) + \int_{\mathbb{R}^d \times \mathbb{R}} (L_- + x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^{R,L}(d(x,x')). \tag{6.1}$$

Lemma 6.1. *$L_- + x'$ is strictly positive $F^L(dx')$-a.e.*

Proof. $(P \otimes \nu^L)\{L_- + x' \leq 0\} = E[1_{\{L_- + x' \leq 0\}} * \nu^L_T] = E[1_{\{L_- + x' \leq 0\}} * \mu^L_T] = E\big[\sum_{s \leq T} 1_{\{L_s \leq 0\}} 1_{\{\Delta L_s \neq 0\}}\big] = 0$ as $L > 0$ by Lemma 2.1. $\square$

Fix (ω, t) and let $l := L_{t-}(\omega)$. Furthermore, let F be any Lévy measure on $\mathbb{R}^{d+1}$ which is equivalent to $F^{R,L}_t(\omega)$ and satisfies (2.5). Equivalence implies that $\mathscr{C}^0_t(\omega), \mathscr{C}^{0,*}_t(\omega)$, and $\mathscr{N}_t(\omega)$ are the same if defined with respect to F instead of F^R. Given $\varepsilon > 0$, let

$$I^F_\varepsilon(y) := \int_{\{|x|+|x'| \leq \varepsilon\}} (l + x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F(d(x,x')),$$

$$I^F_{>\varepsilon}(y) := \int_{\{|x|+|x'| > \varepsilon\}} (l + x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F(d(x,x')),$$

so that

$$I^F(y) := I^F_\varepsilon(y) + I^F_{>\varepsilon}(y)$$

is the last integral in (6.1) when $F = F^{R,L}_t(\omega)$. We know from the proof of Lemma 3.4 that $I^{F^{R,L}}(\pi)$ is well defined and finite for any $\pi \in \mathcal{A}^{fE}$ (of course, when $p > 0$, this is essentially due to the assumption (2.2)). For general F, I^F has the following properties.

Lemma 6.2. *Consider a sequence $y_n \to y_\infty$ in $\mathscr{C}^0$.*

(i) For any $y \in \mathscr{C}^0$, the integral $I^F(y)$ is well defined in $\mathbb{R} \cup \{\mathrm{sign}(p)\infty\}$.

(ii) For $\varepsilon \leq (2 \sup_n |y_n|)^{-1}$ we have $I^F_\varepsilon(y_n) \to I^F_\varepsilon(y_\infty)$.

(iii) If $p \in (0,1)$, I^F is l.s.c., that is, $\liminf_n I^F(y_n) \geq I^F(y_\infty)$.

(iv) If $p < 0$, I^F is u.s.c., that is, $\limsup_n I^F(y_n) \leq I^F(y_\infty)$. Moreover, $y \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$ implies $I^F(y) = -\infty$.

Proof. The first item follows from the subsequent considerations.

(ii) We may assume that h is the identity on $\{|x| \leq \varepsilon\}$; then on this set $p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x) =: \psi(z)|_{z = y^\top x}$, where the function ψ is smooth on $\{|z| \leq 1/2\} \subseteq \mathbb{R}$ satisfying

$$\psi(z) = p^{-1}(1+z)^p - p^{-1} - z = \tfrac{p-1}{2} z^2 + o(|z|^3)$$

because $1+z$ is bounded away from 0. Thus $\psi(z) = z^2\tilde{\psi}(z)$ with a function $\tilde{\psi}$ that is continuous and in particular bounded on $\{|z| \leq 1/2\}$.

As a Lévy measure, F integrates $(|x'|^2 + |x|^2)$ on compacts; in particular, $G(d(x,x')) := |x|^2\, F(d(x,x'))$ defines a finite measure on $\{|x| + |x'| \leq \varepsilon\}$. Hence $I_\varepsilon^F(y)$ is well defined and finite for $|y| \leq (2\varepsilon)^{-1}$, and dominated convergence shows that $I_\varepsilon^F(y) = \int_{\{|x|+|x'|\leq\varepsilon\}} (l+x')\tilde{\psi}(y^\top x)\, G(d(x,x'))$ is continuous in y on $\{|y| \leq (2\varepsilon)^{-1}\}$.

(iii) For $|y|$ bounded by a constant C, the integrand in I^F is bounded from below by $C' + |x'|$ for some constant C' depending on y only through C. We choose ε as before. As $C' + |x'|$ is F-integrable on $\{|x| + |x'| > \varepsilon\}$ by (2.5), $I^F(y)$ is well defined in $\mathbb{R} \cup \{\infty\}$ and l.s.c. by Fatou's lemma.

(iv) The first part follows as in (iii), now the integrand is bounded from above by $C' + |x'|$. If $y \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$, Lemma 6.1 shows that the integrand equals $-\infty$ on a set of positive F-measure. $\square$

Lemma 6.3. *The function g is concave. If $\mathscr{C}$ is convex, g has at most one maximum on $\mathscr{C} \cap \mathscr{C}^0$, modulo $\mathscr{N}$.*

Proof. We first remark that the assertion is not trivial because g need not be strictly concave on $\mathscr{N}^\perp$, for example, the process $R_t = t(1,\dots,1)^\top$ was not excluded.

Note that g is of the form $g(y) = Hy + J(y)$, where $Hy = L_- y^\top b^R + y^\top c^{RL} + \int x' y^\top h(x)\, F^{R,L}$ is linear and $J(y) = \frac{(p-1)}{2} L_- y^\top c^R y + I^{F^{R,L}}(y)$ is concave. We may assume that $h(x) = x 1_{\{|x|\leq 1\}}$.

Let $y_1, y_2 \in \mathscr{C} \cap \mathscr{C}^0$ be such that $g(y_1) = g(y_2) = \sup g =: g^* < \infty$, our aim is to show $y_1 - y_2 \in \mathscr{N}$. By concavity, $g^* = g((y_1 + y_2)/2)) = [g(y_1) + g(y_2)]/2$, which implies $J((y_1 + y_2)/2)) = [J(y_1) + J(y_2)]/2$ due to the linearity of H. Using the definition of J, this shows that J is constant on the line segment connecting y_1 and y_2. A first consequence is that $y_1 - y_2$ lies in the set $\{y : y^\top c^R = 0,\ F^R\{x : y^\top x \neq 0\} = 0\}$ and a second is that $Hy_1 = Hy_2$. It remains to show $(y_1 - y_2)^\top b^R = 0$ to have $y_1 - y_2 \in \mathscr{N}$.

Note that $F^R\{x : y^\top x \neq 0\} = 0$ implies $F^{R,L}\{x : y^\top h(x) \neq 0\} = 0$. Moreover, $y^\top c^R = 0$ implies $y^\top c^{RL} = 0$ due to the absolute continuity $\langle R^{c,i}, L^c\rangle \ll \langle R^{c,i}\rangle$ which follows from the Kunita-Watanabe inequality. Therefore, the first consequence above implies $\int x'(y_1 - y_2)^\top h(x)\, F^{R,L} = 0$ and $(y_1 - y_2)^\top c^{RL} = 0$, and now the second consequence and the definition of H yield $0 = H(y_1 - y_2) = L_-(y_1 - y_2)^\top b^R$. Thus $(y_1 - y_2)^\top b^R = 0$ as $L_- > 0$ and this ends the proof. $\square$

We can now move toward the main goal of this section. Clearly we need some variant of the "Measurable Maximum Theorem" (see, e.g., [1, 18.19], [41, Theorem 9.5], [64, 2K]). We state a version that is tailored to our needs and has a simple proof; the technique is used also in Proposition 4.3.

Lemma 6.4. *Let $\mathscr{D}$ be a predictable set-valued process with nonempty compact values in $2^{\mathbb{R}^d}$. Let $f(y) = f(\omega, t, y)$ be a proper function on $\mathscr{D}$ with values in $\mathbb{R} \cup \{-\infty\}$ such that*

(i) $f(\varphi)$ is predictable whenever φ is a $\mathscr{D}$-valued predictable process,

(ii) $y \mapsto f(y)$ is upper semicontinuous on $\mathscr{D}$ for fixed (ω, t).

Then there exists a $\mathscr{D}$-valued predictable process π such that $f(\pi) = \max_{\mathscr{D}} f$.

Proof. We start with the Castaing representation [64, 1B] of $\mathscr{D}$: there exist $\mathscr{D}$-valued predictable processes $(\varphi_n)_{n\geq 1}$ such that $\overline{\{\varphi_n : n \geq 1\}} = \mathscr{D}$ for each (ω, t). By (i), $f^* := \max_n f(\varphi_n)$ is predictable, and $f^* = \max_{\mathscr{D}} f$ by (ii). Fix $k \geq 1$ and let $\Lambda_n := \{f^* - f(\varphi_n) \leq 1/k\}$, $\Lambda^n := \Lambda_n \setminus (\Lambda_1 \cup \dots \cup \Lambda_{n-1})$. Define $\pi^k := \sum_n \varphi_n 1_{\Lambda^n}$, then $f^* - f(\pi^k) \leq 1/k$ and $\pi^k \in \mathscr{D}$.

It remains to select a cluster point: By compactness, $(\pi^k)_{k\geq 1}$ is bounded for each (ω, t), so there is a convergent subsequence along "random indices" τ_k. More precisely, there exists a strictly increasing sequence of integer-valued predictable processes $\tau_k = \{\tau_k(\omega, t)\}$ and a predictable process π^* such that $\lim_k \pi_t^{\tau_k(\omega,t)}(\omega) = \pi_t^*(\omega)$ for all (ω, t). See, e.g., the proof of Föllmer and Schied [22, Lemma 1.63]. We have $f^* = f(\pi^*)$ by (ii). $\square$

Our random function g satisfies property (i) of Lemma 6.4 because the characteristics are predictable (recall the definition [34, II.1.6]). We also note that the intersection of closed predictable processes is predictable [64, 1M]. The sign of p is important as it switches the semicontinuity of g; we start with the immediate case $p < 0$ and denote $B_r(\mathbb{R}^d) = \{x \in \mathbb{R}^d : |x| \leq r\}$.

Proof of Lemma 3.8 for $p < 0$. In this case g is u.s.c. on $\mathscr{C} \cap \mathscr{C}^0$ (Lemma 6.2). Let $\mathscr{D}(n) := \mathscr{C} \cap \mathscr{C}^0 \cap B_n(\mathbb{R}^d)$. Lemma 6.4 yields a predictable process $\pi^n \in \arg\max_{\mathscr{D}(n)} g$ for each $n \geq 1$, and clearly $\lim_n g(\pi^n) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g$. As $g(\pi^n) \geq g(0) = 0$, we have $\pi^n \in \mathscr{C}^{0,*}$ by Lemma 6.2. $\square$

III.6.1 Measurable Maximizing Sequence for $p \in (0, 1)$

Fix $p \in (0, 1)$. Since the continuity properties of g are not clear, we will use an approximating sequence of continuous functions. (See also Appendix III.7, where an alternative approach is discussed and the continuity is clarified under an additional assumption on $\mathscr{C}$.) We will approximate g using Lévy measures with enhanced integrability, a method suggested by [41] in a similar problem. This preserves monotonicity properties that will be useful to pass to the limit.

All this is not necessary if R is locally bounded, or more generally if $F^{R,L}$ satisfies the following condition. We start with fixed (ω, t).

Definition 6.5. Let F be a Lévy measure on $\mathbb{R}^{d+1}$ which is equivalent to $F^{R,L}$ and satisfies (2.5). (i) We say that F is *p-suitable* if

$$\int (1+|x'|)(1+|x|)^p 1_{\{|x|>1\}}\, F(d(x,x')) < \infty.$$

(ii) The *p-suitable approximating sequence for* F is the sequence $(F_n)_{n\geq 1}$ of Lévy measures defined by $dF_n/dF = f_n$, where

$$f_n(x) = 1_{\{|x|\leq 1\}} + e^{-|x|/n} 1_{\{|x|>1\}}.$$

It is easy to see that each F_n in (ii) shares the properties of F, while in addition being p-suitable because $(1+|x|)^p e^{-|x|/n}$ is bounded. As the sequence f_n is increasing, monotone convergence shows that $\int V dF_n \uparrow \int V\, dF$ for any measurable function $V \geq 0$ on $\mathbb{R}^{d+1}$. We denote by g^F the function which is defined as in (6.1) but with $F^{R,L}$ replaced by F.

Lemma 6.6. *If F is p-suitable, g^F is real-valued and continuous on $\mathscr{C}^0$.*

Proof. Pick $y_n \to y$ in $\mathscr{C}^0$. The only term in (6.1) for which continuity is not evident, is the integral $I^F = I^F_\varepsilon + I^F_{>\varepsilon}$, where we choose ε as in Lemma 6.2. We have $I^F_\varepsilon(y_n) \to I^F_\varepsilon(y)$ by that lemma. When F is p-suitable, the continuity of $I^F_{>\varepsilon}$ follows from the dominated convergence theorem. □

Remark 6.7. Define the set

$$(\mathscr{C} \cap \mathscr{C}^0)^\diamond := \bigcup_{\eta\in[0,1)} \eta(\mathscr{C} \cap \mathscr{C}^0).$$

Its elements y have the property that $1 + y^\top x$ is $F^R(dx)$-essentially *bounded away* from zero. Indeed, $y = \eta y_0$ with $\eta \in [0,1)$ and $F^R\{y_0^\top x \geq -1\} = 0$, hence $1 + y^\top x \geq 1 - \eta$, F^R-a.e. In particular, $(\mathscr{C} \cap \mathscr{C}^0)^\diamond \subseteq \mathscr{C}^{0,*}$. If $\mathscr{C}$ is star-shaped with respect to the origin, we also have $(\mathscr{C} \cap \mathscr{C}^0)^\diamond \subseteq \mathscr{C}$.

We introduce the compact-valued process $\mathscr{D}(r) := \mathscr{C} \cap \mathscr{C}^0 \cap B_r(\mathbb{R}^d)$.

Lemma 6.8. *Let F be p-suitable. Under (C3),* $\arg\max_{\mathscr{D}(r)} g^F \subseteq \mathscr{C}^{0,*}$.

More generally, this holds whenever F is a Lévy measure equivalent to $F^{R,L}$ satisfying (2.5) *and g^F is finite-valued.*

Proof. Assume that $\check{y} \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$ is a maximum of g^F. Let $\eta \in (\underline{\eta}, 1)$ be as in the definition of (C3) and $y_0 := \eta\check{y}$. By Lemma 5.14, the directional derivative $D_{\check{y},y_0} g$ can be calculated by differentiating under the integral sign. For the integrand of I^F we have

$$D_{\check{y},y_0}\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\} = (1-\eta)\big\{(1+\check{y}^\top x)^{p-1}\check{y}^\top x - \check{y}^\top h(x)\big\}.$$

But this is infinite on a set of positive measure as $\check{y} \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$ means that $F\{\check{y}^\top x = -1\} > 0$, contradicting the last assertion of Lemma 5.14. □

Let F be a Lévy measure on $\mathbb{R}^{d+1}$ which is equivalent to $F^{R,L}$ and satisfies (2.5). The crucial step is

Lemma 6.9. *Let (F_n) be the p-suitable approximating sequence for F and fix $r > 0$. For each n, $\arg\max_{\mathscr{D}(r)} g^{F_n} \neq \emptyset$, and for any $y_n^* \in \arg\max_{\mathscr{D}(r)} g^{F_n}$ it holds that $\limsup_n g^F(y_n^*) = \sup_{\mathscr{D}(r)} g^F$.*

Proof. We first show that

$$I^{F_n}(y) \to I^F(y) \quad \text{for any } y \in \mathscr{C}^0. \tag{6.2}$$

Recall that $I^{F_n}(y) = \int (l+x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\} f_n(x)\, F(d(x,x'))$, where f_n is nonnegative and increasing in n. As $f_n = 1$ in a neighborhood of the origin, we need to consider only $I^{F_n}_{>\varepsilon}$ (for $\varepsilon = 1$, say). Its integrand is bounded below, simultaneously for all n, by a negative constant times $(1 + |x'|)$, which is F-integrable on the relevant domain. As (f_n) is increasing, we can apply monotone convergence on the set $\big\{(x,x') : p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x) \geq 0\big\}$ and dominated convergence on the complement to deduce (6.2).

Existence of $y_n^* \in \arg\max_{\mathscr{D}(r)} g^{F_n}$ is clear by compactness of $\mathscr{D}(r)$ and continuity of g^{F_n} (Lemma 6.6). Let $y \in \mathscr{D}(r)$ be arbitrary. By definition of y_n^* and (6.2),

$$\limsup_n g^{F_n}(y_n^*) \geq \limsup_n g^{F_n}(y) = g^F(y).$$

We show $\limsup_n g^F(y_n^*) \geq \limsup_n g^{F_n}(y_n^*)$. We can split the integral $I^{F_n}(y)$ into a sum of three terms: The integral over $\{|x| \leq 1\}$ is the same as for I^F, since $f_n = 1$ on this set. We can assume that the cut-off h vanishes outside $\{|x| \leq 1\}$. The second term is then

$$\int_{\{|x|>1\}} (l + x')p^{-1}(1 + y^\top x)^p f_n \, dF,$$

here the integrand is nonnegative and hence increasing in n, for all y; and the third term is

$$\int_{\{|x|>1\}} (l + x')(-p^{-1}) f_n \, dF,$$

which is decreasing in n but converges to $\int_{\{|x|>1\}} (l + x')(-p^{-1})\, dF$. Thus

$$g^F(y_n^*) \geq g^{F_n}(y_n^*) - \varepsilon_n$$

with the sequence $\varepsilon_n := \int_{\{|x|>1\}} (l + x')(-p^{-1})(f_n - 1)\, dF \downarrow 0$. Together, we conclude $\sup_{\mathscr{D}(r)} g^F \geq \limsup_n g^F(y_n^*) \geq \limsup_n g^{F_n}(y_n^*) \geq \sup_{\mathscr{D}(r)} g^F$. □

Proof of Lemma 3.8 for $p \in (0,1)$. Fix $r > 0$. By Lemma 6.4 we can find measurable selectors $\pi^{n,r}$ for $\arg\max_{\mathscr{D}(r)} g^{F_n}$, i.e., $\pi^{n,r}_t(\omega)$ plays the role of y^*_n in Lemma 6.9. Taking $\pi^n := \pi^{n,n}$ and noting $\mathscr{D}(n) \uparrow \mathscr{C} \cap \mathscr{C}^0$, Lemma 6.9 shows that π^n are $\mathscr{C} \cap \mathscr{C}^0$-valued predictable processes such that $\limsup_n g(\pi^n) = \sup_{\mathscr{C}\cap\mathscr{C}^0} g$ $P \otimes A$-a.e. Lemma 6.8 shows that π^n takes values in $\mathscr{C}^{0,*}$. □

III.7 Appendix B: Parametrization by Representative Portfolios

This appendix introduces an equivalent transformation of the model $(R, \mathscr{C})$ with specific properties (Theorem 7.3); the main idea is to substitute the given assets by wealth processes that represent the investment opportunities of the model. While the result is of independent interest, the main conclusion in our context is that the approximation technique from Appendix III.6.1 for the case $p \in (0,1)$ can be avoided, at least under slightly stronger assumptions on $\mathscr{C}$: If the utility maximization problem is finite, the corresponding Lévy measure in the transformed model is p-suitable (cf. Definition 6.5) and hence the corresponding function g is continuous. This is not only an alternative argument to prove Lemma 3.8. In applications, continuity can be useful to construct a maximizer for g (rather than a maximizing sequence) if one does not know *a priori* that there exists an optimal strategy. A static version of our construction is carried out in Chapter IV.

In this appendix we use the following assumptions on the set-valued process $\mathscr{C}$ of constraints:

(C1) $\mathscr{C}$ is predictable.

(C2) $\mathscr{C}$ is closed.

(C4) $\mathscr{C}$ is star-shaped with respect to the origin: $\eta\mathscr{C} \subseteq \mathscr{C}$ for all $\eta \in [0,1]$.

Since we already obtained a proof of Lemma 3.8, we do not strive for minimal conditions here. Clearly (C4) implies condition (C3) from Section III.2.4, but its main implication is that we can select a bounded (hence R-integrable) process in the subsequent lemma. The following result is the construction of the jth *representative portfolio*, a portfolio with the property that it invests in the jth asset *whenever* this is feasible.

Lemma 7.1. *Fix $1 \leq j \leq d$ and let $H^j = \{x \in \mathbb{R}^d : x^j \neq 0\}$. There exists a bounded predictable $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process ϕ satisfying*

$$\{\phi^j = 0\} = \big\{\mathscr{C} \cap \mathscr{C}^{0,*} \cap H^j = \emptyset\big\}.$$

Proof. Let $B_1 = B_1(\mathbb{R}^d)$ be the closed unit ball and $H := H^j$. Condition (C4) implies $\big\{\mathscr{C} \cap \mathscr{C}^{0,*} \cap H = \emptyset\big\} = \big\{\mathscr{C} \cap B_1 \cap \mathscr{C}^{0,*} \cap H = \emptyset\big\}$, hence we may substitute $\mathscr{C}$ by $\mathscr{C} \cap B_1$. Define the closed sets $H_k = \{x \in \mathbb{R}^d : |x^j| \geq k^{-1}\}$

for $k \geq 1$, then $\bigcup_k H_k = H$. Moreover, let $\mathscr{D}_k = \mathscr{C} \cap \mathscr{C}^0 \cap H_k$. This is a compact-valued predictable process, so there exists a predictable process ϕ_k such that $\phi_k \in \mathscr{D}_k$ (hence $\phi_k^j \neq 0$) on the set $\Lambda_k := \{\mathscr{D}_k \neq \emptyset\}$ and $\phi_k = 0$ on the complement. Define $\Lambda^k := \Lambda_k \setminus (\Lambda_1 \cup \cdots \cup \Lambda_{k-1})$ and $\phi' := \sum_k \phi_k 1_{\Lambda^k}$. Then $|\phi'| \leq 1$ and $\{\phi'^j = 0\} = \big\{\mathscr{C} \cap \mathscr{C}^0 \cap H = \emptyset\big\} = \big\{\mathscr{C} \cap \mathscr{C}^{0,*} \cap H = \emptyset\big\}$; the second equality uses (C4) and Remark 6.7. These two facts also show that $\phi := \frac{1}{2}\phi'$ has the same property while in addition being $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued. $\square$

Remark 7.2. The previous proof also applies if instead of (C4), e.g., the diameter of $\mathscr{C}$ is uniformly bounded and $\mathscr{C}^0 = \mathscr{C}^{0,*}$.

If Φ is a $d \times d$-matrix with columns $\phi_1, \dots, \phi_d \in L(R)$, the matrix stochastic integral $\widetilde{R} = \Phi \bullet R$ is the $\mathbb{R}^d$-valued process given by $\widetilde{R}^j = \phi_j \bullet R$. If $\psi \in L(\Phi \bullet R)$ is $\mathbb{R}^d$-valued, then $\Phi\psi \in L(R)$ and

$$\psi \bullet (\Phi \bullet R) = (\Phi\psi) \bullet R. \tag{7.1}$$

If $\mathscr{D}$ is a set-valued process which is predictable, closed and contains the origin, then the preimage $\Phi^{-1}\mathscr{D}$ shares these properties (cf. [64, 1Q]). Convexity and star-shape are also preserved.

We obtain the following model if we sequentially replace the given assets by representative portfolios; here e_j denotes the jth unit vector in $\mathbb{R}^d$ for $1 \leq j \leq d$ (i.e., $e_j^i = \delta_{ij}$).

Theorem 7.3. *There exists a predictable $\mathbb{R}^{d\times d}$-valued uniformly bounded process Φ such that the financial market model with returns*

$$\widetilde{R} := \Phi \bullet R$$

and constraints $\widetilde{\mathscr{C}} := \Phi^{-1}\mathscr{C}$ has the following properties: for all $1 \leq j \leq d$,

(i) $\Delta\widetilde{R}^j > -1$ *(positive prices),*

(ii) $e_j \in \widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^{0,*}$, *where $\widetilde{\mathscr{C}}^{0,*} = \Phi^{-1}\mathscr{C}^{0,*}$ (entire wealth can be invested in each asset),*

(iii) the model $(\widetilde{R}, \widetilde{\mathscr{C}})$ admits the same wealth processes as $(R, \mathscr{C})$.

Proof. We treat the components one by one. Let $j = 1$ and let $\phi = \phi(1)$ be as in Lemma 7.1. We replace the first asset R^1 by the process $\phi \bullet R$, or equivalently, we replace R by $\Phi \bullet R$, where $\Phi = \Phi(1)$ is the $d \times d$-matrix

$$\Phi = \begin{pmatrix} \phi^1 & & & \\ \phi^2 & 1 & & \\ \vdots & & \ddots & \\ \phi^d & & & 1 \end{pmatrix}.$$

The new natural constraints are $\Phi^{-1}\mathscr{C}^0$ and we replace $\mathscr{C}$ by $\Phi^{-1}\mathscr{C}$. Note that $e_1 \in \Phi^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$ because $\Phi e_1 = \phi \in \mathscr{C} \cap \mathscr{C}^{0,*}$ by construction.

We show that for every $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process $\pi \in L(R)$ there exists ψ predictable such that $\Phi\psi = \pi$. In view of (7.1), this will imply that the new model admits the same wealth processes as the old one. On the set $\{\phi^1 \neq 0\} = \{\Phi \text{ is invertible}\}$ we take $\psi = \Phi^{-1}\pi$ and on the complement we choose $\psi^1 \equiv 0$ and $\psi^j = \pi^j$ for $j \geq 2$; this is the same as inverting Φ on its image. Note that $\{\phi^1 = 0\} \subseteq \{\pi^1 = 0\}$ by the choice of ϕ.

We proceed with the second component of the new model in the same way, and then continue until the last one. We obtain matrices $\Phi(j)$ for $1 \leq j \leq d$ and set $\hat{\Phi} = \Phi(1) \cdots \Phi(d)$. Then $\hat{\Phi}$ has the required properties. Indeed, the construction and $\Phi(i)e_j = e_j$ for $i \neq j$ imply $e_j \in \hat{\Phi}^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$. This is (ii), and (i) is a consequence of (ii). $\square$

Coming back to the utility maximization problem, note that property (iii) implies that the value functions and the opportunity processes for the models $(R, \mathscr{C})$ and $(\widetilde{R}, \widetilde{\mathscr{C}})$ coincide up to evanescence; we identify them in the sequel. Furthermore, if $\tilde{g}$ denotes the analogue of g in the model $(\widetilde{R}, \widetilde{\mathscr{C}})$, cf. (6.1), we have the relation

$$\tilde{g}(y) = g(\Phi y), \quad y \in \widetilde{\mathscr{C}^0}.$$

Finding a maximizer for $\tilde{g}$ is equivalent to finding one for g and if $(\tilde{\pi}, \kappa)$ is an optimal strategy for $(\widetilde{R}, \widetilde{\mathscr{C}})$ then $(\Phi\tilde{\pi}, \kappa)$ is optimal for $(R, \mathscr{C})$. In fact, most properties of interest carry over from $(R, \mathscr{C})$ to $(\widetilde{R}, \widetilde{\mathscr{C}})$, in particular any no-arbitrage property that is defined via the set of admissible (positive) wealth processes.

Remark 7.4. A classical no-arbitrage condition defined in a slightly different way is that there exist a probability measure $Q \approx P$ under which $\mathcal{E}(R)$ is a σ-martingale; cf. Delbaen and Schachermayer [17]. In this case, $\mathcal{E}(\widetilde{R})$ is even a local martingale under Q, as it is a σ-martingale with positive components.

Property (ii) from Theorem 7.3 is useful to apply the following result.

Lemma 7.5. *Let $p \in (0, 1)$ and assume $e_j \in \mathscr{C} \cap \mathscr{C}^{0,*}$ for $1 \leq j \leq d$. Then $u(x_0) < \infty$ implies that $F^{R,L}$ is p-suitable. If, in addition, there exists a constant k_1 such that $D \geq k_1 > 0$, it follows that $\int_{\{|x|>1\}} |x|^p \, F^R(dx) < \infty$.*

Proof. As $p > 0$ and $u(x_0) < \infty$, L is well defined and $L, L_- > 0$ by Section III.2.2. No further properties were used to establish Lemma 3.4, whose formula shows that $g(\pi)$ is finite $P \otimes A$-a.e. for all $\pi \in \mathcal{A} = \mathcal{A}^{fE}$. In particular, from the definition of g, it follows that $\int (L_- + x')\{p^{-1}(1 + \pi^\top x)^p - p^{-1} - \pi^\top h(x)\} \, F^{R,L}(d(x, x'))$ is finite. If $D \geq k_1$, Lemma II.3.5 shows that $L \geq k_1$, hence $L_- + x' \geq k_1$ $F^L(dx')$-a.e. and $\int \{p^{-1}(1 + \pi^\top x)^p - p^{-1} - \pi^\top h(x)\} \, F^R(dx) < \infty$. We choose $\pi = e_j$ (and κ arbitrary) for $1 \leq j \leq d$ to deduce the result. $\square$

In general, the condition $u(x_0) < \infty$ does not imply any properties of R; for instance, in the trivial cases $\mathscr{C} = \{0\}$ or $\mathscr{C}^{0,*} = \{0\}$. The transformation changes the geometry of $\mathscr{C}$ and $\mathscr{C}^{0,*}$ such that Theorem 7.3(ii) holds, and then the situation is different.

Corollary 7.6. *Let $p \in (0,1)$ and $u(x_0) < \infty$. In the model $(\widetilde{R}, \widetilde{\mathscr{C}})$ of Theorem 7.3, $F^{\widetilde{R},L}$ is p-suitable and hence $\tilde{g}$ is continuous.*

Therefore, to prove Lemma 3.8 under (C4), we may substitute $(R, \mathscr{C})$ by $(\widetilde{R}, \widetilde{\mathscr{C}})$ and avoid the use of p-suitable approximating sequences. In some cases, Lemma 7.5 applies directly in $(R, \mathscr{C})$. In particular, if the asset prices are strictly positive ($\Delta R^j > -1$ for $1 \leq j \leq d$), then the positive orthant of $\mathbb{R}^d$ is contained in $\mathscr{C}^{0,*}$ and the condition of Lemma 7.5 is satisfied as soon as $e_j \in \mathscr{C}$ for $1 \leq j \leq d$.

III.8 Appendix C: Omitted Calculation

This appendix contains a calculation which was omitted in the proof of Proposition 5.12.

Lemma 8.1. *Let $(\ell, \check{\pi}, \check{\kappa})$ be a solution of the Bellman equation, $(\pi, \kappa) \in \mathcal{A}$, $X := X(\pi, \kappa)$ and $\check{X} := X(\check{\pi}, \check{\kappa})$. Define $\bar{R} = R - (x - h(x)) * \mu^R$ as well as $\bar{\pi} := (p-1)\check{\pi} + \pi$ and $\bar{\kappa} := (p-1)\check{\kappa} + \kappa$. Then $\xi := \ell \check{X}^{p-1} X$ satisfies*

$$\begin{aligned}
&\big(\check{X}_-^{p-1} X_-\big)^{-1} \bullet \xi = \\
&\ell - \ell_0 + \ell_- \bar{\pi} \bullet \bar{R} - \ell_- \bar{\kappa} \bullet \mu + \ell_-(p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A + \bar{\pi}^\top c^{R\ell} \bullet A \\
&+ \bar{\pi}^\top x' h(x) * \mu^{R,\ell} + (\ell_- + x')\big\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top h(x)\big\} * \mu^{R,\ell}.
\end{aligned}$$

Proof. We may assume $x_0 = 1$. This calculation is similar to the one in the proof of Lemma 3.4 and therefore we shall be brief. By Itô's formula we have $\check{X}^{p-1} = \mathcal{E}(\zeta)$ for

$$\begin{aligned}
\zeta = (p-1)(\check{\pi} \bullet R - \check{\kappa} \bullet \mu) + \tfrac{(p-1)(p-2)}{2} \check{\pi}^\top c^R \check{\pi} \bullet A \\
+ \big\{(1 + \check{\pi}^\top x)^{p-1} - 1 - (p-1)\check{\pi}^\top x\big\} * \mu^R.
\end{aligned}$$

Thus $\check{X}^{p-1} X = \mathcal{E}\big(\zeta + \pi \bullet R - \kappa \bullet \mu + [\zeta, \pi \bullet R]\big) =: \mathcal{E}(\Psi)$ with

$$\begin{aligned}
[R, \zeta] &= [R^c, \zeta^c] + \sum \Delta R \Delta \zeta \\
&= (p-1) c^R \check{\pi} \bullet A + (p-1)\check{\pi}^\top x x * \mu^R \\
&\quad + x\big\{(1 + \check{\pi}^\top x)^{p-1} - 1 - \check{\pi}^\top x\big\} * \mu^R
\end{aligned}$$

and recombining the terms yields

$$\begin{aligned}
\Psi = \bar{\pi} \bullet R - \bar{\kappa} \bullet \mu + (p-1)\big(\tfrac{p-2}{2}\check{\pi} + \pi\big)^\top c^R \check{\pi} \bullet A \\
+ \big\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top x\big\} * \mu^R.
\end{aligned}$$

Then $\left(\check{X}_-^{p-1}X_-\right)^{-1} \bullet \xi = \ell - \ell_0 + \ell_- \bullet \Psi + [\ell, \Psi]$, where

$$\begin{aligned}[\ell, \Psi] &= [\ell^c, \Psi^c] + \sum \Delta\ell\Delta\Psi \\ &= \bar{\pi}^\top c^{R\ell} \bullet A + \bar{\pi}^\top x' x * \mu^{R,\ell} \\ &\quad + x'\left\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top x\right\} * \mu^{R,\ell}.\end{aligned}$$

We arrive at

$$\begin{aligned}&\left(\check{X}_-^{p-1}X_-\right)^{-1} \bullet \xi = \\ &\ell - \ell_0 + \ell_-\bar{\pi} \bullet R - \ell_-\bar{\kappa} \bullet \mu + \ell_-(p-1)\left(\tfrac{p-2}{2}\check{\pi} + \pi\right)^\top c^R\check{\pi} \bullet A + \bar{\pi}^\top c^{R\ell} \bullet A \\ &\quad + \bar{\pi}^\top x' x * \mu^{R,\ell} + (\ell_- + x')\left\{(1 + \check{\pi}^\top x)^{p-1}(1 + \pi^\top x) - 1 - \bar{\pi}^\top x\right\} * \mu^{R,\ell}.\end{aligned}$$

The result follows by writing $x = h(x) + x - h(x)$. $\square$

Chapter IV

Lévy Models

In this chapter, which corresponds to the article [60], we study power utility maximization for exponential Lévy models with portfolio constraints and construct an explicit solution in terms of the Lévy triplet.

IV.1 Introduction

We consider the case when the asset prices follow an exponential *Lévy process* and the investor's preferences are given by a *power utility* function. This problem was first studied by Merton [55] for drifted geometric Brownian motion and by Mossin [57] and Samuelson [65] for the discrete-time analogues. A consistent observation was that when the asset returns are i.i.d., the optimal portfolio and consumption are given by a constant and a deterministic function, respectively. This result was subsequently extended to various classes of Lévy models and its general validity was readily conjectured—we note that the existence of an optimal strategy is known also for much more general models (see Karatzas and Žitković [43]), but *a priori* that strategy is *some* stochastic process without a constructive description.

We prove this conjecture for general Lévy models under *minimal* assumptions; in addition, we consider the case where the choice of the portfolio is constrained to a convex set. The optimal *investment portfolio* is characterized as the maximizer of a deterministic concave function $\mathfrak{g}$ defined in terms of the Lévy triplet; and the maximum of $\mathfrak{g}$ yields the optimal *consumption*. Moreover, the Lévy triplet characterizes the finiteness of the value function, i.e., the maximal expected utility. We also draw the conclusions for the *q-optimal equivalent martingale measures* that are linked to utility maximization by convex duality ($q \in (-\infty, 1) \setminus \{0\}$); this results in an explicit existence characterization and a formula for the density process. Finally, some generalizations to non-convex constraints are studied.

Our method consists in solving the *Bellman equation*, which was introduced for general semimartingale models in Chapter III. In the Lévy setting, this equation reduces to a Bernoulli ordinary differential equation. There are two main mathematical difficulties. The first one is to construct the maximizer for $\mathfrak{g}$, i.e., the optimal portfolio. The necessary compactness is obtained from a minimal no-free-lunch condition ("no unbounded increasing profit") via scaling arguments which were developed by Kardaras [44] for log-utility. In our setting these arguments require certain integrability properties of the asset returns. Without compromising the generality, integrability is achieved by a linear transformation of the model which replaces the given assets by certain portfolios. We construct the maximizer for $\mathfrak{g}$ in the transformed model and then revert to the original one.

The second difficulty is to *verify* the optimality of the constructed consumption and investment portfolio. Here we use the general verification theory of Chapter III and exploit a well-known property of Lévy processes, namely that any Lévy local martingale is a true martingale.

This chapter is organized as follows. The next section specifies the optimization problem and the notation related to the Lévy triplet. We also recall the no-free-lunch condition NUIP$_{\mathscr{C}}$ and the opportunity process. Section IV.3 states the main result for utility maximization under convex constraints and relates the triplet to the finiteness of the value function. The transformation of the model is described in Section IV.4 and the main theorem is proved in Section IV.5. Section IV.6 gives the application to q-optimal measures while non-convex constraints are studied in Section IV.7. Related literature is discussed in the concluding Section IV.8 as this necessitates technical terminology introduced in the body of the chapter.

IV.2 Preliminaries

The following notation is used. If $x, y \in \mathbb{R}$ are reals, $x^+ = \max\{x, 0\}$ and $x \wedge y = \min\{x, y\}$. We set $1/0 := \infty$ where necessary. If $z \in \mathbb{R}^d$ is a d-dimensional vector, z^i is its ith coordinate and $z^\top$ its transpose. Given $A \subseteq \mathbb{R}^d$, $A^\perp$ denotes the Euclidean orthogonal complement and A is said to be *star-shaped* (with respect to the origin) if $\lambda A \subseteq A$ for all $\lambda \in [0,1]$. If X is an $\mathbb{R}^d$-valued semimartingale and $\pi \in L(X)$ is an $\mathbb{R}^d$-valued predictable integrand, the vector stochastic integral is a scalar semimartingale with initial value zero and denoted by $\int \pi\, dX$ or by $\pi \bullet X$. Relations between measurable functions hold almost everywhere unless otherwise stated. Our reference for any unexplained notion or notation from stochastic calculus is Jacod and Shiryaev [34].

IV.2.1 The Optimization Problem

We fix the time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual assumptions of right-continuity and completeness, as well as $\mathcal{F}_0 = \{\emptyset, \Omega\}$ P-a.s. We consider an $\mathbb{R}^d$-valued *Lévy* *process* $R = (R^1, \dots, R^d)$ with $R_0 = 0$. That is, R is a càdlàg semimartingale with stationary independent increments as defined in [34, II.4.1(b)]. It is not relevant for us whether R generates the filtration. The stochastic exponential $S = \mathcal{E}(R) = (\mathcal{E}(R^1), \dots, \mathcal{E}(R^d))$ represents the discounted price processes of d risky assets, while R stands for their returns. If one wants to model only positive prices, one can equivalently use the ordinary exponential (see, e.g., Kallsen [38, Lemma 4.2]). Our agent also has a bank account paying zero interest at his disposal.

The agent is endowed with a deterministic initial capital $x_0 > 0$. A *trading strategy* is a predictable R-integrable $\mathbb{R}^d$-valued process π, where the ith component is interpreted as the fraction of wealth (or the portfolio proportion) invested in the ith risky asset.

We want to consider two cases. Either consumption occurs only at the terminal time T (utility from "terminal wealth" only); or there is intermediate consumption plus a bulk consumption at the time horizon. To unify the notation, we define[1]

$$\delta := \begin{cases} 1 & \text{in the case with intermediate consumption,} \\ 0 & \text{otherwise.} \end{cases}$$

It will be convenient to parametrize the consumption strategies as a fraction of the current wealth. A *propensity to consume* is a nonnegative optional process κ satisfying $\int_0^T \kappa_s \, ds < \infty$ P-a.s. The *wealth process* $X(\pi, \kappa)$ corresponding to a pair (π, κ) is defined by the stochastic exponential

$$X(\pi, \kappa) = x_0 \mathcal{E}\big(\pi \bullet R - \delta \textstyle\int \kappa_s \, ds\big).$$

Let $\mathscr{C} \subseteq \mathbb{R}^d$ be a (constant) set containing the origin; we refer to $\mathscr{C}$ as "the constraints". The set of (constrained) *admissible* strategies is

$$\mathcal{A}(x_0) := \big\{(\pi, \kappa) : X(\pi, \kappa) > 0 \text{ and } \pi_t(\omega) \in \mathscr{C} \text{ for all } (\omega, t) \in \Omega \times [0, T]\big\}.$$

We fix the initial capital x_0 and usually write $\mathcal{A}$ for $\mathcal{A}(x_0)$. Given $(\pi, \kappa) \in \mathcal{A}$, $c := \kappa X(\pi, \kappa)$ is the corresponding *consumption rate* and $X = X(\pi, \kappa)$ satisfies the self-financing condition $X_t = x_0 + \int_0^t X_{s-} \pi_s \, dR_s - \delta \int_0^t c_s \, ds$ as well as $X_- > 0$.

Let $p \in (-\infty, 0) \cup (0, 1)$. We use the power utility function

$$U(x) := \tfrac{1}{p} x^p, \quad x \in (0, \infty)$$

to model the preferences of the agent. Note that we exclude the well-studied logarithmic utility (see [44]) which corresponds to $p = 0$. The constant $\beta := (1 - p)^{-1} > 0$ is the relative risk tolerance of U.

[1] In this chapter, it is more convenient to use the notation $\delta \, dt$ instead of $\mu(dt)$ as in the other chapters.

Let $(\pi, \kappa) \in \mathcal{A}$ and $X = X(\pi, \kappa)$, $c = \kappa X$. The corresponding *expected utility* is $E\big[\delta \int_0^T U(c_t)\,dt + U(X_T)\big]$. The *value function* is given by

$$u(x_0) := \sup_{\mathcal{A}(x_0)} E\Big[\delta \int_0^T U(c_t)\,dt + U(X_T)\Big],$$

where the supremum is taken over all (c, X) which correspond to some $(\pi, \kappa) \in \mathcal{A}(x_0)$. We say that the utility maximization problem is *finite* if $u(x_0) < \infty$. This always holds if $p < 0$ as then $U < 0$. If $u(x_0) < \infty$, (π, κ) is *optimal* if the corresponding (c, X) satisfy $E\big[\delta \int_0^T U(c_t)\,dt + U(X_T)\big] = u(x_0)$.

IV.2.2 Lévy Triplet, Constraints, No-Free-Lunch Condition

Let (b^R, c^R, F^R) be the Lévy triplet of R with respect to some fixed cut-off function $h : \mathbb{R}^d \to \mathbb{R}^d$ (i.e., h is bounded and $h(x) = x$ in a neighborhood of $x = 0$). This means that $b^R \in \mathbb{R}^d$, $c^R \in \mathbb{R}^{d\times d}$ is a nonnegative definite matrix, and F^R is a Lévy measure on $\mathbb{R}^d$, i.e., $F^R\{0\} = 0$ and

$$\int_{\mathbb{R}^d} 1 \wedge |x|^2 \, F^R(dx) < \infty. \tag{2.1}$$

The process R can be represented as

$$R_t = b^R t + R^c_t + h(x) * (\mu^R_t - \nu^R_t) + (x - h(x)) * \mu^R_t.$$

Here μ^R is the integer-valued random measure associated with the jumps of R and $\nu^R_t = tF^R$ is its compensator. Moreover, R^c is the continuous martingale part, in fact, $R^c_t = \sigma W_t$, where $\sigma \in \mathbb{R}^{d\times d}$ satisfies $\sigma\sigma^\top = c^R$ and W is a d-dimensional standard Brownian motion. We refer to [34, II.4] or Sato [68] for background material concerning Lévy processes.

We introduce some subsets of $\mathbb{R}^d$ to be used in the sequel; the terminology follows [44]. The first two are related to the "budget constraint" $X(\pi, \kappa) > 0$. The *natural constraints* are given by

$$\mathscr{C}^0 := \Big\{y \in \mathbb{R}^d : \, F^R\big[x \in \mathbb{R}^d : \, y^\top x < -1\big] = 0\Big\};$$

clearly $\mathscr{C}^0$ is closed, convex, and contains the origin. We also consider the slightly smaller set

$$\mathscr{C}^{0,*} := \Big\{y \in \mathbb{R}^d : \, F^R\big[x \in \mathbb{R}^d : \, y^\top x \le -1\big] = 0\Big\}.$$

It is convex, contains the origin, and its closure equals $\mathscr{C}^0$, but it is a proper subset in general. The meaning of these sets is explained by

Lemma 2.1. *A process $\pi \in L(R)$ satisfies $\mathcal{E}(\pi \bullet R) \ge 0\,(> 0)$ if and only if π takes values in $\mathscr{C}^0\,(\mathscr{C}^{0,*})$ $P \otimes dt$-a.e.*

See, e.g., Lemma III.2.5 for the proof. The linear space of *null-investments* is defined by

$$\mathscr{N} := \big\{y \in \mathbb{R}^d : y^\top b^R = 0,\, y^\top c^R = 0,\, F^R[x : y^\top x \neq 0] = 0\big\}.$$

Then $H \in L(R)$ satisfies $H \bullet R \equiv 0$ if and only if H takes values in $\mathscr{N}$ $P \otimes dt$-a.e. In particular, two portfolios π and π' generate the same wealth process (for given κ) if and only if $\pi - \pi'$ is $\mathscr{N}$-valued.

We recall the set $0 \in \mathscr{C} \subseteq \mathbb{R}^d$ of portfolio constraints. The set $\mathscr{J} \subseteq \mathbb{R}^d$ of *immediate arbitrage opportunities* is defined by

$$\mathscr{J} = \Big\{y : y^\top c^R = 0,\ F^R[y^\top x < 0] = 0,\ y^\top b^R - \int y^\top h(x)\, F^R(dx) \geq 0\Big\} \backslash \mathscr{N}.$$

Note that for $y \in \mathscr{J}$, the process $y^\top R$ is increasing and nonconstant. For a subset G of $\mathbb{R}^d$, its recession cone is given by $\check{G} := \bigcap_{a \geq 0} aG$. Now the condition NUIP$_\mathscr{C}$ (no unbounded increasing profit) can be defined by

$$\mathrm{NUIP}_\mathscr{C} \quad \Longleftrightarrow \quad \mathscr{J} \cap \check{\mathscr{C}} = \emptyset$$

(cf. [44, Theorem 4.5]). This is equivalent to $\mathscr{J} \cap (\mathscr{C} \cap \mathscr{C}^0)\check{} = \emptyset$ because $\mathscr{J} \subseteq \check{\mathscr{C}}^0$, and it means that if a strategy leads to an increasing nonconstant wealth process, then that strategy cannot be scaled arbitrarily. This is a very weak no-free-lunch condition; we refer to [44] for more information about free lunches in exponential Lévy models. We give a simple example to illustrate the objects.

Example 2.2. Assume there is only one asset ($d = 1$), that its price is strictly positive, and that it can jump arbitrarily close to zero and arbitrarily high. In formulas, $F^R(-\infty, -1] = 0$ and for all $\varepsilon > 0$, $F^R(-1, -1 + \varepsilon] > 0$ and $F^R[\varepsilon^{-1}, \infty) > 0$.

Then $\mathscr{C}^0 = \mathscr{C}^{0,*} = [0, 1]$ and $\mathscr{N} = \{0\}$. In this situation NUIP$_\mathscr{C}$ is satisfied for any set $\mathscr{C}$, both because $\mathscr{J} = \emptyset$ and because $\check{\mathscr{C}}^0 = \{0\}$. If the price process is merely nonnegative and $F^R\{-1\} > 0$, then $\mathscr{C}^{0,*} = [0, 1)$ while the rest stays the same.

In fact, most of the scalar models presented in Schoutens [70] correspond to the first part of Example 2.2 for nondegenerate choices of the parameters.

IV.2.3 Opportunity Process

Assume $u(x_0) < \infty$ and let $(\pi, \kappa) \in \mathcal{A}$. For fixed $t \in [0, T]$, the set of "compatible" controls is $\mathcal{A}(\pi, \kappa, t) := \big\{(\tilde{\pi}, \tilde{\kappa}) \in \mathcal{A} : (\tilde{\pi}, \tilde{\kappa}) = (\pi, \kappa) \text{ on } [0, t]\big\}$. By Proposition II.3.1 and Remark II.3.7 there exists a unique càdlàg semimartingale L, called *opportunity process*, such that

$$L_t\, \tfrac{1}{p}\big(X_t(\pi, \kappa)\big)^p = \operatorname*{ess\,sup}_{\mathcal{A}(\pi,\kappa,t)} E\Big[\delta \int_t^T U(\tilde{c}_s)\, ds + U(\tilde{X}_T)\Big|\mathcal{F}_t\Big],$$

where the supremum is taken over all consumption and wealth pairs $(\tilde{c}, \widetilde{X})$ corresponding to some $(\tilde{\pi}, \tilde{\kappa}) \in \mathcal{A}(\pi, \kappa, t)$. We shall see that in the present Lévy setting, the opportunity process is simply a deterministic function. The right hand side above is known as the *value process* of our control problem; in particular the "dynamic value function" at time t is $u_t(x) = L_t \frac{1}{p} x^p$.

IV.3 Main Result

We can now formulate the main theorem for the convex-constrained case; the proofs are given in the two subsequent sections. We consider the following conditions.

Assumptions 3.1.

(i) $\mathscr{C}$ is convex.

(ii) The orthogonal projection of $\mathscr{C} \cap \mathscr{C}^0$ onto $\mathscr{N}^\perp$ is closed.

(iii) $\mathrm{NUIP}_{\mathscr{C}}$ holds.

(iv) $u(x_0) < \infty$, i.e., the utility maximization problem is finite.

To state the result, we define for $y \in \mathscr{C}^0$ the deterministic function

$$\mathfrak{g}(y) := y^\top b^R + \tfrac{(p-1)}{2} y^\top c^R y + \int_{\mathbb{R}^d} \left\{ p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x) \right\} F^R(dx). \tag{3.1}$$

As we will see later, this concave function is well defined with values in $\mathbb{R} \cup \{\operatorname{sign}(p)\infty\}$.

Theorem 3.2. *Under Assumptions 3.1, there exists an optimal strategy $(\hat{\pi}, \hat{\kappa})$ such that $\hat{\pi}$ is a constant vector and $\hat{\kappa}$ is deterministic. Here $\hat{\pi}$ is characterized by*

$$\hat{\pi} \in \arg\max_{\mathscr{C} \cap \mathscr{C}^0} \mathfrak{g}$$

and, in the case with intermediate consumption,

$$\hat{\kappa}_t = a\big((1+a)e^{a(T-t)} - 1\big)^{-1},$$

where $a := \frac{p}{1-p} \max_{\mathscr{C} \cap \mathscr{C}^0} \mathfrak{g}$. The opportunity process is given by

$$L_t = \begin{cases} \exp\big(a(1-p)(T-t)\big) & \textit{without intermediate consumption,} \\ a^{p-1}\big[(1+a)e^{a(T-t)} - 1\big]^{1-p} & \textit{with intermediate consumption.} \end{cases}$$

Concerning the question of uniqueness, we recall the following from Remark III.3.3.

Remark 3.3. The propensity to consume $\hat{\kappa}$ is unique. The optimal portfolio and $\arg\max_{\mathscr{C} \cap \mathscr{C}^0} \mathfrak{g}$ are unique modulo $\mathscr{N}$; i.e., if π^* is another optimal portfolio (or maximizer), then $\hat{\pi} - \pi^*$ takes values in $\mathscr{N}$. Equivalently, the wealth processes coincide.

We comment on Assumptions 3.1.

Remark 3.4. (a) Convexity of $\mathscr{C}$ is of course not necessary to have a solution. We give some generalizations in Section IV.7.

(b) Without the closedness in (ii), there are examples with non-existence of an optimal strategy even for drifted Brownian motion and closed convex cone constraints; see Example 3.5 below. One can note that closedness of $\mathscr{C}$ implies (ii) if $\mathscr{N} \subseteq \mathscr{C}$ and $\mathscr{C}$ is convex (as this implies $\mathscr{C} = \mathscr{C} + \mathscr{N}$, see [44, Remark 2.4]). Similarly, (ii) holds whenever the projection of $\mathscr{C}$ to $\mathscr{N}^\perp$ is closed: if Π denotes the projector, $\mathscr{C}^0 = \mathscr{C}^0 + \mathscr{N}$ yields $\Pi(\mathscr{C} \cap \mathscr{C}^0) = (\Pi\mathscr{C}) \cap \mathscr{C}^0$ and $\mathscr{C}^0$ is closed. This includes the cases where $\mathscr{C}$ is closed and polyhedral, or compact.

(c) Suppose that $\mathrm{NUIP}_{\mathscr{C}}$ does not hold. If $p \in (0,1)$, it is obvious that $u(x_0) = \infty$. If $p < 0$, there exists no optimal strategy, essentially because adding a suitable arbitrage strategy would always yield a higher expected utility. See Karatzas and Kardaras [41, Proposition 4.19] for a proof.

(d) If $u(x_0) = \infty$, either there is no optimal strategy, or there are infinitely many strategies yielding infinite expected utility. It would be inconvenient to call the latter optimal. Indeed, using that $u(x_0/2) = \infty$, one can typically construct such strategies which also exhibit intuitively suboptimal behavior (such as throwing away money by a "suicide strategy"; see Harrison and Pliska [32, §6.1]). Hence we require (iv) to have a meaningful solution to our problem—the relevant question is how to characterize this condition in terms of the model.

The following example is based on Czichowsky et al. [15, §2.2] and illustrates how non-existence of an optimal portfolio may occur when Assumption 3.1(ii) is violated. We denote by e_j, $1 \le j \le d$ the unit vectors in $\mathbb{R}^d$, i.e., $e^i_j = \delta_{ij}$.

Example 3.5 ($\delta = 0$)**.** Let W be a standard Brownian motion in $\mathbb{R}^3$ and

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}; \quad \mathscr{C} = \Big\{ y \in \mathbb{R}^3 : \, |y^1|^2 + |y^2|^2 \le |y^3|^2, \, y^3 \ge 0 \Big\}.$$

Let $R_t = bt + \sigma W_t$, where $b := e_1$ is orthogonal to $\ker \sigma^\top = \mathbb{R}(0,1,1)^\top$. Thus $\mathscr{N} = \ker \sigma^\top$ and $\mathscr{N}^\perp$ is spanned by e_1 and $e_2 - e_3$. The closed convex cone $\mathscr{C}$ is "leaning" against the plane $\mathscr{N}^\perp$ and the orthogonal projection of $\mathscr{C}$ onto $\mathscr{N}^\perp$ is an open half-plane plus the origin. The vectors αe_1 with $\alpha \in \mathbb{R} \setminus \{0\}$ are not contained in this half-plane but in its closure.

The optimal portfolio $\hat{\pi}$ for the *unconstrained* problem lies on this boundary. Indeed, $\mathrm{NUIP}_{\mathbb{R}^3}$ holds and Theorem 3.2 yields $\hat{\pi} = \beta(\sigma\sigma^\top)^{-1} e_1 = \beta e_1$, where $\beta = (1-p)^{-1}$. This is simply Merton's optimal portfolio in the market consisting only of the first asset. By construction we find vectors

$\pi^n \in \mathscr{C}$ whose projections to $\mathscr{N}^\perp$ converge to $\hat{\pi}$ and it is easy to see that $E[U(X_T(\pi^n))] \to E[U(X_T(\hat{\pi}))]$. Hence the value functions for the constrained and the unconstrained problem are identical. Since the solution $\hat{\pi}$ of the unconstrained problem is unique modulo $\mathscr{N}$, this implies that if the constrained problem has a solution, it has to agree with $\hat{\pi}$, modulo $\mathscr{N}$. But $(\{\hat{\pi}\} + \mathscr{N}) \cap \mathscr{C} = \emptyset$, so there is no solution.

The rest of the section is devoted to the characterization of Assumption 3.1(iv) by the jump characteristic F^R and the set $\mathscr{C}$; this is intimately related to the moment condition

$$\int_{\{|x|>1\}} |x|^p \, F^R(dx) < \infty. \tag{3.2}$$

We start with a partial result; again e_j, $1 \leq j \leq d$ denote the unit vectors.

Proposition 3.6. *Let $p \in (0,1)$.*

(i) Under Assumptions 3.1(i)-(iii), (3.2) *implies $u(x_0) < \infty$.*

(ii) If $e_j \in \mathscr{C} \cap \mathscr{C}^{0,*}$ for all $1 \leq j \leq d$, then $u(x_0) < \infty$ implies (3.2).

By Lemma 2.1 the jth asset has a positive price if and only if $e_j \in \mathscr{C}^{0,*}$. Hence we have the following consequence of Proposition 3.6.

Corollary 3.7. *In an unconstrained exponential Lévy model with positive asset prices satisfying* $\mathrm{NUIP}_{\mathbb{R}^d}$*, $u(x_0) < \infty$ is equivalent to* (3.2).

The implication $u(x_0) < \infty \Rightarrow$ (3.2) is essentially true also in the general case; more precisely, it holds in an equivalent model. As a motivation, consider the case where either $\mathscr{C} = \{0\}$ or $\mathscr{C}^0 = \{0\}$. The latter occurs, e.g., if $d = 1$ and the asset has jumps which are unbounded in both directions. Then the statement $u(x_0) < \infty$ carries no information about R because $\pi \equiv 0$ is the only admissible portfolio. On the other hand, we are not interested in assets that cannot be traded, and may as well remove them from the model. This is part of the following result.

Proposition 3.8. *There exists a linear transformation $(\widetilde{R}, \widetilde{\mathscr{C}})$ of the model $(R, \mathscr{C})$, which is equivalent in that it admits the same wealth processes, and has the following properties:*

(i) the prices are strictly positive,

(ii) the wealth can be invested in each asset (i.e., $\pi \equiv e_j$ is admissible),

(iii) if $u(x_0) < \infty$ holds for $(R, \mathscr{C})$, it holds also in the model $(\widetilde{R}, \widetilde{\mathscr{C}})$ and $\int_{\{|x|>1\}} |x|^p \, F^{\widetilde{R}}(dx) < \infty$.

The details of the construction are given in the next section, where we also show that Assumptions 3.1 carry over to $(\widetilde{R}, \widetilde{\mathscr{C}})$.

IV.4 Transformation of the Model

This section contains the announced linear transformation of the market model. Assumptions 3.1 are not used. We first describe how any linear transformation affects our objects.

Lemma 4.1. *Let Λ be a $d \times d$-matrix and define $\widetilde{R} := \Lambda R$. Then $\widetilde{R}$ is a Lévy process with triplet $b^{\widetilde{R}} = \Lambda b^R$, $c^{\widetilde{R}} = \Lambda c^R \Lambda^\top$ and $F^{\widetilde{R}}(\cdot) = F^R(\Lambda^{-1}\cdot)$. Moreover, the corresponding natural constraints and null-investments are given by $\widetilde{\mathscr{C}^0} := (\Lambda^T)^{-1}\mathscr{C}^0$ and $\widetilde{\mathscr{N}} := (\Lambda^T)^{-1}\mathscr{N}$ and the corresponding function $\tilde{\mathfrak{g}}$ satisfies $\tilde{\mathfrak{g}}(z) = \mathfrak{g}(\Lambda^\top z)$.*

The proof is straightforward and omitted. Of course, Λ^{-1} refers to the preimage if Λ is not invertible. Given Λ, we keep the notation from Lemma 4.1 and introduce also $\widetilde{\mathscr{C}} := (\Lambda^T)^{-1}\mathscr{C}$ as well as $\widetilde{\mathscr{C}^{0,*}} := (\Lambda^T)^{-1}\mathscr{C}^{0,*}$ (which is consistent with Section IV.2.2).

Theorem 4.2. *There exists a matrix $\Lambda \in \mathbb{R}^{d\times d}$ such that for $1 \leq j \leq d$,*

(i) $\Delta \widetilde{R}^j > -1$ *up to evanescence,*

(ii) $e_j \in \widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}^{0,*}}$,

(iii) the model $(\widetilde{R}, \widetilde{\mathscr{C}})$ admits the same wealth processes as $(R, \mathscr{C})$.

Proof. We treat the components one by one. Pick any vector $y_1 \in \mathscr{C} \cap \mathscr{C}^{0,*}$ such that $y_1^1 \neq 0$, if there is no such vector, set $y_1 = 0$. We replace the first asset R^1 by the process $y_1^\top R$. In other words, we replace R by $\Lambda_1 R$, where Λ_1 is the matrix

$$\Lambda_1 = \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^d \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The new natural constraints are $(\Lambda_1^\top)^{-1}\mathscr{C}^0$ and we replace $\mathscr{C}$ by $(\Lambda_1^\top)^{-1}\mathscr{C}$. Note that $e_1 \in (\Lambda_1^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$ because $\Lambda_1^\top e_1 = y_1 \in \mathscr{C} \cap \mathscr{C}^{0,*}$ by construction. Similarly, $(\Lambda_1^\top \psi) \bullet R = \psi \bullet (\Lambda_1 R)$ whenever $\Lambda_1^\top \psi \in L(R)$. Therefore, to show that the new model admits the same wealth processes as the old one, we have to show that for every $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued process $\pi \in L(R)$ there exists a predictable ψ such that $\Lambda_1^\top \psi = \pi$; note that this already implies $\psi \in L(\Lambda_1 R)$ and that ψ takes values in $(\Lambda_1^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$. If $\Lambda_1^\top$ is invertible, we take $\psi := (\Lambda_1^\top)^{-1}\pi$. Otherwise $\pi^1 \equiv 0$ by construction and we choose $\psi^1 \equiv 0$ and $\psi^j = \pi^j$ for $j \geq 2$; this is the same as inverting $\Lambda_1^\top$ on its image.

We proceed with the second component of the new model in the same way, and then continue until the last one. We obtain matrices Λ_j, $1 \leq j \leq d$ and set $\Lambda = \Lambda_d \cdots \Lambda_1$. The construction and $\Lambda_i^\top e_j = e_j$ for $i \neq j$ imply $e_j \in (\Lambda^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*})$, which is (ii), and (i) is a consequence of (ii). $\square$

From now on let Λ and $\widetilde{R}$ be as in Theorem 4.2.

Corollary 4.3. *(i) The value functions for $(R,\mathscr{C})$ and $(\widetilde{R},\widetilde{\mathscr{C}})$ coincide.*

(ii) The opportunity processes for $(R,\mathscr{C})$ and $(\widetilde{R},\widetilde{\mathscr{C}})$ coincide.

(iii) $\sup_{\widetilde{\mathscr{C}}\cap\widetilde{\mathscr{C}}^{0,*}} \tilde{\mathfrak{g}} = \sup_{\mathscr{C}\cap\mathscr{C}^{0,*}} \mathfrak{g}$.

(iv) $z \in \arg\max_{\widetilde{\mathscr{C}}\cap\widetilde{\mathscr{C}}^{0,*}} \tilde{\mathfrak{g}}$ *if and only if* $\Lambda^\top z \in \arg\max_{\mathscr{C}\cap\mathscr{C}^{0,*}} \mathfrak{g}$.

(v) (π,κ) is an optimal strategy for $(\widetilde{R},\widetilde{\mathscr{C}})$ if and only if $(\Lambda^\top\pi,\kappa)$ is optimal for $(R,\mathscr{C})$.

(vi) $\mathrm{NUIP}_{\widetilde{\mathscr{C}}}$ *holds for $\widetilde{R}$ if and only if* $\mathrm{NUIP}_{\mathscr{C}}$ *holds for R.*

Proof. This follows from Theorem 4.2(iii) and Lemma 4.1. □

The transformation also preserves certain properties of the constraints.

Remark 4.4. (a) If $\mathscr{C}$ is closed (star-shaped, convex), then $\widetilde{\mathscr{C}}$ is also closed (star-shaped, convex).

(b) Let $\mathscr{C}$ be compact, then $\widetilde{\mathscr{C}}$ is compact only if Λ is invertible. However, the relevant properties for Theorem 4.2 are that $\Lambda^\top\widetilde{\mathscr{C}} = \mathscr{C}\cap(\Lambda^\top\mathbb{R}^d)$ and that $e_j \in \widetilde{\mathscr{C}}$ for $1\le j\le d$; we can equivalently substitute $\widetilde{\mathscr{C}}$ by a compact set having these properties. If $\Lambda^\top$ is considered as a mapping $\mathbb{R}^d\to\Lambda^\top\mathbb{R}^d$, it admits a continuous right-inverse f, and $(\Lambda^\top)^{-1}\mathscr{C} = f(\mathscr{C}\cap\Lambda^\top\mathbb{R}^d)+\ker(\Lambda^\top)$. Here $f(\mathscr{C}\cap\Lambda^\top\mathbb{R}^d)$ is compact and contained in $B_r=\{x\in\mathbb{R}^d : |x|\le r\}$ for some $r\ge 1$. The set $\widehat{\mathscr{C}} := \big[f(\mathscr{C}\cap\Lambda^\top\mathbb{R}^d)+\ker(\Lambda^\top)\big]\cap B_r$ has the two desired properties.

Next, we deal with the projection of $\widetilde{\mathscr{C}}\cap\widetilde{\mathscr{C}}^0$ onto $\widetilde{\mathscr{N}}^\perp$. We begin with a "coordinate-free" description for its closedness; it can be seen as a simple static version of Czichowsky and Schweizer [14].

Lemma 4.5. *Let $\mathscr{D}\subseteq\mathbb{R}^d$ be a nonempty set and let $\mathscr{D}'$ be its orthogonal projection onto $\mathscr{N}^\perp$. Then $\mathscr{D}'$ is closed in $\mathbb{R}^d$ if and only if $\{y^\top R_T : y\in\mathscr{D}\}$ is closed for convergence in probability.*

Proof. Recalling the definition of $\mathscr{N}$, we may assume that $\mathscr{D}=\mathscr{D}'$. If (y_n) is a sequence in $\mathscr{D}$ with some limit y_*, clearly $y_n^\top R_T\to y_*^\top R_T$ in probability. If $\{y^\top R_T : y\in\mathscr{D}\}$ is closed, it follows that $y_*\in\mathscr{D}$ because $\mathscr{D}\cap\mathscr{N}=\{0\}$; hence $\mathscr{D}$ is closed.

Conversely, let $y_n\in\mathscr{D}$ and assume $y_n^\top R_T\to Y$ in probability for some $Y\in L^0(\mathcal{F})$. With $\mathscr{D}\cap\mathscr{N}=\{0\}$ it follows that (y_n) is bounded, therefore, it has a subsequence which converges to some y_*. If $\mathscr{D}$ is closed, then $y_*\in\mathscr{D}$ and we conclude that $Y=y_*^\top R_T$, showing closedness in probability. □

Lemma 4.6. *Assume that $\mathscr{C}\cap\mathscr{C}^{0,*}$ is dense in $\mathscr{C}\cap\mathscr{C}^0$. Then the orthogonal projection of $\mathscr{C}\cap\mathscr{C}^0$ onto $\mathscr{N}^\perp$ is closed if and only if this holds for $\widetilde{\mathscr{C}}\cap\widetilde{\mathscr{C}}^0$ and $\widetilde{\mathscr{N}}^\perp$.*

Proof. (i) Recall the construction of $\Lambda = \Lambda_d \cdots \Lambda_1$ from the proof of Theorem 4.2. Assume first that $\Lambda = \Lambda_i$ for some $1 \leq i \leq d$. In a first step, we show

$$\Lambda^\top(\widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^0) = \mathscr{C} \cap \mathscr{C}^0. \tag{4.1}$$

By construction, either Λ is invertible, in which case the claim is clear, or otherwise $\Lambda^\top$ is the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane $H_i = \{y \in \mathbb{R}^d : y^i = 0\}$ and $\mathscr{C} \cap \mathscr{C}^{0,*} \subseteq H_i$. By the density assumption it follows that $\mathscr{C} \cap \mathscr{C}^0 \subseteq H_i$. Thus $(\Lambda^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^0) = \widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^0 + H_i^\perp$, the sum being orthogonal, and $\Lambda^\top[(\Lambda^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^0)] = \mathscr{C} \cap \mathscr{C}^0$ as claimed. We also note that

$$(\Lambda^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^{0,*}) \subseteq (\Lambda^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^0) \quad \text{is dense.} \tag{4.2}$$

Using (4.1) we have $\{y^\top R_T : y \in \mathscr{C} \cap \mathscr{C}^0\} = \{\tilde{y}^\top \widetilde{R}_T : \tilde{y} \in \widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^0\}$ and now the result follows from Lemma 4.5, for the special case $\Lambda = \Lambda_i$.

(ii) In the general case, we have $\widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^0 = (\Lambda_d^\top)^{-1} \circ \cdots \circ (\Lambda_1^\top)^{-1}(\mathscr{C} \cap \mathscr{C}^0)$. We apply part (i) successively to $\Lambda_1, \dots, \Lambda_d$ to obtain the result, here (4.2) ensures that the density assumption is satisfied in each step. □

Lemma 4.7. *Let $p \in (0,1)$ and assume $e_j \in \mathscr{C} \cap \mathscr{C}^{0,*}$. Then $u(x_0) < \infty$ implies $\int_{\{|x|>1\}} |x^j|^p \, F^R(dx) < \infty$.*

Proof. Note that $e_j \in \mathscr{C}^{0,*}$ implies $\Delta R^j > -1$, i.e., $\int_{\{|x|>1\}} |x^j|^p \, F^R(dx) = \int_{\{|x|>1\}} ((x^j)^+)^p \, F^R(dx)$. Moreover, we have $E\big[x_0^p \mathcal{E}(R^j)_T^p\big] \leq u(x_0)$. There exists a Lévy process Z such that $\mathcal{E}(R^j)^p = e^Z$, hence $E\big[\mathcal{E}(R^j)_T^p\big] < \infty$ implies that $\mathcal{E}(R^j)^p$ is of class (D) (cf. [38, Lemma 4.4]). In particular, $\mathcal{E}(R^j)^p$ has a Doob-Meyer decomposition with a well defined drift (predictable finite variation part). The stochastic logarithm Y of $\mathcal{E}(R^j)^p$ is given by $Y = \mathcal{E}(R^j)_-^{-p} \bullet \mathcal{E}(R^j)^p$ and the drift of Y is again well defined because the integrand $\mathcal{E}(R^j)_-^{-p}$ is locally bounded. Itô's formula shows that Y is a Lévy process with drift

$$A_t^Y = t\bigg(p(b^R)^j + \tfrac{p(p-1)}{2}(c^R)^{jj} + \int_{\mathbb{R}^d} \big\{(1+x^j)^p - 1 - pe_j^\top h(x)\big\} \, F^R(dx)\bigg).$$

In particular, $\int_{\{|x|>1\}} ((x^j)^+)^p \, F^R(dx) < \infty$. □

Note that the previous lemma shows Proposition 3.6(ii); moreover, in the general case, we obtain the desired integrability in the transformed model.

Corollary 4.8. *$u(x_0) < \infty$ implies $\int_{\{|x|>1\}} |x|^p \, F^{\widetilde{R}}(dx) < \infty$.*

Proof. By Theorem 4.2(ii) and Corollary 4.3(i) we can apply Lemma 4.7 in the model $(\widetilde{R}, \widetilde{\mathscr{C}})$. □

Remark 4.9. The transformation in Theorem 4.2 preserves the Lévy structure. Theorem 4.2 and Lemma 4.7 were generalized to semimartingale models in Appendix B of Chapter III. There, additional assumptions are required for measurable selections and particular structures of the model are not preserved in general.

IV.5 Proof of Theorem 3.2

Our aim is to prove Theorem 3.2 and Proposition 3.6(i). We shall see that we may replace Assumption 3.1(iv) by the integrability condition (3.2). Under (3.2), we will obtain the fact that $u(x_0) < \infty$ as we construct the optimal strategies, and that will yield the proof for both results.

IV.5.1 Solution of the Bellman Equation

We start with informal considerations to construct a candidate solution, which we then verify. If there is an optimal strategy, Theorem III.3.2 states that the drift rate a^L of the opportunity process L (a special semimartingale in general) satisfies the Bellman equation

$$a^L\,dt = \delta(p-1)L_-^{p/(p-1)}\,dt - p \max_{y\in\mathscr{C}\cap\mathscr{C}^0} g(y)\,dt; \quad L_T = 1, \tag{5.1}$$

where g is the following function, stated in terms of the joint differential semimartingale characteristics of (R, L):

$$\begin{aligned} g(y) = L_- y^\top \Big(b^R + \tfrac{c^{RL}}{L_-} + \tfrac{(p-1)}{2}c^R y\Big) &+ \int_{\mathbb{R}^d\times\mathbb{R}} x' y^\top h(x)\,F^{R,L}(d(x,x')) \\ + \int_{\mathbb{R}^d\times\mathbb{R}} (L_- + x')\big\{p^{-1}(1+y^\top x)^p - p^{-1} &- y^\top h(x)\big\}\,F^{R,L}(d(x,x')). \end{aligned}$$

In this equation one should see the characteristics of R as the driving terms and L as the solution. In the present Lévy case, the differential characteristics of R are given by the Lévy triplet, in particular, they are deterministic. To wit, there is *no exogenous stochasticity* in (5.1). Therefore we can expect that the opportunity process is deterministic. We make a smooth *Ansatz* ℓ for L. As ℓ has no jumps and vanishing martingale part, g reduces to $g(y) = \ell\mathfrak{g}(y)$, where $\mathfrak{g}$ is as (3.1). We show later that this deterministic function is well defined. For the maximization in (5.1), we have the following result.

Lemma 5.1. *Suppose that Assumptions 3.1(i)-(iv) hold, or alternatively that Assumptions 3.1(i)-(iii) and* (3.2) *hold. Then* $\mathfrak{g}^* := \sup_{\mathscr{C}\cap\mathscr{C}^0}\mathfrak{g} < \infty$ *and there exists a nonrandom vector* $\check{\pi} \in \mathscr{C}\cap\mathscr{C}^{0,*}$ *such that* $\mathfrak{g}(\check{\pi}) = \mathfrak{g}^*$.

The proof is given in the subsequent section. As ℓ is a smooth function, its drift rate is simply the time derivative, hence we deduce from (5.1)

$$d\ell_t = \delta(p-1)\ell_t^{p/(p-1)}\,dt - p\mathfrak{g}^*\ell_t\,dt; \quad \ell_T = 1.$$

This is a Bernoulli ODE. If we denote $\beta := 1/(1-p)$, the transformation $f(t) := \ell_{T-t}^{\beta}$ produces the forward linear equation

$$\tfrac{d}{dt}f(t) = \delta + \big(\tfrac{p}{1-p}\mathfrak{g}^*\big)f(t); \quad f(0) = 1,$$

which has, with $a = \frac{p}{1-p}\mathfrak{g}^*$, the unique solution $f(t) = -\delta/a + (1+\delta/a)e^{at}$. Therefore,

$$\ell_t = \begin{cases} e^{(a/\beta)(T-t)} = e^{p\mathfrak{g}^*(T-t)} & \text{if } \delta = 0,\\ a^{-1/\beta}\big((1+a)e^{a(T-t)} - 1\big)^{1/\beta} & \text{if } \delta = 1. \end{cases}$$

If we define $\check{\kappa}_t := \ell_t^{-\beta} = a\big((1+a)e^{a(T-t)} - 1\big)^{-1}$, then $(\ell, \check{\pi}, \check{\kappa})$ is a solution of the Bellman equation in the sense of Definition III.4.1; note that the martingale part vanishes. Let also $\check{X} = X(\check{\pi}, \check{\kappa})$ be the corresponding wealth process. We want to *verify* this solution, i.e., to show that ℓ is the opportunity process and that $(\check{\pi}, \check{\kappa})$ is optimal. We shall use the following result; it is a special case of Proposition III.4.7 and Theorem III.5.15.

Lemma 5.2. *The process*

$$\Gamma := \ell\check{X}^p + \delta\int \check{\kappa}_s\ell_s\check{X}_s^p\,ds \tag{5.2}$$

is a local martingale. If $\mathscr{C}$ is convex, then Γ is a martingale if and only if $u(x_0) < \infty$ and $(\check{\pi}, \check{\kappa})$ is optimal and ℓ is the opportunity process.

To check that Γ is a martingale, it is convenient to consider the closely related process

$$\Psi := p\check{\pi}^\top R^c + \big\{(1+\check{\pi}^\top x)^p - 1\big\} * (\mu^R - \nu^R).$$

Remark III.5.9 shows that $\mathcal{E}(\Psi)$ is a positive local martingale and that Γ is a martingale as soon as $\mathcal{E}(\Psi)$ is. Now Ψ has an advantageous structure: as $\check{\pi}$ is constant, Ψ is a semimartingale with constant characteristics. In other words, $\mathcal{E}(\Psi)$ is an exponential Lévy local martingale, therefore automatically a true martingale (e.g., [38, Lemmata 4.2, 4.4]). Hence we can apply Lemma 5.2 to finish the proofs of Theorem 3.2 and Proposition 3.6(i), modulo Lemma 5.1.

IV.5.2 Proof of Lemma 5.1: Construction of the Maximizer

Our goal is to show Lemma 5.1. In this section we will use that $\mathscr{C}$ is star-shaped, but not its convexity. For convenience, we state again the definition

$$\mathfrak{g}(y) = y^\top b^R + \tfrac{(p-1)}{2} y^\top c^R y + \int_{\mathbb{R}^d} \big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx). \tag{5.3}$$

The following lemma is a direct consequence of this formula and does not depend on Assumptions 3.1; it is a simplified version of Lemma III.6.2.

Lemma 5.3. *(i) If $p \in (0,1)$, $\mathfrak{g}$ is well defined with values in $(-\infty,\infty]$ and lower semicontinuous on $\mathscr{C}^0$. If (3.2) holds, $\mathfrak{g}$ is finite and continuous on $\mathscr{C}^0$.*

(ii) If $p<0$, $\mathfrak{g}$ is well defined with values in $[-\infty,\infty)$ and upper semicontinuous on $\mathscr{C}^0$. Moreover, $\mathfrak{g}$ is finite on $\check{\mathscr{C}}$ and $\mathfrak{g}(y)=-\infty$ for $y \in \mathscr{C}^0 \setminus \mathscr{C}^{0,*}$.

Proof. Fix a sequence $y_n \to y$ in $\mathscr{C}^0$. A Taylor expansion and (2.1) show that $\int_{|x|\le\varepsilon} \big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx)$ is finite and continuous along (y_n) for ε small enough, e.g., $\varepsilon = (2\sup_n |y_n|)^{-1}$.

If $p<0$, we have $p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x) \le -p^{-1} - y^\top h(x)$ and Fatou's lemma shows that $\int_{|x|>\varepsilon} \big\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\big\}\, F^R(dx)$ is upper semicontinuous of with respect to y. For $p>0$ we have the converse inequality and the same argument yields lower semicontinuity. If $p>0$ and (3.2) holds, the integral is finite and dominated convergence yields continuity.

Let $p<0$. For finiteness on $\check{\mathscr{C}}$ we note that $\mathfrak{g}$ is even finite on $\lambda\mathscr{C}^0$ for any $\lambda\in[0,1)$. Indeed, $y \in \lambda\mathscr{C}^0$ implies $y^\top x \ge -\lambda > -1$ $F^R(dx)$-a.e., hence the integrand in (5.3) is bounded F^R-a.e. and we conclude by (2.1). The last claim is immediate from the definitions of $\mathscr{C}^0$ and $\mathscr{C}^{0,*}$ as well as (5.3). □

Assume the alternative version of Lemma 5.1 under Assumptions 3.1(i)-(iii) and (3.2) has already been proved; we argue that the complete claim of that lemma then follows. Indeed, suppose that Assumptions 3.1(i)-(iv) hold. We first observe that $\mathscr{C}\cap\mathscr{C}^{0,*}$ is dense in $\mathscr{C}\cap\mathscr{C}^0$. To see this, note that for $y\in\mathscr{C}\cap\mathscr{C}^0\setminus\mathscr{C}^{0,*}$ and $n\in\mathbb{N}$ we have $y_n := (1-n^{-1})y \to y$ and y_n is in $\mathscr{C}^{0,*}$ (by the definition) and also in $\mathscr{C}$, due to the star-shape. Using Section IV.4 and its notation, Assumptions 3.1(i)-(iv) now imply that the transformed model $(\widetilde{R},\widetilde{\mathscr{C}})$ satisfies Assumptions 3.1(i)-(iii) and (3.2). We apply the above alternative version of Lemma 5.1 in that model to obtain $\tilde{\mathfrak{g}}^* := \sup_{\widetilde{\mathscr{C}}\cap\widetilde{\mathscr{C}}^0} \tilde{\mathfrak{g}} < \infty$ and a vector $\tilde{\pi} \in \widetilde{\mathscr{C}}\cap\widetilde{\mathscr{C}}^{0,*}$ such that $\tilde{\mathfrak{g}}(\tilde{\pi}) = \tilde{\mathfrak{g}}^*$. The density of $\mathscr{C}\cap\mathscr{C}^{0,*}$ observed above and the semicontinuity from Lemma 5.3(i) imply that $\mathfrak{g}^* := \sup_{\mathscr{C}\cap\mathscr{C}^0}\mathfrak{g} = \sup_{\mathscr{C}\cap\mathscr{C}^{0,*}}\mathfrak{g}$. Using this argument also for $(\widetilde{R},\widetilde{\mathscr{C}})$, Corollary 4.3(iii) yields $\mathfrak{g}^* = \tilde{\mathfrak{g}}^* < \infty$. Moreover, Corollary 4.3(iv) states that $\check{\pi} := \Lambda^\top\tilde{\pi} \in \mathscr{C}\cap\mathscr{C}^{0,*}$ is a maximizer for $\mathfrak{g}$.

Summarizing this discussion, it suffices to prove Lemma 5.1 under Assumptions 3.1(i)-(iii) and (3.2); hence these will be our assumptions for the rest of the section.

Formally, by differentiation under the integral, the directional derivatives of $\mathfrak{g}$ are given by $(\tilde{y}-y)^\top \nabla \mathfrak{g}(y) = \mathfrak{G}(\tilde{y},y)$, with

$$\begin{aligned}\mathfrak{G}(\tilde{y},y) &:= (\tilde{y}-y)^\top \big(b^R + (p-1)c^R y\big) \\ &\quad + \int_{\mathbb{R}^d} \Big\{ \frac{(\tilde{y}-y)^\top x}{(1+y^\top x)^{1-p}} - (\tilde{y}-y)^\top h(x) \Big\}\, F^R(dx).\end{aligned} \tag{5.4}$$

We take this as the definition of $\mathfrak{G}(\tilde{y},y)$ whenever the integral makes sense.

Remark 5.4. Formally setting $p=0$, we see that $\mathfrak{G}$ corresponds to the *relative rate of return* of two portfolios in the theory of log-utility [41, Eq. (3.2)].

Lemma 5.5. *Let $\tilde{y} \in \mathscr{C}^0$. On the set $\mathscr{C}^0 \cap \{\mathfrak{g} > -\infty\}$, $\mathfrak{G}(\tilde{y},\cdot)$ is well defined with values in $(-\infty,\infty]$. Moreover, $\mathfrak{G}(0,\cdot)$ is lower semicontinuous on $\mathscr{C}^0 \cap \{\mathfrak{g} > -\infty\}$.*

Proof. The first part follows by rewriting $\mathfrak{G}(\tilde{y},y)$ as

$$\begin{aligned}(\tilde{y}-y)^\top \big(b^R + (p-1)c^R y\big) - \int \big\{(1+y^\top x)^p - 1 - p y^\top h(x)\big\}\, F^R(dx) \\ + \int \Big\{ \frac{1+\tilde{y}^\top x}{(1+y^\top x)^{1-p}} - 1 - (\tilde{y}+(p-1)y)^\top h(x) \Big\}\, F^R(dx)\end{aligned}$$

because the first integral occurs in (5.3) and $1+\tilde{y}^\top x \geq 0$ F^R-a.e. by definition of $\mathscr{C}^0$. Let $p \in (0,1)$ and $\tilde{y}=0$ in the definition of $\mathfrak{G}$. Using

$$\frac{-y^\top x}{(1+y^\top x)^{1-p}} \geq -\frac{1+y^\top x}{(1+y^\top x)^{1-p}} = -(1+y^\top x)^p$$

and (3.2), Fatou's lemma yields that $\mathfrak{G}(0,\cdot)$ is l.s.c. on $\mathscr{C}^0$. If $p<0$, then $z/(1+z)^{1-p} \leq 1$ for $z \geq 0$ implies $\frac{-y^\top x}{(1+y^\top x)^{1-p}} \geq -1$. Again, Fatou's lemma yields the claim. □

As our goal is to prove Lemma 5.1, we may assume in the following that

$$\mathscr{C} \cap \mathscr{C}^0 \subseteq \mathscr{N}^\perp.$$

Indeed, noting that $\mathfrak{g}(y) = \mathfrak{g}(y+y')$ for $y' \in \mathscr{N}$, we may substitute $\mathscr{C} \cap \mathscr{C}^0$ by its projection to $\mathscr{N}^\perp$. The remainder of the section parallels the case of log-utility as treated in [44, Lemmata 5.2,5.1]. By Lemmata 5.3 and 5.5, $\mathfrak{G}(0,y)$ is well defined with values in $(-\infty,\infty]$ for $y \in \check{\mathscr{C}}$, so the following statement makes sense.

Lemma 5.6. *Let $y \in (\mathscr{C} \cap \mathscr{C}^0)\check{}$, then $y \in \mathscr{J}$ if and only if $\mathfrak{G}(0,ay) \leq 0$ for all $a \geq 0$.*

Proof. If $y \in \mathscr{J}$, then $\mathfrak{G}(0, ay) \leq 0$ by the definitions of $\mathscr{J}$ and $\mathfrak{G}$; we prove the converse. As $y \in (\mathscr{C} \cap \mathscr{C}^0)\check{}$ implies $F^R[ay^\top x < -1] = 0$ for all a, we have $F^R[y^\top x < 0] = 0$. Since $y^\top x \geq 0$ entails $|\frac{y^\top x}{(1+y^\top x)^{1-p}}| \leq 1$, dominated convergence yields

$$\lim_{a \to \infty} \int \Big\{ \frac{y^\top x}{(1+ay^\top x)^{1-p}} - y^\top h(x) \Big\} \, F^R(dx) = - \int y^\top h(x) \, F^R(dx).$$

By assumption, $-a^{-1}\mathfrak{G}(0, ay) \geq 0$, i.e,

$$y^\top b^R + a(p-1) y^\top c^R y + \int \Big\{ \frac{y^\top x}{(1+ay^\top x)^{1-p}} - y^\top h(x) \Big\} \, F^R(dx) \geq 0.$$

As $(p-1)y^\top c^R y \leq 0$, taking $a \to \infty$ shows $y^\top c^R = 0$ and then we also see $y^\top b^R - \int y^\top h(x) \, F(dx) \geq 0$. □

Proof of Lemma 5.1. Let $(y_n) \subset \mathscr{C} \cap \mathscr{C}^0$ be such that $\mathfrak{g}(y_n) \to \mathfrak{g}^*$. We may assume $\mathfrak{g}(y_n) > -\infty$ and $y_n \in \mathscr{C}^{0,*}$ by Lemma 5.3, since $\mathscr{C}^{0,*} \subseteq \mathscr{C}^0$ is dense.

We claim that (y_n) has a bounded subsequence. By way of contradiction, suppose that (y_n) is unbounded. Without loss of generality, $\xi_n := y_n/|y_n|$ converges to some ξ. Moreover, we may assume by redefining y_n that $\mathfrak{g}(y_n) = \max_{\lambda \in [0,1]} \mathfrak{g}(\lambda y_n)$, because $\mathfrak{g}$ is continuous on each of the compact sets $C_n = \{\lambda y_n : \lambda \in [0,1]\}$. Indeed, if $p < 0$, continuity follows by dominated convergence using $1 + \lambda y^\top x \geq 1 + y^\top x$ on $\{x : y^\top x \leq 0\}$; while for $p \in (0,1)$, $\mathfrak{g}$ is continuous by Lemma 5.3.

Using concavity one can check that $\mathfrak{G}(0, a\xi_n)$ is indeed the directional derivative of the function $\mathfrak{g}$ at $a\xi_n$ (cf. Lemma III.5.14). In particular, $\mathfrak{g}(y_n) = \max_{\lambda \in [0,1]} \mathfrak{g}(\lambda y_n)$ implies that $\mathfrak{G}(0, a\xi_n) \leq 0$ for $a > 0$ and all n such that $|y_n| \geq a$ (and hence $a\xi_n \in C_n$). By the star-shape and closedness of $\mathscr{C} \cap \mathscr{C}^0$ we have that $\xi \in (\mathscr{C} \cap \mathscr{C}^0)\check{}$. Lemmata 5.3 and 5.5 yield the semi-continuity to pass from $\mathfrak{G}(0, a\xi_n) \leq 0$ to $\mathfrak{G}(0, a\xi) \leq 0$ and now Lemma 5.6 shows $\xi \in \mathscr{J}$, contradicting the $\mathrm{NUIP}_{\mathscr{C}}$ condition that $\mathscr{J} \cap (\mathscr{C} \cap \mathscr{C}^0)\check{} = \emptyset$.

We have shown that after passing to a subsequence, there exists a limit $y^* = \lim_n y_n$. Lemma 5.3 shows $\mathfrak{g}^* = \lim_n \mathfrak{g}(y_n) = \mathfrak{g}(y^*) < \infty$; and $y^* \in \mathscr{C}^{0,*}$ for $p < 0$. For $p \in (0,1)$, $y^* \in \mathscr{C}^{0,*}$ follows as in Lemma III.6.8. □

IV.6 q-Optimal Martingale Measures

In this section we consider $\delta = 0$ (no consumption) and $\mathscr{C} = \mathbb{R}^d$. Then Assumptions 3.1 are equivalent to

$$\mathrm{NUIP}_{\mathbb{R}^d} \text{ holds} \quad \text{and} \quad u(x_0) < \infty \tag{6.1}$$

and these conditions are in force for the following discussion. Let $\mathscr{M}$ be the set of all equivalent local martingale measures for $S = \mathcal{E}(R)$. Then $\mathrm{NUIP}_{\mathbb{R}^d}$

is equivalent to $\mathscr{M} \neq \emptyset$, more precisely, there exists $Q \in \mathscr{M}$ under which R is a Lévy martingale (see [44, Remark 3.8]). In particular, we are in the setting of Kramkov and Schachermayer [49].

Let $q = p/(p-1) \in (-\infty, 0) \cup (0,1)$ be the exponent conjugate to p, then $Q \in \mathscr{M}$ is called *q-optimal* if $E[-q^{-1}(dQ/dP)^q]$ is finite and minimal over $\mathscr{M}$. If $q < 0$, i.e., $p \in (0,1)$, then $u(x_0) < \infty$ is equivalent to the existence of some $Q \in \mathscr{M}$ such that $E[-q^{-1}(dQ/dP)^q] < \infty$ (cf. Kramkov and Schachermayer [50]).

This minimization problem over $\mathscr{M}$ is linked to power utility maximization by convex duality in the sense of [49]. More precisely, that article considers a "dual problem" over an enlarged domain of certain supermartingales. We recall from Proposition II.4.2 that the solution to that dual problem is given by the positive supermartingale $\widehat{Y} = L\widehat{X}^{p-1}$, where L is the opportunity process and $\widehat{X} = x_0\mathcal{E}(\hat{\pi} \bullet R)$ is the optimal wealth process corresponding to $\hat{\pi}$ as in Theorem 3.2. It follows from [49, Theorem 2.2(iv)] that the q-optimal martingale measure $\widehat{Q}$ exists if and only if $\widehat{Y}$ is a martingale, and in that case $\widehat{Y}/\widehat{Y}_0$ is the P-density process of $\widehat{Q}$. Recall the functions $\mathfrak{g}$ and $\mathfrak{G}$ from (3.1) and (5.4). A direct calculation (or Remark III.5.18) shows

$$\widehat{Y}/\widehat{Y}_0 = \mathcal{E}\Big(\mathfrak{G}(0,\hat{\pi})t + (p-1)\hat{\pi}^\top R^c + \big\{(1+\hat{\pi}^\top x)^{p-1} - 1\big\} * (\mu^R - \nu^R)\Big).$$

Here absence of drift is equivalent to $\mathfrak{G}(0,\hat{\pi}) = 0$, or more explicitly,

$$\hat{\pi}^\top b^R + (p-1)\hat{\pi}^\top c^R \hat{\pi} + \int_{\mathbb{R}^d} \Big\{\frac{\hat{\pi}^\top x}{(1+\hat{\pi}^\top x)^{1-p}} - \hat{\pi}^\top h(x)\Big\} F^R(dx) = 0, \quad (6.2)$$

and in that case

$$\widehat{Y}/\widehat{Y}_0 = \mathcal{E}\Big((p-1)\hat{\pi}^\top R^c + \big\{(1+\hat{\pi}^\top x)^{p-1} - 1\big\} * (\mu^R - \nu^R)\Big). \quad (6.3)$$

This is an exponential Lévy martingale because $\hat{\pi}$ is a constant vector; in particular, it is indeed a true martingale.

Theorem 6.1. *Let $q \in (-\infty, 0) \cup (0,1)$. The following are equivalent:*

(i) The q-optimal martingale measure $\widehat{Q}$ exists,

(ii) (6.1) *and* (6.2) *hold,*

(iii) there exists $\hat{\pi} \in \mathscr{C}^0$ such that $\mathfrak{g}(\hat{\pi}) = \max_{\mathscr{C}^0} \mathfrak{g} < \infty$ and (6.2) *holds.*

Under these equivalent conditions, (6.3) *is the P-density process of $\widehat{Q}$.*

Proof. We have just argued the equivalence of (i) and (ii). Under (6.1), there exists $\hat{\pi}$ satisfying (iii) by Theorem 3.2. Conversely, given (iii) we construct the solution to the utility maximization problem as before and (6.1) follows; recall Remark 3.4(c). □

Remark 6.2. (i) If $\widehat{Q}$ exists, (6.3) shows that the change of measure from P to $\widehat{Q}$ has constant (deterministic and time-independent) Girsanov parameters $\big((p-1)\hat{\pi}, (1+\hat{\pi}^\top x)^{p-1}\big)$; compare [34, III.3.24] or Jeanblanc et al. [35, §A.1, §A.2]. Therefore, *R is again a Lévy process under $\widehat{Q}$.* This result was previously obtained in [35] by an abstract argument (cf. Section IV.8 below).

(ii) Existence of $\widehat{Q}$ is a fairly delicate question compared to the existence of the supermartingale $\widehat{Y}$. Recalling the definition of $\mathfrak{G}$, (6.2) essentially expresses that the budget constraint $\mathscr{C}^0$ in the maximization of $\mathfrak{g}$ is "not binding". Theorem 6.1 gives an explicit and sharp description for the existence of $\widehat{Q}$; this appears to be missing in the previous literature.

IV.7 Extensions to Non-Convex Constraints

In this section we consider the utility maximization problem for some cases where the constraints $0 \in \mathscr{C} \subseteq \mathbb{R}^d$ are not convex. Let us first recapitulate where the convexity assumption was used above. The proof of Lemma 5.1 used the star-shape of $\mathscr{C}$, but not convexity. In the rest of Section IV.5.1, the shape of $\mathscr{C}$ was irrelevant except in Lemma 5.2.

We denote by $\overline{\mathrm{co}}\,(\mathscr{C})$ the closed convex hull of $\mathscr{C}$.

Corollary 7.1. *Let $p<0$ and suppose that either (i) or (ii) below hold:*

(i) *(a) $\mathscr{C}$ is star-shaped,*

(b) the orthogonal projection of $\overline{\mathrm{co}}\,(\mathscr{C}) \cap \mathscr{C}^0$ onto $\mathscr{N}^\perp$ is closed,

(c) $\mathrm{NUIP}_{\overline{\mathrm{co}}\,(\mathscr{C})}$ *holds.*

(ii) $\mathscr{C} \cap \mathscr{C}^0$ is compact.

Then the assertion of Theorem 3.2 remains valid.

Proof. (i) The construction of $(\ell, \check{\pi}, \check{\kappa})$ is as above; we have to substitute the verification step which used Lemma 5.2. The model $(R, \overline{\mathrm{co}}\,(\mathscr{C}))$ satisfies the assumptions of Theorem 3.2. Hence the corresponding opportunity process $L^{\overline{\mathrm{co}}\,(\mathscr{C})}$ is deterministic and bounded away from zero. The definition of the opportunity process and the inclusion $\mathscr{C} \subseteq \overline{\mathrm{co}}\,(\mathscr{C})$ imply that the opportunity process $L = L^{\mathscr{C}}$ for $(R, \mathscr{C})$ is also bounded away from zero. Hence ℓ/L is bounded and we can verify $(\ell, \check{\pi}, \check{\kappa})$ by Corollary III.5.4, which makes no assumptions about the shape of $\mathscr{C}$.

(ii) We may assume without loss of generality that $\mathscr{C} = \mathscr{C} \cap \mathscr{C}^0$. In (i), the star-shape was used only to construct a maximizer for $\mathfrak{g}$. When $\mathscr{C} \cap \mathscr{C}^0$ is compact, its existence is clear by the upper semicontinuity from Lemma 5.3, which also shows that any maximizer is necessarily in $\mathscr{C}^{0,*}$. To proceed as in (i), it remains to note that the projection of the compact set $\overline{\mathrm{co}}\,(\mathscr{C}) \cap \mathscr{C}^0$ onto $\mathscr{N}^\perp$ is compact, and $\mathrm{NUIP}_{\overline{\mathrm{co}}\,(\mathscr{C})}$ holds because $(\overline{\mathrm{co}}\,(\mathscr{C}))\check{} = \{0\}$ since $\overline{\mathrm{co}}\,(\mathscr{C})$ is bounded. $\square$

When the constraints are not star-shaped and $p > 0$, an additional condition is necessary to ensure that the maximum of $\mathfrak{g}$ is not attained on $\mathscr{C}^0 \setminus \mathscr{C}^{0,*}$, or equivalently, to obtain a positive optimal wealth process. In Section III.2.4 we introduced the following condition:

(C3) There exists $\underline{\eta} \in (0,1)$ such that $y \in (\mathscr{C} \cap \mathscr{C}^0) \setminus \mathscr{C}^{0,*}$ implies $\eta y \in \mathscr{C}$ for all $\eta \in (\underline{\eta}, 1)$.

This is clearly satisfied if $\mathscr{C}$ is star-shaped or if $\mathscr{C}^{0,*} = \mathscr{C}^0$.

Corollary 7.2. *Let $p \in (0,1)$ and suppose that either (i) or (ii) below hold:*

(i) Assumptions 3.1 hold except that $\mathscr{C}$ is star-shaped instead of being convex.

(ii) $\mathscr{C} \cap \mathscr{C}^0$ is compact and satisfies (C3) and $u(x_0) < \infty$.

Then the assertion of Theorem 3.2 remains valid.

Proof. (i) The assumptions carry over to the transformed model as before, hence again we only need to substitute the verification argument. In view of $p \in (0,1)$, we can use Theorem III.5.2, which makes no assumptions about the shape of $\mathscr{C}$. Note that we have already checked its condition (cf. Remark III.5.16).

(ii) We may again assume $\mathscr{C} = \mathscr{C} \cap \mathscr{C}^0$ and Remark 4.4 shows that we can choose $\widetilde{\mathscr{C}} \cap \widetilde{\mathscr{C}}^0$ to be compact in the transformed model satisfying (3.2). That is, we can again assume (3.2) without loss of generality. Then $\mathfrak{g}$ is continuous and hence existence of a maximizer on $\mathscr{C} \cap \mathscr{C}^0$ is clear. Under (C3), any maximizer is in $\mathscr{C}^{0,*}$ by the same argument as in the proof of Lemma 5.1. □

The following result covers *all* closed constraints and applies to most of the standard models (cf. Example 2.2).

Corollary 7.3. *Let $\mathscr{C}$ be closed and assume that $\mathscr{C}^0$ is compact and that $u(x_0) < \infty$. Then the assertion of Theorem 3.2 remains valid.*

Proof. Note that (C3) holds for all sets $\mathscr{C}$ when $\mathscr{C}^{0,*}$ is closed (and hence equal to $\mathscr{C}^0$). It remains to apply part (ii) of the two previous corollaries. □

Remark 7.4. (i) For $p \in (0,1)$ we also have the analogue of Proposition 3.6(i): under the assumptions of Corollary 7.2 excluding $u(x_0) < \infty$, (3.2) implies $u(x_0) < \infty$.

(ii) The optimal propensity to consume $\hat{\kappa}$ remains unique even when the constraints are not convex (cf. Theorem III.3.2). However, there is no uniqueness for the optimal portfolio. In fact, in the setting of the above corollaries, any constant vector $\pi \in \arg\max_{\mathscr{C} \cap \mathscr{C}^0} \mathfrak{g}$ is an optimal portfolio (by the same proofs); and when $\mathscr{C}$ is not convex, the difference of two such π need not be in $\mathscr{N}$. See also Remark III.3.3 for statements about dynamic portfolios. Conversely, by Theorem III.3.2 any optimal portfolio, possibly dynamic, takes values in $\arg\max_{\mathscr{C} \cap \mathscr{C}^0} \mathfrak{g}$.

IV.8 Related Literature

We discuss some related literature in a highly selective manner; an exhaustive overview is beyond our scope. For the unconstrained utility maximization problem with general Lévy processes, Kallsen [38] gave a result of verification type: If there exists a vector π satisfying a certain equation, π is the optimal portfolio. This equation is essentially our (6.2) and therefore holds only if the corresponding dual element $\widehat{Y}$ is the density process of a measure. Muhle-Karbe [58, Example 4.24] showed that this condition fails in a particular model, in other words, the supermartingale $\widehat{Y}$ is not a martingale in that example. In the one-dimensional case, he introduced a weaker inequality condition [58, Corollary 4.21], but again existence of π was not discussed. (In fact, our proofs show the necessity of that inequality condition; cf. Remark III.5.16.)

Numerous variants of our utility maximization problem were also studied along more traditional lines of dynamic programming. E.g., Benth et al. [6] solve a similar problem with infinite time horizon when the Lévy process satisfies additional integrability properties and the portfolios are chosen in $[0, 1]$. This part of the literature generally requires technical conditions, which we sought to avoid.

Jeanblanc et al. [35] study the q-optimal measure $\widehat{Q}$ for Lévy processes when $q < 0$ or $q > 1$ (note that the considered parameter range does not coincide with ours). They show that the Lévy structure is preserved under $\widehat{Q}$, if the latter exists; a result we recovered in Remark 6.2 above for our values of q. In [35] this is established by showing that starting from any equivalent change of measure, a suitable choice of constant Girsanov parameters reduces the q-divergence of the density. This argument does not seem to extend to our general dual problem which involves supermartingales rather than measures; in particular, it cannot be used to show that the optimal portfolio is a constant vector. A deterministic, but not explicit characterization of $\widehat{Q}$ is given in [35, Theorem 2.7]. The authors also provide a more explicit candidate for the q-optimal measure [35, Theorem 2.9], but the condition of that theorem fails in general (see Bender and Niethammer [5]).

In the Lévy setting the q-optimal measures ($q \in \mathbb{R}$) coincide with the minimal Hellinger measures and hence the pertinent results apply. See Choulli and Stricker [12] and in particular their general sufficient condition [12, Theorem 2.3]. We refer to [35, p. 1623] for a discussion. Our result differs in that both the existence of $\widehat{Q}$ and its density process are described explicitly in terms of the Lévy triplet.

Chapter V

Risk Aversion Asymptotics

In this chapter, which corresponds to the article [62], we use the tools from Chapters II and III to study the optimal strategy as the relative risk aversion tends to infinity or to one.

V.1 Introduction

We study preferences given by power utility random fields for an agent who can invest in a financial market which is modeled by a general semimartingale. We defer the precise formulation to the next section to allow for a brief presentation of the contents and focus on the power utility function $U^{(p)}(x) = \frac{1}{p}x^p$, where $p \in (-\infty, 0) \cup (0, 1)$. Under standard assumptions, there exists for each p an optimal trading and consumption strategy that maximizes the expected utility corresponding to $U^{(p)}$. Our main interest concerns the *behavior of these strategies in the limits* $p \to -\infty$ *and* $p \to 0$.

The relative risk aversion of $U^{(p)}$ tends to infinity for $p \to -\infty$. Hence economic intuition suggests that the agent should become reluctant to take risks and, in the limit, not invest in the risky assets. Our first main result confirms this intuition. More precisely, we prove in a general semimartingale model that the optimal consumption, expressed as a proportion of current wealth, converges pointwise to a deterministic function. This function corresponds to the consumption which would be optimal in the case where trading is not allowed. In the continuous semimartingale case, we show that the optimal trading strategy tends to zero in a local L^2-sense and that the corresponding wealth process converges in the semimartingale topology.

Our second result pertains to the same limit $p \to -\infty$ but concerns the problem without intermediate consumption. In the continuous case, we show that the optimal trading strategy scaled by $1-p$ converges to a strategy which is optimal for exponential utility. We provide economic intuition for this fact via a sequence of auxiliary power utility functions with shifted domains.

The limit $p \to 0$ is related to the logarithmic utility function. Our third main result is the convergence of the corresponding optimal consumption for the general semimartingale case, and the convergence of the trading strategy and the wealth process in the continuous case.

All these results are readily observed for special models where the optimal strategies can be calculated explicitly. While the corresponding economic intuition extends to general models, it is *a priori* unclear how to go about proving the results. Indeed, the problem is to *get our hands on the optimal controls*, which is a notorious question in stochastic optimal control.

Our main tool is the opportunity process. We prove its convergence using control-theoretic arguments and convex analysis. On the one hand, this yields the convergence of the value function. On the other hand, we deduce the convergence of the optimal consumption, which is directly related to the opportunity process. The optimal trading strategy is also linked to this process, by the Bellman equation. We study the asymptotics of this backward stochastic differential equation (BSDE) to obtain the convergence of the strategy. This involves nonstandard arguments to deal with nonuniform quadratic growth in the driver and solutions that are not locally bounded.

To derive the results in the stated generality, it is important to *combine* ideas from optimal control, convex analysis and BSDE theory rather than to rely on only one of these ingredients; and one may see the problem at hand as a *model problem of control* in a semimartingale setting.

The chapter is organized as follows. In the next section, we specify the optimization problem in detail. Section V.3 summarizes the main results on the risk aversion asymptotics of the optimal strategies and indicates connections to the literature. Section V.4 introduces the main tools, the opportunity process and the Bellman equation, and explains the general approach for the proofs. In Section V.5 we study the dependence of the opportunity process on p and establish some related estimates. Sections V.6 deals with the limit $p \to -\infty$; we prove the main results stated in Section V.3 and, in addition, the convergence of the opportunity process and the solution to the dual problem (in the sense of convex duality). Similarly, Section V.7 contains the proof of the main theorem for $p \to 0$ and additional refinements.

V.2 Preliminaries

The following notation is used. If $x, y \in \mathbb{R}$ are reals, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. We use $1/0 := \infty$ where necessary. If $z \in \mathbb{R}^d$ is a d-dimensional vector, z^i is its ith coordinate, $z^\top$ its transpose, and $|z| = (z^\top z)^{1/2}$ the Euclidean norm. If X is an $\mathbb{R}^d$-valued semimartingale and π is an $\mathbb{R}^d$-valued predictable integrand, the vector stochastic integral, denoted by $\int \pi \, dX$ or $\pi \bullet X$, is a scalar semimartingale with initial value zero. Relations between measurable functions hold almost everywhere unless otherwise mentioned. Dellacherie and Meyer [18] and Jacod and Shiryaev [34] are references for unexplained notions from stochastic calculus.

V.2.1 The Optimization Problem

We consider a fixed time horizon $T \in (0, \infty)$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual assumptions of right-continuity and completeness, as well as $\mathcal{F}_0 = \{\emptyset, \Omega\}$ P-a.s. Let R be an $\mathbb{R}^d$-valued càdlàg semimartingale with $R_0 = 0$. Its components are interpreted as the returns of d risky assets and the stochastic exponential $S = (\mathcal{E}(R^1), \dots, \mathcal{E}(R^d))$ represents their prices. Let $\mathscr{M}$ be the set of equivalent σ-martingale measures for S. We assume

$$\mathscr{M} \neq \emptyset, \tag{2.1}$$

so that arbitrage is excluded in the sense of the NFLVR condition (see Delbaen and Schachermayer [17]). Our agent also has a bank account at his disposal. As usual in mathematical finance, the interest rate is assumed to be zero.

The agent is endowed with a deterministic initial capital $x_0 > 0$. A *trading strategy* is a predictable R-integrable $\mathbb{R}^d$-valued process π, where π^i is interpreted as the fraction of the current wealth (or the portfolio proportion) invested in the ith risky asset. A *consumption rate* is an optional process $c \geq 0$ such that $\int_0^T c_t\,dt < \infty$ P-a.s. We want to consider two cases simultaneously: Either consumption occurs only at the terminal time T (utility from "terminal wealth" only); or there is intermediate and a bulk consumption at the time horizon. To unify the notation, we define the measure μ on $[0,T]$,

$$\mu(dt) := \begin{cases} 0 & \text{in the case without intermediate consumption,} \\ dt & \text{in the case with intermediate consumption.} \end{cases}$$

Moreover, let $\mu^\circ := \mu + \delta_{\{T\}}$, where $\delta_{\{T\}}$ is the unit Dirac measure at T. The *wealth process* $X(\pi, c)$ of a pair (π, c) is defined by the linear equation

$$X_t(\pi, c) = x_0 + \int_0^t X_{s-}(\pi, c)\pi_s\,dR_s - \int_0^t c_s\,\mu(ds), \quad 0 \leq t \leq T.$$

The set of *admissible* trading and consumption pairs is

$$\mathcal{A}(x_0) = \big\{(\pi, c) :\, X(\pi, c) > 0 \text{ and } c_T = X_T(\pi, c)\big\}.$$

The convention $c_T = X_T(\pi, c)$ is merely for notational convenience and means that all the remaining wealth is consumed at time T. We fix the initial capital x_0 and usually write $\mathcal{A}$ for $\mathcal{A}(x_0)$. Moreover, $c \in \mathcal{A}$ indicates that there exists π such that $(\pi, c) \in \mathcal{A}$; an analogous convention is used for similar expressions.

It will be convenient to parametrize the consumption strategies as fractions of the wealth. Let $(\pi, c) \in \mathcal{A}$ and let $X = X(\pi, c)$ be the corresponding wealth process. Then

$$\kappa := \frac{c}{X}$$

is called the *propensity to consume* corresponding to (π, c). In general, a propensity to consume is an optional process $\kappa \geq 0$ such that $\int_0^T \kappa_s \, ds < \infty$ P-a.s. and $\kappa_T = 1$. The parametrizations by c and by κ are equivalent (see Remark II.2.1) and we abuse the notation by identifying c and κ when π is given. Note that the wealth process can be expressed as

$$X(\pi, \kappa) = x_0 \mathcal{E}\big(\pi \bullet R - \kappa \bullet \mu\big). \tag{2.2}$$

The preferences of the agent are modeled by a random utility function with constant relative risk aversion. More precisely, let D be a càdlàg adapted positive process and fix $p \in (-\infty, 0) \cup (0, 1)$. We define the utility random field

$$U_t(x) := U_t^{(p)}(x) := D_t \tfrac{1}{p} x^p, \quad x \in (0, \infty),\ t \in [0, T], \tag{2.3}$$

where we assume that there are constants $0 < k_1 \leq k_2 < \infty$ such that

$$k_1 \leq D_t \leq k_2, \quad 0 \leq t \leq T. \tag{2.4}$$

The process D is taken to be independent of p; interpretations are discussed in Remark II.2.2. The parameter p in $U^{(p)}$ will sometimes be suppressed in the notation and made explicit when we want to recall the dependence. The same applies to other quantities in this paper.

The constant $1 - p > 0$ is called the *relative risk aversion* of U. The *expected utility* corresponding to a consumption rate $c \in \mathcal{A}$ is given by $E\big[\int_0^T U_t(c_t) \, \mu^\circ(dt)\big]$, which is either $E[U_T(c_T)]$ or $E[\int_0^T U_t(c_t) \, dt + U_T(c_T)]$. We will always assume that the optimization problem is nondegenerate, i.e.,

$$u_p(x_0) := \sup_{c \in \mathcal{A}(x_0)} E\Big[\int_0^T U_t^{(p)}(c_t) \, \mu^\circ(dt)\Big] < \infty. \tag{2.5}$$

This condition depends on the choice of p, but not on x_0. Note that $u_{p_0}(x_0) < \infty$ implies $u_p(x_0) < \infty$ for any $p < p_0$; and for $p < 0$ the condition (2.5) is void since then $U^{(p)} < 0$. A strategy $(\pi, c) \in \mathcal{A}(x_0)$ is *optimal* if $E\big[\int_0^T U_t(c_t) \, \mu^\circ(dt)\big] = u(x_0)$. Note that U_t is irrelevant for $t < T$ when there is no intermediate consumption. We recall the following existence result.

Proposition 2.1 (Karatzas and Žitković [43])**.** *For each p, if $u_p(x_0) < \infty$, there exists an optimal strategy $(\hat{\pi}, \hat{c}) \in \mathcal{A}$. The corresponding wealth process $\widehat{X} = X(\hat{\pi}, \hat{c})$ is unique. The consumption rate $\hat{c}$ can be chosen to be càdlàg and is unique $P \otimes \mu^\circ$-a.e.*

In the sequel, $\hat{c}$ denotes this càdlàg version, $\widehat{X} = X(\hat{\pi}, \hat{c})$ is the optimal wealth process and $\hat{\kappa} = \hat{c}/\widehat{X}$ is the optimal propensity to consume.

V.2.2 Decompositions and Spaces of Processes

In some of the statements, we will assume that the price process S (or equivalently R) is continuous. In this case, it follows from (2.1) and Schweizer [71] that R satisfies the *structure condition*, i.e.,

$$R = M + \int d\langle M\rangle\lambda, \tag{2.6}$$

where M is a continuous local martingale with $M_0 = 0$ and $\lambda \in L^2_{loc}(M)$.

Let ξ be a scalar special semimartingale, i.e., there exists a (unique) canonical decomposition $\xi = \xi_0 + M^\xi + A^\xi$, where $\xi_0 \in \mathbb{R}$, M^ξ is a local martingale, A^ξ is predictable of finite variation, and $M^\xi_0 = A^\xi_0 = 0$. As M is continuous, M^ξ has a Kunita-Watanabe (KW) decomposition with respect to M,

$$\xi = \xi_0 + Z^\xi \bullet M + N^\xi + A^\xi, \tag{2.7}$$

where $[M^i, N^\xi] = 0$ for $1 \leq i \leq d$ and $Z^\xi \in L^2_{loc}(M)$; see Ansel and Stricker [2, *cas* 3]. Analogous notation will be used for other special semimartingales and, with a slight abuse of terminology, we will refer to (2.7) as the KW decomposition of ξ.

Let $\mathcal{S}$ be the space of all càdlàg P-semimartingales and $r \in [1,\infty)$. If $X \in \mathcal{S}$ has the canonical decomposition $X = X_0 + M^X + A^X$, we define

$$\|X\|_{\mathcal{H}^r} := |X_0| + \big\|\textstyle\int_0^T |dA^X|\big\|_{L^r} + \big\|[M^X]_T^{1/2}\big\|_{L^r}.$$

In particular, we will often use that $\|N\|^2_{\mathcal{H}^2} = E\big[[N]_T\big]$ for a local martingale N with $N_0 = 0$. If X is a non-special semimartingale, $\|X\|_{\mathcal{H}^r} := \infty$. We can now define $\mathcal{H}^r := \{X \in \mathcal{S} : \|X\|_{\mathcal{H}^r} < \infty\}$. The same space is sometimes denoted by $\mathcal{S}^r$ in the literature; moreover, there are many equivalent definitions for $\mathcal{H}^r$ (see [18, VII.98]). The localized spaces $\mathcal{H}^r_{loc}$ are defined in the usual way. In particular, if $X, X^n \in \mathcal{S}$ we say that $X^n \to X$ in $\mathcal{H}^r_{loc}$ if there exists a localizing sequence of stopping times $(\tau_m)_{m\geq 1}$ such that $\lim_n \|(X^n - X)^{\tau_m}\|_{\mathcal{H}^r} = 0$ for all m. The localizing sequence may depend on the sequence (X^n), causing this convergence to be non-metrizable. On $\mathcal{S}$, the Émery distance is defined by

$$d(X,Y) := |X_0 - Y_0| + \sup_{|H|\leq 1} E\bigg[\sup_{t\in[0,T]} 1 \wedge |H \bullet (X-Y)_t|\bigg],$$

where the supremum is taken over all predictable processes bounded by one in absolute value. This complete metric induces on $\mathcal{S}$ the *semimartingale topology* (cf. Émery [20]).

An optional process X satisfies a certain property *prelocally* if there exists a localizing sequence of stopping times τ_m such that $X^{\tau_m -} := X1_{[0,\tau_m)} + X_{\tau_m -}1_{[\tau_m,T]}$ satisfies this property for each m. When X is continuous, prelocal simply means local.

Proposition 2.2 ([20]). *Let $X, X^n \in \mathcal{S}$ and $r \in [1,\infty)$. Then $X^n \to X$ in the semimartingale topology if and only if every subsequence of (X^n) has a subsequence which converges to X prelocally in $\mathcal{H}^r$.*

We denote by BMO the space of martingales N with $N_0 = 0$ satisfying

$$\|N\|_{BMO}^2 := \Big\| \sup_\tau E\big[[N]_T - [N]_{\tau-}\big|\mathcal{F}_\tau\big]\Big\|_{L^\infty} < \infty,$$

where τ ranges over all stopping times (more precisely, this is the BMO_2-norm). There exists a similar notion for semimartingales: let $\mathcal{H}^\omega$ be the subspace of $\mathcal{H}^1$ consisting of all special semimartingales X with $X_0 = 0$ and

$$\|X\|_{\mathcal{H}^\omega}^2 := \Big\| \sup_\tau E\Big[\big([M^X]_T - [M^X]_{\tau-}\big)^{1/2} + \textstyle\int_{\tau-}^T |dA^X| \Big|\mathcal{F}_\tau\Big]\Big\|_{L^\infty} < \infty.$$

Finally, let $\mathcal{R}^r$ be the space of scalar adapted processes which are right-continuous and such that

$$\|X\|_{\mathcal{R}^r} := \Big\| \sup_{0\leq t\leq T} |X_t| \Big\|_{L^r} < \infty.$$

With a mild abuse of notation, we will use the same norm also for left-continuous processes.

V.3 Main Results

In this section we present the main results about the limits of the optimal strategies. To state an assumption in the results, we first recall the *opportunity process* $L(p)$. Fix p such that $u_p(x_0) < \infty$. Then by Proposition II.3.1 there exists a unique càdlàg semimartingale $L(p)$ such that

$$L_t(p)\,\tfrac{1}{p}\big(X_t(\pi,c)\big)^p = \operatorname*{ess\,sup}_{\tilde{c}\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big], \quad 0\leq t\leq T \quad (3.1)$$

for all $(\pi,c)\in\mathcal{A}$, where $\mathcal{A}(\pi,c,t) := \big\{(\tilde{\pi},\tilde{c})\in\mathcal{A} :\ (\tilde{\pi},\tilde{c}) = (\pi,c) \text{ on } [0,t]\big\}$.

We can now proceed to state the main results. The proofs are postponed to Sections V.6 and V.7. Those sections also contain statements about the convergence of the opportunity processes and the solutions to the dual problems, as well as some refinements of the results below.

V.3.1 The Limit $p \to -\infty$

The relative risk aversion $1-p$ of $U^{(p)}$ increases to infinity as $p\to-\infty$. Therefore we expect that in the limit, the agent does not invest at all. In that situation the optimal propensity to consume is $\kappa_t = (1+T-t)^{-1}$ since this corresponds to a constant consumption rate. Our first result shows that this coincides with the limit of the $U^{(p)}$-optimal propensities to consume.

Theorem 3.1. *The following convergences hold as $p \to -\infty$.*

(i) Let $t \in [0,T]$. In the case with intermediate consumption,

$$\hat{\kappa}_t(p) \to \frac{1}{1+T-t} \quad P\text{-a.s.}$$

If $\mathbb{F}$ is continuous, the convergence is uniform in t, P-a.s.; and holds also in $\mathcal{R}^r_{loc}$ for all $r \in [1,\infty)$.

(ii) If S is continuous and $L(p)$ is continuous for all $p<0$, then

$$\hat{\pi}(p) \to 0 \text{ in } L^2_{loc}(M)$$

and $\widehat{X}(p) \to x_0 \exp\big(-\int_0^\cdot \frac{\mu(ds)}{1+T-s}\big)$ in the semimartingale topology.

The continuity assumptions in (ii) are always satisfied if the filtration $\mathbb{F}$ is generated by a Brownian motion; see also Remark 4.2.

Literature. We are not aware of a similar result in the continuous-time literature, with the exception that when the strategies can be calculated explicitly, the convergences mentioned in this section are often straightforward to obtain. E.g., Grasselli [31] carries out such a construction in a complete market model. There are also related systematic results. Carassus and Rásonyi [10] and Grandits and Summer [30] study convergence to the superreplication problem for increasing (absolute) risk aversion of general utility functions in discrete models. Note that superreplicating the contingent claim $B \equiv 0$ corresponds to not trading at all. For the maximization of exponential utility $-\exp(-\alpha x)$ without claim, the optimal strategy is proportional to the inverse of the absolute risk aversion α and hence trivially converges to zero in the limit $\alpha \to \infty$. The case with claim has also been studied. See, e.g., Mania and Schweizer [53] for a continuous model, and Becherer [4] for a related result. The references given here and later in this section do not consider intermediate consumption.

We continue with our second main result, which concerns only the case without intermediate consumption. We first introduce in detail the exponential hedging problem already mentioned above. Let $B \in L^\infty(\mathcal{F}_T)$ be a contingent claim. Then the aim is to maximize the expected exponential utility (here with $\alpha = 1$) of the terminal wealth including the claim,

$$\max_{\vartheta\in\Theta} E\big[-\exp\big(B - x_0 - (\vartheta \bullet R)_T\big)\big], \tag{3.2}$$

where ϑ is the trading strategy parametrized by the *monetary amounts* invested in the assets (setting $\overline{\vartheta}^i := 1_{\{S^i_- \neq 0\}}\vartheta^i/S^i_-$ yields $\overline{\vartheta} \bullet S = \vartheta \bullet R$ and corresponds to the more customary *number* of shares of the assets).

To describe the set Θ, we define the *entropy* of $Q \in \mathscr{M}$ relative to P by

$$H(Q|P) := E\Big[\frac{dQ}{dP}\log\Big(\frac{dQ}{dP}\Big)\Big] = E^Q\Big[\log\Big(\frac{dQ}{dP}\Big)\Big]$$

and let $\mathscr{M}^{ent} = \{Q \in \mathscr{M} : H(Q|P) < \infty\}$. We assume in the following that

$$\mathscr{M}^{ent} \neq \emptyset. \tag{3.3}$$

Now $\Theta := \{\vartheta \in L(R) : \vartheta \bullet R \text{ is a } Q\text{-supermartingale for all } Q \in \mathscr{M}^{ent}\}$ is the class of admissible strategies for (3.2). If S is locally bounded, there exists an optimal strategy $\hat{\vartheta} \in \Theta$ for (3.2) by Kabanov and Stricker [37, Theorem 2.1]. (See Biagini and Frittelli [7, 8] for the unbounded case.)

As there is no intermediate consumption, the process D in (2.3) reduces to a random variable $D_T \in L^\infty(\mathcal{F}_T)$. If we choose

$$B := \log(D_T), \tag{3.4}$$

we have the following result.

Theorem 3.2. *Let S be continuous and assume that $L(p)$ is continuous for all $p < 0$. Under* (3.3) *and* (3.4)*,*

$$(1-p)\,\hat{\pi}(p) \to \hat{\vartheta} \quad \textit{in } L^2_{loc}(M).$$

Here $\hat{\pi}(p)$ is in the fractions of wealth parametrization, while $\hat{\vartheta}$ denotes the monetary amounts invested for the exponential utility.

As this convergence may seem surprising at first glance, we give the following heuristics.

Remark 3.3. Assume $B = \log(D_T) = 0$ for simplicity. The preferences induced by $U^{(p)}(x) = \frac{1}{p}x^p$ on $\mathbb{R}_+$ are not directly comparable to the ones given by the exponential utility, which are defined on $\mathbb{R}$. We consider the shifted power utility functions

$$\widetilde{U}^{(p)}(x) := U^{(p)}\big(x + 1 - p\big), \quad x \in (p-1, \infty).$$

Then $\widetilde{U}^{(p)}$ again has relative risk aversion $1 - p > 0$ and its domain of definition increases to $\mathbb{R}$ as $p \to -\infty$. Moreover,

$$(1-p)^{1-p}\,\widetilde{U}^{(p)}(x) = \tfrac{1-p}{p}\Big(\frac{x}{1-p} + 1\Big)^p \to -e^{-x}, \quad p \to -\infty, \tag{3.5}$$

and the multiplicative constant does not affect the preferences.

Let the agent with utility function $\widetilde{U}^{(p)}$ be endowed with some initial capital $x_0^* \in \mathbb{R}$ independent of p. (If $x_0^* < 0$, we consider only values of p such that $p - 1 < x_0^*$.) The change of variables $x = \tilde{x} + 1 - p$ yields $U^{(p)}(x) = \widetilde{U}^{(p)}(\tilde{x})$. Hence the corresponding optimal wealth processes $\widehat{X}(p)$ and $\widetilde{X}(p)$ are related by $\widetilde{X}(p) = \widehat{X}(p) - 1 + p$ if we choose the initial capital $x_0 := x_0^* + 1 - p > 0$ for the agent with $U^{(p)}$. We conclude

$$d\widetilde{X}(p) = d\widehat{X}(p) = \widehat{X}(p)\hat{\pi}(p)\,dR = \big(\widetilde{X}(p) + 1 - p\big)\hat{\pi}(p)\,dR,$$

i.e., the optimal monetary investment $\tilde{\vartheta}(p)$ for $\widetilde{U}^{(p)}$ is given by

$$\tilde{\vartheta}(p) = \big(\widetilde{X}(p) + 1 - p\big)\hat{\pi}(p).$$

In view of (3.5), it is reasonable that $\tilde{\vartheta}(p)$ should converge to $\hat{\vartheta}$, the optimal monetary investment for the exponential utility. We recall that $\hat{\pi}(p)$ (in fractions of wealth) does not depend on x_0 and converges to zero under the conditions of Theorem 3.1. Thus, loosely speaking, $\widetilde{X}(p)\hat{\pi}(p) \approx 0$ for $-p$ large, and hence

$$\tilde{\vartheta}(p) \approx (1-p)\hat{\pi}(p).$$

More precisely, one can show that $\lim_{p\to-\infty}\big(\widetilde{X}(p)\hat{\pi}(p)\big) \bullet R = 0$ in the semimartingale topology, using arguments as in Appendix V.8.

Literature. To the best of our knowledge, the statement of Theorem 3.2 is new in the systematic literature. However, there are known results on the dual side for the case $B = 0$. The problem dual to exponential utility maximization is the minimization of $H(Q|P)$ over $\mathscr{M}^{ent}$ and the optimal $Q^E \in \mathscr{M}^{ent}$ is called minimal entropy martingale measure. Under additional assumptions on the model, the solution $\widehat{Y}(p)$ of the dual problem for power utility (4.3) introduced below is a martingale and then the measure Q^q defined by $dQ^q/dP = \widehat{Y}_T(p)/\widehat{Y}_0(p)$ is called q-optimal martingale measure, where $q < 1$ is conjugate to p. This measure can be defined also for $q > 1$, in which case it is not connected to power utility. The convergence of Q^q to Q^E for $q \to 1+$ was proved by Grandits and Rheinländer [29] for continuous semimartingale models satisfying a reverse Hölder inequality. Under the additional assumption that $\mathbb{F}$ is continuous, the convergence of Q^q to Q^E for $q \to 1$ and more generally the continuity of $q \mapsto Q^q$ for $q \geq 0$ were obtained by Mania and Tevzadze [54] (see also Santacroce [67]) using BSDE convergence together with BMO arguments. The latter are possible due to the reverse Hölder inequality; an assumption which is not present in our results.

V.3.2 The Limit $p \to 0$

As p tends to zero, the relative risk aversion of the power utility tends to 1, which corresponds to the utility function $\log(x)$. Hence we consider

$$u_{\log}(x_0) := \sup_{c\in\mathcal{A}(x_0)} E\Big[\int_0^T \log(c_t)\,\mu^\circ(dt)\Big];$$

here integrals are set to $-\infty$ if they are not well defined in $\overline{\mathbb{R}}$. A log-utility agent exhibits a very special ("myopic") behavior, which allows for an explicit solution of the utility maximization problem (cf. Goll and Kallsen [24, 25]). If in particular S is continuous, the log-optimal strategy is

$$\pi_t = \lambda_t, \quad \kappa_t = \frac{1}{1+T-t}$$

by [24, Theorem 3.1], where λ is defined by (2.6). Our result below shows that the optimal strategy for power utility with $D \equiv 1$ converges to the log-optimal one as $p \to 0$. In general, the randomness of D is an additional source of risk and will cause an excess hedging demand. Consider the bounded semimartingale

$$\eta_t := E\Big[\int_t^T D_s\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big].$$

If S is continuous, $\eta = \eta_0 + Z^\eta \bullet M + N^\eta + A^\eta$ denotes the Kunita-Watanabe decomposition of η with respect to M and the standard case $D \equiv 1$ corresponds to $\eta_t = \mu^\circ[t,T]$ and $Z^\eta = 0$.

Theorem 3.4. *Assume $u_{p_0}(x_0) < \infty$ for some $p_0 \in (0,1)$. As $p \to 0$,*

(i) in the case with intermediate consumption,

$$\hat{\kappa}_t(p) \to \frac{D_t}{\eta_t} \quad \textit{uniformly in } t,\ P\textit{-a.s.}$$

(ii) if S is continuous,

$$\hat{\pi}(p) \to \lambda + \frac{Z^\eta}{\eta_-} \quad \textit{in } L^2_{loc}(M)$$

and the corresponding wealth processes converge in the semimartingale topology.

Remark 3.5. If we consider the limit $p \to 0-$, we need not *a priori* assume that $u_{p_0}(x_0) < \infty$ for some $p_0 > 0$. Without that condition, the assertions of Theorem 3.4 remain valid if (i) is replaced by the weaker statement that $\lim_{p\to 0-} \hat{\kappa}_t(p) \to D_t/\eta_t$ P-a.s. for all t. If $\mathbb{F}$ is continuous, (i) remains valid without changes. In particular, these convergences hold even if $u_{\log}(x_0) = \infty$.

Literature. In the following discussion we assume $D \equiv 1$ for simplicity. It is part of the folklore that the log-optimal strategy can be obtained from $\hat{\pi}(p)$ by formally setting $p = 0$. Initiated by Jouini and Napp [36], a recent branch of the literature studies the stability of the utility maximization problem under perturbations of the utility function (with respect to pointwise convergence) and other ingredients of the problem. To the best of our knowledge, intermediate consumption was not considered so far and the previous results for continuous time concern continuous semimartingale models.

We note that $\log(x) = \lim_{p\to 0}(U^{(p)}(x) - p^{-1})$ and here the additive constant does not influence the optimal strategy, i.e., we have pointwise convergence of utility functions "equivalent" to $U^{(p)}$. Now Larsen [51, Theorem 2.2] implies that the optimal terminal wealth $\widehat{X}_T$ for $U^{(p)}$ converges in probability to the log-optimal one and that the value functions at time zero converge pointwise (in the continuous case without consumption). We use the specific form of our utility functions and obtain a stronger result. Finally, we can mention that on the dual side and for $p \to 0-$, the convergence is related to the continuity of q-optimal measures as mentioned after Remark 3.3.

For general D and p, it seems difficult to determine the precise influence of D on the optimal trading strategy $\hat{\pi}(p)$. We can read Theorem 3.4(ii) as a partial result on the excess hedging demand $\hat{\pi}(p) - \hat{\pi}(p,1)$ due to D; here $\hat{\pi}(p,1)$ denotes the optimal strategy for the case $D \equiv 1$.

Corollary 3.6. *Suppose that the conditions of Theorem 3.4(ii) hold. Then $\hat{\pi}(p) - \hat{\pi}(p,1) \to Z^\eta/\eta_-$ in $L^2_{loc}(M)$ as $p \to 0$; i.e., the asymptotic excess hedging demand due to D is given by Z^η/η_-.*

The stability theory mentioned above considers also perturbations of the probability measure P (see Kardaras and Žitković [45]) and our corollary can be related as follows. In the special case when D is a martingale, $U^{(p)}$ under P corresponds to the standard power utility function optimized under the measure $d\widetilde{P} = (D_T/D_0)\,dP$ (see Remark II.2.2). The excess hedging demand due to D then represents the influence of the “subjective beliefs” $\widetilde{P}$.

V.4 Tools and Ideas for the Proofs

In this section we introduce our main tools and then present the basic ideas how to apply them for the proofs of the theorems.

V.4.1 Opportunity Processes

We fix p and assume $u_p(x_0) < \infty$ throughout this section. We first discuss the properties of the (primal) opportunity process $L = L(p)$ as introduced in (3.1). Directly from that equation we have that $L_T = D_T$ and that $u_p(x_0) = L_0 \frac{1}{p} x_0^p$ is the value function from (2.5). Moreover, L has the following properties by Lemma II.3.5 and the bounds (2.4) for D.

Lemma 4.1. *The opportunity process satisfies $L, L_- > 0$.*

(i) If $p \in (0,1)$, L is a supermartingale satisfying

$$L_t \geq \big(\mu^\circ[t,T]\big)^{-p}\, E\Big[\int_t^T D_s\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \geq k_1.$$

(ii) If $p < 0$, L is a bounded semimartingale satisfying

$$0 < L_t \leq \big(\mu^\circ[t,T]\big)^{-p}\, E\Big[\int_t^T D_s\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \leq k_2\big(\mu^\circ[t,T]\big)^{1-p}.$$

If in addition there is no intermediate consumption, then L is a submartingale.

In particular, L is always a special semimartingale. We denote by

$$\beta := \frac{1}{1-p} > 0, \quad q := \frac{p}{p-1} \in (-\infty,0) \cup (0,1) \tag{4.1}$$

the relative risk tolerance and the exponent conjugate to p, respectively. These constants are of course redundant given p, but turn out to simplify the notation.

In the case with intermediate consumption, the opportunity process and the optimal consumption are related by

$$\hat{c}_t = \Big(\frac{D_t}{L_t}\Big)^{\beta} \widehat{X}_t \qquad \text{and hence} \qquad \hat{\kappa}_t = \Big(\frac{D_t}{L_t}\Big)^{\beta} \tag{4.2}$$

according to Theorem II.5.1. Next, we introduce the convex-dual analogue of L; cf. Section II.4 for the following notions and results. The *dual problem* is

$$\inf_{Y \in \mathscr{Y}} E\Big[\int_0^T U_t^*(Y_t)\,\mu^\circ(dt)\Big], \tag{4.3}$$

where $U_t^*(y) = \sup_{x>0}\big\{U_t(x) - xy\big\} = -\frac{1}{q}y^q D_t^\beta$ is the conjugate of U_t. Only three properties of the domain $\mathscr{Y} = \mathscr{Y}(p)$ are relevant for us. First, each element $Y \in \mathscr{Y}$ is a positive càdlàg supermartingale. Second, the set $\mathscr{Y}$ depends on p only by a normalization: with the constant $y_0(p) := L_0(p)x_0^{p-1}$, the set $\mathscr{Y}' := y_0(p)^{-1}\mathscr{Y}(p)$ does not depend on p. As the elements of $\mathscr{Y}$ will occur only in terms of certain fractions, the constant plays no role. Third, the P-density process of any $Q \in \mathscr{M}$ is contained in $\mathscr{Y}$ (modulo scaling).

The *dual opportunity process* L^* is the analogue of L for the dual problem and can be defined by

$$L_t^* := \begin{cases} \operatorname{ess\,sup}_{Y \in \mathscr{Y}} E\Big[\int_t^T D_s^\beta (Y_s/Y_t)^q\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] & \text{if } p<0\,, \\ \operatorname{ess\,inf}_{Y \in \mathscr{Y}} \; E\Big[\int_t^T D_s^\beta (Y_s/Y_t)^q\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] & \text{if } p \in (0,1). \end{cases} \tag{4.4}$$

Here the extremum is attained at the minimizer $Y \in \mathscr{Y}$ for (4.3), which we denote by $\widehat{Y} = \widehat{Y}(p)$. Finally, we shall use that the primal and the dual opportunity process are related by the power

$$L^* = L^\beta. \tag{4.5}$$

V.4.2 Bellman BSDE

We continue with a fixed p such that $u_p(x_0) < \infty$. We recall (Chapter III) the Bellman BSDE, which in the present chapter will be used only for continuous S. In this case, recall (2.6) and let $L = L_0 + Z^L \bullet M + N^L + A^L$ be the KW decomposition[1] of L with respect to M. Then the triplet (L, Z^L, N^L)

[1] In this chapter, we write Z^L instead of φ^L (as in Chapter III), since this is more in line with the BSDE literature.

satisfies the Bellman BSDE

$$\begin{aligned} dL_t &= \frac{q}{2}\, L_{t-}\Big(\lambda_t + \frac{Z^L_t}{L_{t-}}\Big)^\top d\langle M\rangle_t \,\Big(\lambda_t + \frac{Z^L_t}{L_{t-}}\Big) \;-pU^*_t(L_{t-})\,\mu(dt) \\ &\quad + Z^L_t\, dM_t + dN^L_t; \\ L_T &= D_T. \end{aligned} \tag{4.6}$$

Put differently, the finite variation part of L satisfies

$$A^L_t = \frac{q}{2}\int_0^t L_{s-}\Big(\lambda_s + \frac{Z^L_s}{L_{s-}}\Big)^\top d\langle M\rangle_s \,\Big(\lambda_s + \frac{Z^L_s}{L_{s-}}\Big) \;-p\int_0^t U^*_s(L_{s-})\,\mu(ds). \tag{4.7}$$

Here U^* is defined as after (4.3). Moreover, the optimal trading strategy $\hat\pi$ can be described by

$$\hat\pi_t = \beta\Big(\lambda_t + \frac{Z^L_t}{L_{t-}}\Big). \tag{4.8}$$

See Corollary III.3.12 for these results. Finally, still under the assumption of continuity, the solution to the dual problem (4.3) is given by the local martingale

$$\widehat{Y} = y_0\mathcal{E}\Big(-\lambda \bullet M + \frac{1}{L_-} \bullet N^L\Big), \tag{4.9}$$

with the constant $y_0 = u_p'(x_0) = L_0 x_0^{p-1}$ (cf. Remark III.5.18).

Remark 4.2. Continuity of S does not imply that L is continuous; the local martingale N^L may still have jumps (see also Remark III.3.13(i)). If the filtration $\mathbb{F}$ is continuous (i.e., all $\mathbb{F}$-martingales are continuous), it clearly follows that L and S are continuous. The most important example with this property is the Brownian filtration.

V.4.3 The Strategy for the Proofs

We can now summarize the basic scheme that is common for the proofs of the three theorems.

The first step is to prove the *pointwise convergence* of the opportunity process L or of the dual opportunity process L^*; the choice of the process depends on the theorem. The convergence of the optimal propensity to consume $\hat\kappa$ then follows in view of the feedback formula (4.2). The definitions of L and L^* via the value processes lend themselves to control-theoretic arguments, and of course Jensen's inequality will be the basic tool to derive estimates. In view of the relation $L^* = L^\beta$ from (4.5), it is essentially equivalent whether one works with L or L^*, as long as p is fixed. However, the dual problem has the advantage of being defined over a set of supermartingales, which are easier to handle than consumption and wealth processes. This is particularly useful when passing to the limit.

The second step is the convergence of the trading strategy $\hat{\pi}$. Note that its formula (4.8) contains the integrand Z^L from the KW decomposition of L with respect to M. Therefore, the convergence of $\hat{\pi}$ is related to the *convergence of the martingale part* M^L (resp. M^{L^*}). In general, the pointwise convergence of a semimartingale is not enough to deduce the convergence of its martingale part; this requires some control over the semimartingale decomposition. In our case, this control is given by the Bellman BSDE (4.6), which can be seen as a description for the dependence of the finite variation part A^L on the martingale part M^L. As we use the BSDE to show the convergence of M^L, we benefit from techniques from the theory of quadratic BSDEs. However, we cannot apply standard results from that theory since our assumptions are not strong enough.

In general, our approach is to extract as much information as possible by basic control arguments and convex analysis *before* tackling the BSDE, rather than to rely exclusively on (typically delicate) BSDE arguments. For instance, we use the BSDE only after establishing the pointwise convergence of its left hand side, i.e., the opportunity process. This essentially eliminates the need for an *a priori* estimate or a comparison principle and constitutes a key reason for the generality of our results. Our procedure shares basic features of the viscosity approach to Markovian control problems, where one also works directly with the value function before tackling the Hamilton-Jacobi-Bellman equation.

V.5 Auxiliary Results

We start by collecting inequalities for the dependence of the opportunity processes on p. The precise formulations are motivated by the applications in the proofs of the previous theorems, but the comparison results are also of independent interest.

V.5.1 Comparison Results

We assume in the entire section that $u_{p_0}(x_0) < \infty$ for a given exponent p_0. For convenience, we recall the quantities $\beta = 1/(1-p) > 0$ and $q = p/(p-1)$ defined in (4.1). It is useful to note that $q \in (-\infty, 0)$ for $p \in (0,1)$ and vice versa. When there is a second exponent p_0 under consideration, β_0 and q_0 have the obvious definition. We also recall from (2.4) the bounds k_1 and k_2 for D.

Proposition 5.1. *Let* $0 < p < p_0 < 1$. *For each* $t \in [0,T]$,

$$L^*_t(p) \;\leq\; E\Big[\int_t^T D_s^\beta \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]^{1-q/q_0} \Big(k_1^{\beta-\beta_0} L^*_t(p_0)\Big)^{q/q_0}, \tag{5.1}$$

$$L_t(p) \;\leq\; \big(k_2\mu^\circ[t,T]\big)^{1-p/p_0} L_t(p_0)^{p/p_0}. \tag{5.2}$$

If $p < p_0 < 0$, the converse inequalities hold, if in (5.1) k_1 *is replaced by* k_2. *If $p < 0 < p_0 < 1$, the converse inequalities hold, if in* (5.2) k_2 *is replaced by* k_1.

Proof. We fix t and begin with (5.1). To unify the proofs, we first argue a Jensen inequality: if $X = (X_s)_{s\in[t,T]} > 0$ is optional and $\alpha \in (0,1)$, then

$$E\Big[\int_t^T D_s^\beta X_s^\alpha \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \le E\Big[\int_t^T D_s^\beta \mu^\circ(ds)\Big|\mathcal{F}_t\Big]^{1-\alpha} E\Big[\int_t^T D_s^\beta X_s \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]^\alpha. \tag{5.3}$$

To see this, introduce the probability space $\big([t,T]\times\Omega, \mathcal{B}([t,T])\otimes\mathcal{F}, \nu\big)$, where

$$\nu(I\times G) := E\Big[\xi^{-1}\int_I 1_G D_s^\beta \,\mu^\circ(ds)\Big], \quad G\in\mathcal{F},\, I\in\mathcal{B}([t,T]),$$

with the normalizing factor $\xi := E[\int_t^T D_s^\beta \mu^\circ(ds)|\mathcal{F}_t]$. On this space, X is a random variable and we have the conditional Jensen inequality

$$E^\nu\big[X^\alpha\big|[t,T]\times\mathcal{F}_t\big] \le E^\nu\big[X\big|[t,T]\times\mathcal{F}_t\big]^\alpha$$

for the σ-field $[t,T]\times\mathcal{F}_t := \{[t,T]\times A : A\in\mathcal{F}_t\}$. But this inequality coincides with (5.3) if we identify $L^0([t,T]\times\Omega, [t,T]\times\mathcal{F}_t)$ and $L^0(\Omega,\mathcal{F}_t)$ by using that an element of the first space is necessarily constant in its time variable.

Let $0 < p \le p_0 < 1$ and let $\widehat{Y} := \widehat{Y}(p_0)$ be the solution of the dual problem for p_0. Using (4.4) and then (5.3) with $\alpha := q/q_0 \in (0,1)$ and $X_s^\alpha := \big((\widehat{Y}_s/\widehat{Y}_t)^{q_0}\big)^\alpha = (\widehat{Y}_s/\widehat{Y}_t)^q$,

$$\begin{aligned} L_t^*(p) &\le E\Big[\int_t^T D_s^\beta \big(\widehat{Y}_s/\widehat{Y}_t\big)^q \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \\ &\le E\Big[\int_t^T D_s^\beta \mu^\circ(ds)\Big|\mathcal{F}_t\Big]^{1-q/q_0} E\Big[\int_t^T D_s^\beta (\widehat{Y}_s/\widehat{Y}_t)^{q_0} \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]^{q/q_0}. \end{aligned}$$

Now $D_s^\beta \le k_1^{\beta-\beta_0} D_s^{\beta_0}$ since $\beta - \beta_0 < 0$, which completes the proof of the first claim in view of (4.4). In the cases with $p < 0$, the infimum in (4.4) is replaced by a supremum and $\alpha = q/q_0$ is either > 1 or < 0, reversing the direction of Jensen's inequality.

We turn to (5.2). Let $0 < p \le p_0 < 1$ and $\widehat{X} = \widehat{X}(p)$, $\hat{c} = \hat{c}(p)$. Using (3.1) and the usual Jensen inequality twice,

$$\begin{aligned} L_t(p_0)\widehat{X}_t^{p_0} &\ge E\Big[\int_t^T D_s \hat{c}_s^{p_0} \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] \\ &\ge \mu^\circ[t,T]^{1-p_0/p} E\Big[\int_t^T D_s^{p/p_0} \hat{c}_s^p \,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]^{p_0/p} \\ &\ge \big(k_2\mu^\circ[t,T]\big)^{1-p_0/p}\big(L_t(p)\widehat{X}_t^p\big)^{p_0/p} \end{aligned}$$

and the claim follows. The other cases are similar. □

A useful consequence is that $L(p)$ gains moments as p moves away from the possibly critical exponent p_0.

Corollary 5.2. *(i) Let $0 < p < p_0 < 1$. Then*

$$L(p) \leq C L(p_0) \tag{5.4}$$

with a constant C independent of p_0 and p. In the case without intermediate consumption we can take $C = 1$.

(ii) Let $r \geq 1$ and $0 < p \leq p_0/r$. Then

$$E\big[(L_\tau(p))^r\big] \leq C_r$$

for all stopping times τ, with a constant C_r independent of p_0, p, τ. In particular, $L(p)$ is of class (D) for all $p \in (0, p_0)$.

Proof. (i) Denote $L = L(p_0)$. By Lemma 4.1, $L/k_1 \geq 1$, hence $L^{p/p_0} = k_1^{p/p_0}(L/k_1)^{p/p_0} \leq k_1^{p/p_0}(L/k_1)$ as $p/p_0 \in (0,1)$. Proposition 5.1 yields the result with $C = \big(\mu^\circ[0,T]k_2/k_1\big)^{1-p/p_0}$; note that $C \leq 1 \vee (1+T)k_2/k_1$. In the absence of intermediate consumption we may assume $k_1 = k_2 = 1$ by the subsequent Remark 5.3 and then $C = 1$.

(ii) Let $r \geq 1$, $0 < p \leq p_0/r$, and $L = L(p_0)$. Proposition 5.1 shows

$$L_t(p)^r \leq \big(k_2 \mu^\circ[t,T]\big)^{r(1-p/p_0)} L_t^{rp/p_0} \leq \big((1 \vee k_2)(1+T)\big)^r L_t^{rp/p_0}.$$

Note $rp/p_0 \in (0,1)$, thus L^{rp/p_0} is a supermartingale by Lemma 4.1 and $E[L_\tau^{rp/p_0}] \leq L_0^{rp/p_0} \leq 1 \vee k_2$. □

Remark 5.3. In the case without intermediate consumption we may assume $D \equiv 1$ in the proof of Corollary 5.2(i). Indeed, D reduces to the random variable D_T and can be absorbed into the measure P as follows. Under the measure $\widetilde{P}$ with P-density process $\xi_t = E[D_T|\mathcal{F}_t]/E[D_T]$, the opportunity process for the utility function $\widetilde{U}(x) = \frac{1}{p}x^p$ is $\widetilde{L} = L/\xi$ by Remark II.3.2. If Corollary 5.2(i) is proved for $D \equiv 1$, we conclude $\widetilde{L}(p) \leq \widetilde{L}(p_0)$ and then the inequality for L follows.

Inequality (5.4) is stated for reference as it has a simple form; however, note that it was deduced using the very poor estimate $a^b \geq a$ for $a, b \geq 1$. In the pure investment case, we have $C = 1$ and so (5.4) is a direct comparison result. Intermediate consumption destroys this monotonicity property: (5.4) fails for $C = 1$ in that case, e.g., if $D \equiv 1$ and $R_t = t + W_t$, where W is a standard Brownian motion, and $p = 0.1$ and $p_0 = 0.2$, as can be seen by explicit calculation. This is not surprising from a BSDE perspective, because the driver of (4.6) is not monotone with respect to p in the presence of the $d\mu$-term. In the pure investment case, the driver is monotone and so the comparison result can be expected, even for the entire parameter range. This is confirmed by the next result; note that the inequality is *converse* to (5.2) for the considered parameters.

Proposition 5.4. *Let $p < p_0 < 0$, then*

$$L_t(p) \leq \frac{k_2}{k_1} \left(\mu^\circ[t,T]\right)^{p_0-p} L_t(p_0).$$

In the case without intermediate consumption, $L(p) \leq L(p_0)$.

The proof is based on the following auxiliary statement.

Lemma 5.5. *Let $Y > 0$ be a supermartingale. For fixed $0 \leq t \leq s \leq T$,*

$$\phi : (0,1) \to \mathbb{R}_+, \quad q \mapsto \phi(q) := \Big(E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big]\Big)^{\frac{1}{1-q}}$$

is a monotone decreasing function P-a.s. If Y is a martingale, we have $\phi(1) := \lim_{q\to 1-} \phi(q) = \exp\big(-E\big[(Y_s/Y_t)\log(Y_s/Y_t)\big|\mathcal{F}_t\big]\big)$ P-a.s., where the conditional expectation has values in $\mathbb{R} \cup \{+\infty\}$.

Lemma 5.5 can be obtained using Jensen's inequality and a suitable change of measure; see Lemma II.4.10 for details.

Proof of Proposition 5.4. Let $0 < q_0 < q < 1$ be the dual exponents and denote $\widehat{Y} := \widehat{Y}(p)$. By Lemma 5.5 and Jensen's inequality for $\frac{1-q}{1-q_0} \in (0,1)$,

$$\begin{aligned}
\int_t^T E\big[(\widehat{Y}_s/\widehat{Y}_t)^q \big| \mathcal{F}_t\big]\,\mu^\circ(ds) &\leq \int_t^T \Big(E\big[(\widehat{Y}_s/\widehat{Y}_t)^{q_0} \big| \mathcal{F}_t\big]\Big)^{\frac{1-q}{1-q_0}} \mu^\circ(ds) \\
&\leq \mu^\circ[t,T]^{\left(1-\frac{1-q}{1-q_0}\right)} \bigg(\int_t^T E\big[(\widehat{Y}_s/\widehat{Y}_t)^{q_0} \big| \mathcal{F}_t\big]\,\mu^\circ(ds)\bigg)^{\frac{1-q}{1-q_0}}.
\end{aligned}$$

Using (2.4) and (4.4) twice, we conclude that

$$\begin{aligned}
L_t^*(p) &\leq k_2^\beta \int_t^T E\big[(\widehat{Y}_s/\widehat{Y}_t)^q \big| \mathcal{F}_t\big]\,\mu^\circ(ds) \\
&\leq k_2^\beta k_1^{-\beta_0 \frac{1-q}{1-q_0}} \mu^\circ[t,T]^{\left(1-\frac{1-q}{1-q_0}\right)} \bigg(\int_t^T E\big[D_s^{\beta_0}(\widehat{Y}_s/\widehat{Y}_t)^{q_0} \big| \mathcal{F}_t\big]\,\mu^\circ(ds)\bigg)^{\frac{1-q}{1-q_0}} \\
&\leq k_2^\beta k_1^{-\beta_0 \frac{1-q}{1-q_0}} \mu^\circ[t,T]^{\left(1-\frac{1-q}{1-q_0}\right)} L_t^*(p_0)^{\frac{1-q}{1-q_0}}.
\end{aligned}$$

Now (4.5) and $\beta = 1 - q$ yield the first result. In the case without intermediate consumption, we may assume $D \equiv 1$ and hence $k_1 = k_2 = 1$, as in Remark 5.3. □

Remark 5.6. Our argument for Proposition 5.4 extends to $p = -\infty$ (cf. Lemma 6.7 below). The proposition generalizes [54, Proposition 2.2], where the result is proved for the case without intermediate consumption and under the additional condition that $\widehat{Y}(p_0)$ is a martingale (or equivalently, that the q_0-optimal equivalent martingale measure exists).

Propositions 5.1 and 5.4 combine to the following continuity property of $p \mapsto L(p)$ at interior points of $(-\infty, 0)$. We will not pursue this further as we are interested mainly in the boundary points of this interval.

Corollary 5.7. *Assume $D \equiv 1$ and let $C_t := \mu^\circ[t,T]$. If $p \le p_0 < 0$,*

$$C_t^{1-p/p_0} L(p_0)^{p/p_0} \;\le\; L(p) \;\le\; C_t^{p_0-p} L(p_0) \;\le\; C_t^{1-p_0/p+p_0-p} L(p)^{p_0/p}.$$

In particular, $p \mapsto L_t(p)$ is continuous on $(-\infty, 0)$ uniformly in t, P-a.s.

Remark 5.8. The optimal propensity to consume $\hat{\kappa}(p)$ is *not* monotone with respect to p in general. For instance, monotonicity fails for $D \equiv 1$ and $R_t = t + W_t$, where W is a standard Brownian motion, and $p \in \{-1/2, -1, -2\}$. One can note that p determines both the risk aversion and the elasticity of intertemporal substitution (see, e.g., Gollier [27, §15]). As with any time-additive utility specification, it is not possible in our setting to study the dependence on each of these quantities in an isolated way.

V.5.2 *BMO* Estimate

In this section we give BMO estimates for the martingale part of L. The following lemma is well known; we state the proof since the argument will be used also later on.

Lemma 5.9. *Let X be a submartingale satisfying $0 \le X \le \alpha$ for some constant $\alpha > 0$. Then for all stopping times $0 \le \sigma \le \tau \le T$,*

$$E\big[[X]_\tau - [X]_\sigma \big| \mathcal{F}_\sigma\big] \le E\big[X_\tau^2 - X_\sigma^2 \big| \mathcal{F}_\sigma\big].$$

Proof. Let $X = X_0 + M^X + A^X$ be the Doob-Meyer decomposition. As $X_t^2 = X_0^2 + 2\int_0^t X_{s-}\,(dM_s^X + dA_s^X) + [X]_t$ and $2\int_\sigma^\tau X_{s-} dA_s^X \ge 0$,

$$[X]_\tau - [X]_\sigma \le X_\tau^2 - X_\sigma^2 - 2\int_\sigma^\tau X_{s-} dM_s^X.$$

The claim follows by taking conditional expectations because $X_- \bullet M^X$ is a martingale. Indeed, X is bounded and $\sup_t |M_t^X| \le 2\alpha + A_T^X \in L^1$, so the BDG inequalities [18, VII.92] show $[M^X]_T^{1/2} \in L^1$, hence $[X_- \bullet M^X]_T^{1/2} \in L^1$, which by the BDG inequalities implies that $\sup_t |X_- \bullet M_t^X| \in L^1$. $\square$

We wish to apply Lemma 5.9 to $L(p)$ in the case $p < 0$. However, the submartingale property fails in general for the case with intermediate consumption (cf. Lemma 4.1). We introduce instead a closely related process having this property.

Lemma 5.10. *Let $p < 0$ and consider the case with intermediate consumption. Then*

$$B_t := \Big(\frac{1+T-t}{1+T}\Big)^p L_t + \frac{1}{(1+T)^p}\int_0^t D_s\,ds$$

is a submartingale satisfying $0 < B_t \le k_2(1+T)^{1-p}$.

Proof. Choose $(\pi, c) \equiv (0, x_0/(1+T))$ in Proposition II.3.4 to see that B is a submartingale. The bound follows from Lemma 4.1. □

We are now in the position to exploit Lemma 5.9.

Lemma 5.11. *(i) Let $p_1 < 0$. There exists a constant $C = C(p_1)$ such that $\|M^{L(p)}\|_{BMO} \le C$ for all $p \in (p_1, 0)$. In the case without intermediate consumption one can take $p_1 = -\infty$.*

(ii) Assume $u_{p_0}(x_0) < \infty$ for some $p_0 \in (0,1)$ and let σ be a stopping time such that $L(p_0)^\sigma \le \alpha$ for a constant $\alpha > 0$. Then there exists $C' = C'(\alpha)$ such that $\|(M^{L(p)})^\sigma\|_{BMO} \le C'$ for all $p \in (0, p_0]$.

Proof. (i) Let $p_1 < p < 0$ and let τ be a stopping time. We first show that

$$E\big[[L(p)]_T - [L(p)]_\tau \big| \mathcal{F}_\tau\big] \le C. \tag{5.5}$$

In the case without intermediate consumption, $L = L(p)$ is a positive submartingale with $L \le k_2$ (Lemma 4.1), so Lemma 5.9 implies (5.5) with $C = k_2^2$. In the other case, define B as in Lemma 5.10 and $f(t) := (\frac{1+T-t}{1+T})^p$. Then $[L]_t - [L]_0 = \int_0^t f^{-2}(s)\,d[B]_s$ and $f^{-2}(s) \le 1$ as f is increasing with $f(0) = 1$. Thus $[L]_T - [L]_\tau = \int_\tau^T f^{-2}(s)\,d[B]_s \le [B]_T - [B]_\tau$. Now (5.5) follows since $B \le k_2(1+T)^{1-p}$ and Lemma 5.9 imply

$$E\big[[B]_T - [B]_\tau \big| \mathcal{F}_\tau\big] \le k_2^2(1+T)^{2-2p} \le k_2^2(1+T)^{2-2p_1} =: C(p_1).$$

We have $[L] = L_0^2 + [M^L] + [A^L] + 2[M^L, A^L]$. Since A^L is predictable, $N := 2[M^L, A^L]$ is a local martingale with some localizing sequence (σ_n). Moreover, $[M^L]_t - [M^L]_s = [L]_t - [L]_s - ([A]_t - [A]_s) - (N_t - N_s)$ and (5.5) imply

$$E\big[[M^L]_{T\wedge\sigma_n} - [M^L]_{\tau\wedge\sigma_n} \big| \mathcal{F}_{\tau\wedge\sigma_n}\big] \le C.$$

Choosing $\tau = 0$ and $n \to \infty$ we see that $[M]_T \in L^1(P)$ and thus Hunt's lemma [18, V.45] shows the a.s.-convergence in this inequality; i.e., we have $E\big[[M^L]_T - [M^L]_\tau \big| \mathcal{F}_\tau\big] \le C$. If L is bounded by α, the jumps of M^L are bounded by 2α (cf. [34, I.4.24]), therefore

$$\sup_\tau E\big[[M^L]_T - [M^L]_{\tau-} \big| \mathcal{F}_\tau\big] \le C + 4\alpha^2.$$

By Lemma 4.1 we can take $\alpha = k_2(1+T)^{1-p_1}$, and $\alpha = k_2$ when there is no intermediate consumption.

(ii) Let $0 < p \leq p_0 < 1$. The assumption and Corollary 5.2(i) show that $L(p)^\sigma \leq C_\alpha$ for a constant C_α independent of p and p_0. We apply Lemma 5.9 to the nonnegative process $X(p) := C_\alpha - L(p)^\sigma$, which is a submartingale by Lemma 4.1, and obtain $E\big[[L(p)^\sigma]_T - [L(p)^\sigma]_\tau \big| \mathcal{F}_\tau\big] = E\big[[X(p)]_T - [X(p)]_\tau \big| \mathcal{F}_\tau\big] \leq C_\alpha^2$. Now the rest of the proof is as in (i). □

Corollary 5.12. *Let S be continuous and assume that either $p \in (0,1)$ and L is bounded or that $p < 0$ and L is bounded away from zero. Then $\lambda \bullet M \in BMO$, where λ and M are defined by* (2.6).

Proof. In both cases, the assumed bound and Lemma 4.1 imply that L is bounded away from zero and infinity. Taking conditional expectations in (4.6), we obtain a constant $C > 0$ such that

$$E\bigg[\int_t^T L_-\Big(\lambda + \frac{Z^L}{L_-}\Big)^\top d\langle M\rangle \Big(\lambda + \frac{Z^L}{L_-}\Big)\bigg|\mathcal{F}_t\bigg] \leq C, \quad 0 \leq t \leq T.$$

Moreover, we have $M^L \in BMO$ by Lemma 5.11. Using the bounds for L and the Cauchy-Schwarz inequality, it follows that $E[\int_t^T \lambda^\top d\langle M\rangle\, \lambda | \mathcal{F}_t] \leq C'(1 + \|Z^L \bullet M\|_{BMO}) \leq C'(1 + \|M^L\|_{BMO})$ for a constant $C' > 0$. □

We remark that uniform bounds for L (as in the condition of Corollary 5.12) are equivalent to a reverse Hölder inequality $\mathrm{R}_q(P)$ for some element of the dual domain $\mathscr{Y}$; see Proposition II.4.5 for details. Here the index q satisfies $q < 1$. Therefore, our corollary complements well known results stating that $\mathrm{R}_q(P)$ with $q > 1$ implies $\lambda \bullet M \in BMO$ (in a suitable setting); see, e.g., Delbaen et al. [16, Theorems A,B].

V.6 The Limit $p \to -\infty$

The first goal of this section is to prove Theorem 3.1. Recall that the consumption strategy is related to the opportunity processes via (4.2) and (4.5). From these relations and the intuition mentioned before Theorem 3.1, we expect that the dual opportunity process $L^*_t = L^\beta_t$ converges to $\mu^\circ[t,T]$ as $p \to -\infty$. Noting that the exponent $\beta = 1/(1-p) \to 0$, this implies that $L_t(p) \to \infty$ for all $t < T$, in the case with intermediate consumption. Therefore, we shall work here with L^* rather than L. In the pure investment case, the situation is different as then $L \leq k_2$ (Lemma 4.1). There, the limit of L yields additional information; this is examined in Section V.6.1 below.

Proposition 6.1. *For each $t \in [0,T]$,*

$$\lim_{p\to-\infty} L^*_t(p) = \mu^\circ[t,T] \quad P\text{-a.s. and in } L^r(P),\ r \in [1,\infty),$$

with a uniform bound. If $\mathbb{F}$ is continuous, the convergences are uniform in t.

Remark 6.2. We will use later that the same convergences hold if t is replaced by a stopping time, which is an immediate consequence in view of the uniform bound. Of course, we mean by "uniform bound" that there exists a constant $C > 0$, independent of p and t, such that $0 \leq L^*_t(p) \leq C$. Analogous terminology will be used in the sequel.

Proof. We consider $0 > p \to -\infty$ and note that $q \to 1-$ and $\beta \to 0+$. From Lemma 4.1 we have

$$0 \leq L^*_t(p) = L^\beta_t(p) \leq k_2^\beta \mu^\circ[t,T] \to \mu^\circ[t,T], \tag{6.1}$$

uniformly in t. To obtain a lower bound, we consider the density process Y of some $Q \in \mathscr{M}$, which exists by assumption (2.1). From (4.4) we have

$$L^*_t(p) \geq k_1^\beta \int_t^T E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big]\, \mu^\circ(ds).$$

For fixed $s \geq t$, clearly $(Y_s/Y_t)^q \to Y_s/Y_t$ P-a.s. as $q \to 1$, and noting the bound $0 \leq (Y_s/Y_t)^q \leq 1 + Y_s/Y_t \in L^1(P)$ we conclude by dominated convergence that

$$E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] \to E\big[Y_s/Y_t \big| \mathcal{F}_t\big] \equiv 1 \quad P\text{-a.s., for all } s \geq t.$$

Since Y^q is a supermartingale, $0 \leq E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] \leq 1$. Hence, for each t, dominated convergence shows

$$\int_t^T E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big]\, \mu^\circ(ds) \to \mu^\circ[t,T] \;\; P\text{-a.s.}$$

This ends the proof of the first claim. The convergence in $L^r(P)$ follows by the bound (6.1).

Assume that $\mathbb{F}$ is continuous; then the martingale Y is continuous. For fixed $(s,\omega) \in [0,T] \times \Omega$ we consider (a fixed version of)

$$f_q(t) := E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big]^{1/q}(\omega), \quad t \in [0,s].$$

These functions are continuous in t and increasing in q by Jensen's inequality, and converge to 1 for each t. Hence $f_q \to 1$ uniformly in t on the compact $[0,s]$, by Dini's lemma. The same holds for $f_q(t)^q = E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big](\omega)$.

Fix $\omega \in \Omega$ and let $\varepsilon, \varepsilon' > 0$. By Egorov's theorem there exist a measurable set $I = I(\omega) \subseteq [0,T]$ and $\delta = \delta(\omega) \in (0,1)$ such that $\mu^\circ([0,T] \setminus I) < \varepsilon$ and $\sup_{t\in[0,s]} |E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] - 1| < \varepsilon'$ for all $q > 1 - \delta$ and all $s \in I$. For $q > 1 - \delta$ and $t \in [0,T]$ we have

$$\begin{aligned}
&\int_t^T \big|E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] - 1\big|\, \mu^\circ(ds) \\
&\leq \int_I \big|E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] - 1\big|\, \mu^\circ(ds) + \int_{[t,T]\setminus I} \big|E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] - 1\big|\, \mu^\circ(ds) \\
&\leq \varepsilon'(1+T) + \varepsilon.
\end{aligned}$$

We have shown that $\sup_{t\in[0,T]} |L^*_t(p)-\mu^\circ[t,T]| \to 0$ P-a.s., and also in $L^r(P)$ by dominated convergence and the uniform bound resulting from (6.1) in view of $k_2^\beta \mu^\circ[t,T] \leq (1\vee k_2)(1+T)$. □

Under additional continuity assumptions, we will prove that the martingale part of L^* converges to zero in $\mathcal{H}^2_{loc}$. We first need some preparations. For each p, it follows from Lemma 4.1 that L^* has a canonical decomposition $L^* = L^*_0 + M^{L^*} + A^{L^*}$. When S is continuous, we denote the KW decomposition with respect to M by $L^* = L^*_0 + Z^{L^*} \bullet M + N^{L^*} + A^{L^*}$. If in addition L is continuous, we obtain from $L^* = L^\beta$ and (4.7) by Itô's formula that

$$M^{L^*} = \beta L^{\beta-1} \bullet M^L; \quad Z^{L^*}/L^* = \beta Z^L/L; \quad N^{L^*} = \beta L^{\beta-1} \bullet N^L; \tag{6.2}$$
$$A^{L^*} = \tfrac{q}{2} \int \big(\beta\lambda L^* + 2Z^{L^*}\big)^\top d\langle M\rangle\, \lambda + \tfrac{p}{2} \int \big(L^*\big)^{-1} d\langle N^{L^*}\rangle - \int D^\beta\, d\mu.$$

Here $d\mu$ is a shorthand for $\mu(ds)$.

Lemma 6.3. *Let $p_0 < 0$. There exists a localizing sequence (σ_n) such that*

$$\big(L^*(p)\big)^{\sigma_n}_- > 1/n \text{ simultaneously for all } p \in (-\infty, p_0];$$

and moreover, if S and $L(p)$ are continuous, $(M^{L^*(p)})^{\sigma_n} \in BMO$ for $p \leq p_0$.

Proof. Fix $p_0 < 0$ (and corresponding q_0) and a sequence $\varepsilon_n \downarrow 0$ in $(0,1)$. Set $\sigma_n = \inf\{t \geq 0 : L^*_t(p_0) \leq \varepsilon_n\} \wedge T$. Then $\sigma_n \to T$ stationarily because each path of $L^*(p_0)$ is bounded away from zero (Lemma 4.1). Proposition 5.1 implies that there is a constant $\alpha = \alpha(p_0) > 0$ such that $L^*_t(p) \geq \alpha \left(L^*_t(p_0)\right)^{q/q_0}$ for all $p \leq p_0$. It follows that $L^*_{(\sigma_n\wedge t)-}(p) \geq \alpha\varepsilon_n^{1/q_0}$ for all $p \leq p_0$ and we have proved the first claim.

Fix $p \in (-\infty, p_0]$, let S and $L = L(p)$ be continuous and recall that $M^{L^*} = \beta L^{\beta-1} \bullet M^L$ from (6.2). Noting that $\beta - 1 < 0$, we have just shown that the integrand $\beta L^{\beta-1}$ is bounded on $[0, \sigma_n]$. Since $M^L \in BMO$ by Lemma 5.11(i), we conclude that $(M^{L^*})^{\sigma_n} \in BMO$. □

Proposition 6.4. *Assume that S and $L(p)$ are continuous for all $p < 0$. As $p \to -\infty$,*

$$Z^{L^*(p)} \to 0 \text{ in } L^2_{loc}(M) \quad \text{and} \quad N^{L^*(p)} \to 0 \text{ in } \mathcal{H}^2_{loc}.$$

Proof. We fix some $p_0 < 0$ and consider $p \in (-\infty, p_0]$. Using Lemma 6.3, we may assume by localization that $M^{L^*(p)} \in \mathcal{H}^2$ and $\lambda \in L^2(M)$. Define the continuous processes $X = X(p)$ by

$$X_t(p) := k_2^\beta \mu^\circ[t,T] - L^*_t(p),$$

then $0 \leq X(p) \leq (1\vee k_2)(1+T)$ by (6.1). Fix p. We shall apply Itô's formula to $\Phi(X)$, where

$$\Phi(x) := \exp(x) - x.$$

For $x \geq 0$, Φ satisfies

$$\Phi(x) \geq 1, \quad \Phi'(0) = 0, \quad \Phi'(x) \geq 0, \quad \Phi''(x) \geq 1, \quad \Phi''(x) - \Phi'(x) = 1.$$

We have $\frac{1}{2}\int_0^T \Phi''(X)\, d\langle X\rangle = \Phi(X_T) - \Phi(X_0) - \int_0^T \Phi'(X)\,(\,dM^X + dA^X)$. As $\Phi'(X)$ is like X uniformly bounded and $M^X = -M^{L^*} \in \mathcal{H}^2$, the stochastic integral wrt. M^X is a true martingale and

$$E\Big[\int_0^T \Phi''(X)\, d\langle X\rangle\Big] = 2E\big[\Phi(X_T) - \Phi(X_0)\big] - 2E\Big[\int_0^T \Phi'(X)\, dA^X\Big].$$

Note that $dA^X = -k_2^\beta\, d\mu - dA^{L^*}$, so that (6.2) yields

$$2\, dA^X = -q\big(\beta\lambda L^* + 2Z^{L^*}\big)^\top d\langle M\rangle\, \lambda - p\big(L^*\big)^{-1} d\langle N^{L^*}\rangle + 2\big(D^\beta - k_2^\beta\big)\, d\mu.$$

Letting $p \to -\infty$, we have $q \to 1-$ and $\beta \to 0+$. Hence, using that X and L^* are bounded uniformly in p,

$$\begin{aligned}
-q\beta E\Big[\int_0^T \Phi'(X)(\lambda L^*)^\top d\langle M\rangle\, \lambda\Big] &\to 0,\\
E\Big[\int_0^T \Phi'(X)\big(D^\beta - k_2^\beta\big)\, d\mu\Big] &\to 0,\\
E\big[\Phi(X_T) - \Phi(X_0)\big] &\to 0,
\end{aligned}$$

where the last convergence is due to Proposition 6.1 (and the subsequent remark). If o denotes the sum of these three expectations tending to zero,

$$\begin{aligned}
&E\Big[\int_0^T \Phi''(X)\, d\langle X\rangle\Big]\\
&\qquad = E\Big[\int_0^T \Phi'(X)\Big\{2q\big(Z^{L^*}\big)^\top d\langle M\rangle\, \lambda + p\big(L^*\big)^{-1} d\langle N^{L^*}\rangle\Big\}\Big] + o.
\end{aligned}$$

Note $d\langle X\rangle = d\langle L^*\rangle = \big(Z^{L^*}\big)^\top d\langle M\rangle\, Z^{L^*} + d\langle N^{L^*}\rangle$. For the right hand side, we use $\Phi'(X) \geq 0$ and $|q| < 1$ and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
&E\Big[\int_0^T \Phi''(X)\, \Big\{\big(Z^{L^*}\big)^\top d\langle M\rangle\, Z^{L^*} + d\langle N^{L^*}\rangle\Big\}\Big]\\
&\leq E\Big[\int_0^T \Phi'(X)\Big\{\big(Z^{L^*}\big)^\top d\langle M\rangle\, Z^{L^*} + \lambda^\top d\langle M\rangle\, \lambda + p\big(L^*\big)^{-1} d\langle N^{L^*}\rangle\Big\}\Big] + o.
\end{aligned}$$

We bring the terms with Z^{L^*} and N^{L^*} to the left hand side, then

$$\begin{aligned}
&E\Big[\int_0^T \{\Phi''(X) - \Phi'(X)\}\big(Z^{L^*}\big)^\top d\langle M\rangle\, Z^{L^*}\Big]\\
&+ E\Big[\int_0^T \{\Phi''(X) - p\Phi'(X)\big(L^*\big)^{-1}\} d\langle N^{L^*}\rangle\Big] \leq E\Big[\int_0^T \Phi'(X)\lambda^\top d\langle M\rangle\, \lambda\Big] + o.
\end{aligned}$$

As $\Phi'(0)=0$, we have $\lim_{p\to-\infty}\Phi'(X_t)\to 0$ P-a.s. for all t, with a uniform bound, hence $\lambda\in L^2(M)$ implies that the right hand side converges to zero. We recall $\Phi''-\Phi'\equiv 1$ and $\Phi''(X)-p\Phi'(X)\big(L^*\big)^{-1}\geq\Phi''(0)=1$. Thus both expectations on the left hand side are nonnegative and we can conclude that they converge to zero; therefore, $E\big[\int_0^T(Z^{L^*})^\top\,d\langle M\rangle\,Z^{L^*}\big]\to 0$ and $E[\langle N^{L^*}\rangle_T]\to 0$. $\square$

Proof of Theorem 3.1. In view of (4.2), part (i) follows from Proposition 6.1; note that the convergence in $\mathcal{R}^r_{loc}$ is immediate as $\hat\kappa(p)$ is locally bounded uniformly in p by Lemma 6.3 and (4.2). For part (ii), recall from (4.8) and (6.2) that

$$\hat\pi=\beta(\lambda+Z^L/L)=\beta\lambda+Z^{L^*}/L^*$$

for each p. As $\beta\to 0$, clearly $\beta\lambda\to 0$ in $L^2_{loc}(M)$. By Lemma 6.3, $1/L^*$ is locally bounded uniformly in p, hence $\hat\pi(p)\to 0$ in $L^2_{loc}(M)$ follows from Proposition 6.4. As $\hat\kappa(p)$ is locally bounded uniformly in p, Corollary 8.4(i) from the Appendix yields the convergence of the wealth processes $\widehat{X}(p)$. $\square$

V.6.1 Convergence to the Exponential Problem

In this section, we prove Theorem 3.2 and establish the convergence of the corresponding opportunity processes. We assume that there is no intermediate consumption, that S is locally bounded and satisfies (3.3), and that the contingent claim B is bounded (we will choose a specific B later). Hence there exists an (essentially unique) optimal strategy $\hat\vartheta\in\Theta$ for (3.2). It is easy to see that $\hat\vartheta$ does not depend on the initial capital x_0. If $\vartheta\in\Theta$, we denote by $G(\vartheta)=\vartheta\bullet R$ the corresponding gains process and define $\Theta(\vartheta,t)=\{\tilde\vartheta\in\Theta:\ G_t(\tilde\vartheta)=G_t(\vartheta)\}$. We consider the value process (from initial wealth zero) of (3.2),

$$V_t(\vartheta):=\operatorname{ess\,sup}_{\tilde\vartheta\in\Theta(\vartheta,t)}E\big[-\exp\big(B-G_T(\tilde\vartheta)\big)\big|\mathcal{F}_t\big],\quad 0\leq t\leq T.$$

Note the concatenation property $\vartheta^1,\vartheta^2\in\Theta\Rightarrow\vartheta^1 1_{[0,t]}+\vartheta^2 1_{(t,T]}\in\Theta$. With $G_{t,T}(\vartheta):=\int_t^T\vartheta\,dR$, we have $G_T(\tilde\vartheta)=G_t(\vartheta)+G_{t,T}(\tilde\vartheta 1_{(t,T]})$ for $\tilde\vartheta\in\Theta(\vartheta,t)$. Therefore, if we define the exponential opportunity process

$$L^{\exp}_t:=\operatorname{ess\,inf}_{\tilde\vartheta\in\Theta}E\big[\exp\big(B-G_{t,T}(\tilde\vartheta)\big)\big|\mathcal{F}_t\big],\quad 0\leq t\leq T, \tag{6.3}$$

then using standard properties of the essential infimum one can check that

$$V_t(\vartheta)=-\exp(-G_t(\vartheta))\,L^{\exp}_t.$$

Thus $L^{\exp}$ is a reduced form of the value process, analogous to $L(p)$ for power utility. We also note that $L^{\exp}_T=\exp(B)$.

Lemma 6.5. *The exponential opportunity process $L^{\exp}$ is a submartingale satisfying $L^{\exp}\leq\|\exp(B)\|_{L^\infty(P)}$ and $L^{\exp},L^{\exp}_->0$.*

Proof. The martingale optimality principle of dynamic programming, which is proved here exactly as in Proposition II.6.2, yields that $V(\vartheta)$ is a supermartingale for every $\vartheta \in \Theta$ such that $E[V_\cdot(\vartheta)] > -\infty$ and a martingale if and only if ϑ is optimal. As $V(\vartheta) = -\exp(-G(\vartheta))\, L^{\exp}$, we obtain the submartingale property by the choice $\vartheta \equiv 0$. It follows that $L^{\exp} \leq \|L^{\exp}_T\|_{L^\infty} = \|\exp(B)\|_{L^\infty}$.

The optimal strategy $\hat{\vartheta}$ is optimal for all the conditional problems (6.3), hence $L^{\exp}_t = E\big[\exp\big(B - G_{t,T}(\hat{\vartheta})\big)\big|\mathcal{F}_t\big] > 0$. Thus $\xi := \exp(-G(\hat{\vartheta}))\, L^{\exp}$ is a positive martingale, by the optimality principle. In particular, we have $P[\inf_{0 \leq t \leq T} \xi_t > 0] = 1$, and now the same property for $L^{\exp}$ follows. □

Assume that S is continuous and denote the KW decomposition of $L^{\exp}$ with respect to M by $L^{\exp} = L^{\exp}_0 + Z^{L^{\exp}} \bullet M + N^{L^{\exp}} + A^{L^{\exp}}$. Then the triplet $(\ell, z, n) := \big(L^{\exp}, Z^{L^{\exp}}, N^{L^{\exp}}\big)$ satisfies the BSDE

$$d\ell_t = \frac{1}{2}\,\ell_{t-}\Big(\lambda_t + \frac{z_t}{\ell_{t-}}\Big)^\top d\langle M\rangle_t \Big(\lambda_t + \frac{z_t}{\ell_{t-}}\Big) + z_t\, dM_t + dn_t \tag{6.4}$$

with terminal condition $\ell_T = \exp(B)$, and the optimal strategy $\hat{\vartheta}$ is

$$\hat{\vartheta} = \lambda + \frac{Z^{L^{\exp}}}{L^{\exp}_-}. \tag{6.5}$$

This can be derived directly by dynamic programming or inferred, e.g., from Frei and Schweizer [23, Proposition 1]. We will actually reprove the BSDE later, but present it already at this stage for the following motivation.

We observe that (6.4) *coincides with the BSDE* (4.6)*, except that q is replaced by* 1 and the terminal condition is $\exp(B)$ instead of D_T. From now on *we assume* $\exp(B) = D_T$, then one can guess that the solutions $L(p)$ should converge to $L^{\exp}$ as $q \to 1-$, or equivalently $p \to -\infty$.

Theorem 6.6. *Let S be continuous.*

(i) As $p \downarrow -\infty$, $L_t(p) \downarrow L^{\exp}_t$ P-a.s. for all t, with a uniform bound.

(ii) If $L(p)$ is continuous for each $p < 0$, then $L^{\exp}$ is also continuous and the convergence $L(p) \downarrow L^{\exp}$ is uniform in t, P-a.s. Moreover,

$$(1-p)\,\hat{\pi}(p) \to \hat{\vartheta} \quad \text{in } L^2_{loc}(M).$$

We note that (ii) is also a statement about the rate of convergence for $\hat{\pi}(p) \to 0$ in Theorem 3.1(ii) for the case without intermediate consumption. The proof occupies most of the remainder of the section. Part (i) follows from the next two lemmata; recall that the monotonicity of $p \mapsto L_t(p)$ was already established in Proposition 5.4 while the uniform bound is from Lemma 4.1.

Lemma 6.7. *We have $L(p) \geq L^{\exp}$ for all $p < 0$.*

Proof. As is well-known, we may assume that $B = 0$ by a change of measure from P to $dP(B) = (e^B/E[e^B])\,dP$. Let $Q^E \in \mathscr{M}^{ent}$ be the measure with minimal entropy $H(Q|P)$; see, e.g., [37, Theorem 3.5]. Let Y be its P-density process, then

$$-\log(L_t^{\exp}) = E^{Q^E}\big[\log(Y_T/Y_t)\big|\mathcal{F}_t\big] = E\big[(Y_T/Y_t)\log(Y_T/Y_t)\big|\mathcal{F}_t\big]. \tag{6.6}$$

This is merely a dynamic version of the well-known duality relation stated, e.g., in [37, Theorem 2.1] and one can retrieve this version, e.g., from [23, Eq. (8),(10)]. Using the decreasing function ϕ from Lemma 5.5,

$$\begin{aligned} L_t^{\exp} &= \exp\Big(-E\big[(Y_T/Y_t)\log(Y_T/Y_t)\big|\mathcal{F}_t\big]\Big) = \phi(1) \\ &\leq \phi(q) = E\big[(Y_T/Y_t)^q\big|\mathcal{F}_t\big]^{1/\beta} \leq L^*(p)^{1/\beta} = L(p), \end{aligned}$$

where (4.4) was used for the second inequality. $\square$

Lemma 6.8. *Let S be continuous. Then* $\limsup_{p\to-\infty} L_t(p) \leq L_t^{\exp}$.

Proof. Fix $t \in [0,T]$. We denote $\mathcal{E}_{tT}(X) := \mathcal{E}(X)_T/\mathcal{E}(X)_t$ and similarly $X_{tT} := X_T - X_t$.

(i) Let $\vartheta \in L(R)$ be such that $|\vartheta \bullet R| + \langle \vartheta \bullet R\rangle$ is bounded by a constant. Noting that $L(R) \subseteq \mathcal{A}$ because R is continuous, we have from (3.1) that

$$\begin{aligned} L_t(p) &= \operatorname{ess\,inf}_{\pi\in\mathcal{A}} E\big[D_T\mathcal{E}_{tT}^p(\pi \bullet R)\big|\mathcal{F}_t\big] \leq E\big[D_T\mathcal{E}_{tT}^p(|p|^{-1}\vartheta \bullet R)\big|\mathcal{F}_t\big] \\ &= E\big[D_T \exp\big(-(\vartheta \bullet R)_{tT} + \tfrac{1}{2|p|}\langle \vartheta \bullet R\rangle_{tT}\big)\big|\mathcal{F}_t\big]. \end{aligned}$$

The expression under the last conditional expectation is bounded uniformly in p, so the last line converges to $E\big[\exp\big(B - (\vartheta \bullet R)_{tT}\big)\big|\mathcal{F}_t\big]$ P-a.s. when $p \to -\infty$; recall $D_T = \exp(B)$. We have shown

$$\limsup_{p\to-\infty} L_t(p) \leq E\big[\exp\big(B - (\vartheta \bullet R)_{tT}\big)\big|\mathcal{F}_t\big] \quad P\text{-a.s.} \tag{6.7}$$

(ii) Let $\vartheta \in L(R)$ be such that $\exp(-\vartheta \bullet R)$ is of class (D). Defining the stopping times $\tau_n = \inf\{s > 0 : |\vartheta \bullet R_s| + \langle \vartheta \bullet R\rangle_s \geq n\}$, we have

$$\limsup_{p\to-\infty} L_t(p) \leq E\big[\exp\big(B - (\vartheta \bullet R)_{tT}^{\tau_n}\big)\big|\mathcal{F}_t\big] \quad P\text{-a.s.}$$

for each n, by step (i) applied to $\vartheta 1_{(0,\tau_n]}$. Using the class (D) property, the right hand side converges to $E\big[\exp\big(B - (\vartheta \bullet R)_{tT}\big)\big|\mathcal{F}_t\big]$ in $L^1(P)$ as $n \to \infty$, and also P-a.s. along a subsequence. Hence (6.7) again holds.

(iii) The previous step has a trivial extension: Let $g_{tT} \in L^0(\mathcal{F}_T)$ be a random variable such that $g_{tT} \leq (\vartheta \bullet R)_{tT}$ for some ϑ as in (ii). Then

$$\limsup_{p\to-\infty} L_t(p) \leq E\big[\exp(B - g_{tT})\big|\mathcal{F}_t\big] \quad P\text{-a.s.}$$

(iv) Let $\hat{\vartheta} \in \Theta$ be the optimal strategy. We claim that there exists a sequence $g_{tT}^n \in L^0(\mathcal{F}_T)$ of random variables as in (iii) such that

$$\exp\big(B - g_{tT}^n\big) \to \exp\big(B - G_{t,T}(\hat{\vartheta})\big) \text{ in } L^1(P).$$

Indeed, we may assume $B = 0$, as in the previous proof. Then our claim follows by the construction of Schachermayer [69, Theorem 2.2] applied to the time interval $[t, T]$; recall the definitions [69, Eq. (4),(5)]. We conclude that $\limsup_{p\to-\infty} L_t(p) \leq E\big[\exp\big(B - G_{t,T}(\hat{\vartheta})\big)\big|\mathcal{F}_t\big] = L_t^{\exp}$ P-a.s. by the $L^1(P)$-continuity of the conditional expectation. □

Remark 6.9. Recall that $\exp(-G(\hat{\vartheta}))L^{\exp}$ is a martingale, hence of class (D). If $L^{\exp}$ is uniformly bounded away from zero, it follows that $\exp(-G(\hat{\vartheta}))$ is already of class (D) and the last two steps in the previous proof are unnecessary. This situation occurs precisely when the right hand side of (6.6) is bounded uniformly in t. In standard terminology, the latter condition states that the reverse Hölder inequality $\mathrm{R}_{L\log(L)}(P)$ is satisfied by the density process of the minimal entropy martingale measure.

Lemma 6.10. *Let S be continuous and assume that $L(p)$ is continuous for all $p < 0$. Then $L^{\exp}$ is continuous and $L_t(p) \to L_t^{\exp}$ uniformly in t, P-a.s. Moreover, $Z^{L(p)} \to Z^{\exp}$ in $L^2_{loc}(M)$ and $N(p) \to N^{\exp}$ in $\mathcal{H}^2_{loc}$.*

We have already identified the monotone limit $L_t^{\exp} = \lim L_t(p)$. Hence, by uniqueness of the KW decomposition, the above lemma follows from the subsequent one, which we state separately to clarify the argument. The most important input from the control problems is that by stopping, we can bound $L(p)$ away from zero simultaneously for all p (cf. Lemma 6.3).

Lemma 6.11. *Let S be continuous and assume that $L(p)$ is continuous for all $p < 0$. Then $\big(L(p), Z^{L(p)}, N(p)\big)$ converge to a solution $(\widetilde{L}, \widetilde{Z}, \widetilde{N})$ of the BSDE (6.4) as $p \to -\infty$: $\widetilde{L}$ is continuous and $L_t(p) \to \widetilde{L}_t$ uniformly in t, P-a.s.; while $Z^{L(p)} \to \widetilde{Z}$ in $L^2_{loc}(M)$ and $N(p) \to \widetilde{N}$ in $\mathcal{H}^2_{loc}$.*

Proof. For notational simplicity, we write the proof for the one-dimensional case ($d = 1$). We fix a sequence $p_n \downarrow -\infty$ and corresponding $q_n \uparrow 1$. As $p \mapsto L_t(p)$ is monotone and positive, the P-a.s. limit $\widetilde{L}_t := \lim_n L_t(p_n)$ exists.

The sequence $M^{L(p_n)}$ of martingales is bounded in the Hilbert space $\mathcal{H}^2$ by Lemma 5.11(i). Hence it has a subsequence, still denoted by $M^{L(p_n)}$, which converges to some $\widetilde{M} \in \mathcal{H}^2$ in the weak topology of $\mathcal{H}^2$. If we denote the KW decomposition by $\widetilde{M} = \widetilde{Z} \bullet M + \widetilde{N}$, we have by orthogonality that $Z^{L(p_n)} \to \widetilde{Z}$ weakly in $L^2(M)$ and $N^{L(p_n)} \to \widetilde{N}$ weakly in $\mathcal{H}^2$. We shall use the BSDE to deduce the strong convergence.

The drivers in the BSDE (4.6) corresponding to p_n and in (6.4) are

$$f^n(t, l, z) := q_n\, f(t, l, z), \quad f(t, l, z) := \frac{1}{2}\, l\Big(\lambda_t + \frac{z}{l}\Big)^2$$

for $(t, l, z) \in [0, T] \times (0, \infty) \times \mathbb{R}$. For fixed t and any convergent sequence $(l_m, z_m) \to (l, z) \in (0, \infty) \times \mathbb{R}$, we have

$$f^m(t, l_m, z_m) \to f(t, l, z) \;\; P\text{-a.s.}$$

By Lemmata 6.7 and 6.5 we can find a localizing sequence (τ_k) such that

$$1/k < L(p)^{\tau_k} \leq k_2 \quad \text{for all } p < 0,$$

where the upper bound is from Lemma 4.1. For the processes from (2.6) we may assume that $\lambda^{\tau_k} \in L^2(M)$ and $M^{\tau_k} \in \mathcal{H}^2$ for each k.

To relax the notation, let $L^n = L(p_n)$, $Z^n = Z^{L(p_n)}$, $N^n = N^{L(p_n)}$, and $M^n = M^{L(p_n)} = Z^n \bullet M + N^n$. The purpose of the localization is that (f^n) are uniformly quadratic in the relevant domain: As $(L^n, Z^n)^{\tau_k}$ takes values in $[1/k, k_2] \times \mathbb{R}$ and

$$|f^n(t, l, z)| \leq \left| l\lambda_t^2 + \lambda_t z + z^2/l \right| \leq (1 + l)\lambda_t^2 + (1 + 1/l)z^2,$$

we have for all $m, n \in \mathbb{N}$ that

$$\begin{aligned} |f^m(t, L_t^n, Z_t^n)|^{\tau_k} &\leq \xi_t + C_k (Z_{t\wedge\tau_k}^n)^2, \quad \text{where} \\ \xi &:= (1 + k_2)\big(\lambda^{\tau_k}\big)^2 \in L^1_{\tau_k}(M), \quad C_k := 1 + k. \end{aligned} \tag{6.8}$$

Here $L^r_\tau(M) := \{H \in L^2_{loc}(M) : H1_{[0,\tau]} \in L^r(M)\}$ for a stopping time τ and $r \geq 1$. Similarly, we set $\mathcal{H}^2_\tau = \{X \in \mathcal{S} : X^\tau \in \mathcal{H}^2\}$. Now the following can be shown using a technique of Kobylanski [48].

Lemma 6.12. *For fixed k,*

(i) $Z^n \to \widetilde{Z}$ *in* $L^2_{\tau_k}(M)$ *and* $N^n \to \widetilde{N}$ *in* $\mathcal{H}^2_{\tau_k}$*,*

(ii) $\sup_{t\leq T} |L^n_{t\wedge\tau_k} - \widetilde{L}_{t\wedge\tau_k}| \to 0$ *P-a.s.*

The proof is deferred to Appendix V.9. Since (ii) holds for all k, it follows that $\widetilde{L}$ is continuous. Now Dini's lemma shows $\sup_{t\leq T} |L^n_t - \widetilde{L}_t| \to 0$ P-a.s. as claimed. Lemma 6.12 also implies that the limit $(\widetilde{L}, \widetilde{Z}, \widetilde{N})$ satisfies the BSDE (6.4) on $[0, \tau_k]$ for all k, hence on $[0, T]$. The terminal condition is satisfied as $L^n_T = D_T = \exp(B)$ for all n.

To end the proof, note that the convergences hold for the original sequence (p_n), rather than just for a subsequence, since $p \mapsto L(p)$ is monotone and since our choice of (τ_k) does not depend on the subsequence. □

We can now finish the proof of Theorem 6.6 (and Theorem 3.2).

Proof of Theorem 6.6. Part (i) was already proved. For (ii), uniform convergence and continuity were shown in Lemma 6.10. In view of (4.8) and (6.5), the claim for the strategies is that

$$(1 - p)\,\hat{\pi}(p) = \lambda + \frac{Z^{L(p)}}{L(p)} \to \lambda + \frac{Z^{L^{\exp}}}{L^{\exp}} = \hat{\vartheta} \quad \text{in } L^2_{loc}(M).$$

By a localization as in the previous proof, we may assume that $L(p) + (L(p))^{-1} + L^{\exp} + (L^{\exp})^{-1}$ is bounded uniformly in p, and, by Lemma 6.10, that $Z^{L(p)} \bullet M \to Z^{\exp} \bullet M$ in $\mathcal{H}^2$. We have

$$\Big\| \tfrac{Z^{L(p)}}{L(p)} \bullet M - \tfrac{Z^{L^{\exp}}}{L^{\exp}} \bullet M \Big\|_{\mathcal{H}^2} \\ \leq \Big\| \tfrac{1}{L(p)} \big(Z^{L(p)} - Z^{L^{\exp}}\big) \bullet M \Big\|_{\mathcal{H}^2} + \Big\| \Big(\tfrac{1}{L(p)} - \tfrac{1}{L^{\exp}}\Big) Z^{L^{\exp}} \bullet M \Big\|_{\mathcal{H}^2}.$$

Clearly the first norm converges to zero. Noting that $Z^{L^{\exp}} \bullet M \in \mathcal{H}^2$ (even BMO) due to Lemma 5.9, the second norm tends to zero by dominated convergence for stochastic integrals. $\square$

The last result of this section concerns the convergence of the (normalized) solution $\widehat{Y}(p)$ of the dual problem (4.3); see also the comment after Remark 3.3. We recall the assumption (3.3) and that there is no intermediate consumption. To state the result, let $Q^E(B) \in \mathscr{M}$ be the measure which minimizes the relative entropy $H(\,\cdot\,|P(B))$ over $\mathscr{M}$, where $dP(B) := (e^B/E[e^B])\,dP$. For $B = 0$ this is simply the minimal entropy martingale measure, and the existence of $Q^E(B)$ follows from the existence of the latter by a change of measure.

Proposition 6.13. *Let S be continuous and assume that $L(p)$ is continuous for all $p < 0$. Then $\widehat{Y}(p)/\widehat{Y}_0(p)$ converges in the semimartingale topology to the density process of $Q^E(B)$ as $p \to -\infty$.*

Proof. We deduce from Lemma 6.10 that $L^{-1} \bullet N \to (L^{\exp})^{-1} \bullet N^{\exp}$ in $\mathcal{H}^2_{loc}$, as in the previous proof. Since $\widehat{Y}/\widehat{Y}_0 = \mathcal{E}(-\lambda \bullet M + L^{-1} \bullet N)$ by (4.9), Lemma 8.2(ii) shows that $\widehat{Y}/\widehat{Y}_0 \to \mathcal{E}\big(-\lambda \bullet M + (L^{\exp})^{-1} \bullet N^{\exp}\big)$ in the semimartingale topology. The right hand side is the density process of $Q^E(B)$; this follows, e.g., from [23, Proposition 1]. $\square$

V.7 The Limit $p \to 0$

In this section we prove Theorem 3.4, some refinements of that result, as well as the corresponding convergence for the opportunity processes and the dual problem. Due to substantial technical differences, we consider separately the limits $p \to 0$ from below and from above. Recall the semimartingale $\eta_t = E\big[\int_t^T D_s\,\mu^\circ(ds)\big|\mathcal{F}_t\big]$ with canonical decomposition

$$\eta_t = (\eta_0 + M^\eta_t) + A^\eta_t = E\Big[\int_0^T D_s\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big] - \int_0^t D_s\,\mu(ds). \tag{7.1}$$

Clearly η is a supermartingale with continuous finite variation part, and a martingale in the case without intermediate consumption ($\mu = 0$). From (2.4) we have the uniform bounds

$$0 < k_1 \leq \eta \leq (1 + T)k_2. \tag{7.2}$$

V.7.1 The Limit $p \to 0-$

We start with the convergence of the opportunity processes.

Proposition 7.1. *As $p \to 0-$,*

(i) for each $t \in [0,T]$, $L^*_t(p) \to \eta_t$ P-a.s. and in $L^r(P)$ for $r \in [1,\infty)$, with a uniform bound.

(ii) if $\mathbb{F}$ is continuous, then $L^*_t(p) \to \eta_t$ uniformly in t, P-a.s.; and in $\mathcal{R}^r$ for $r \in [1,\infty)$.

(iii) if $u_{p_0}(x_0) < \infty$ for some $p_0 \in (0,1)$, then $L^*_t(p) \to \eta_t$ uniformly in t, P-a.s.; in $\mathcal{R}^r$ for $r \in [1,\infty)$; and prelocally in $\mathcal{R}^\infty$.

The same assertions hold for L^* replaced by L.

Proof. We note that $p \to 0-$ implies $q \to 0+$ and $\beta \to 1-$. In view of $L = (L^*)^{1/\beta}$, it suffices to prove the claims for L^*. From Lemma 4.1,

$$0 \le L^*_t(p) \le \mu^\circ[t,T]^{-\beta p}\, E\Big[\int_t^T D_s\, \mu^\circ(ds)\Big|\mathcal{F}_t\Big]^\beta \to \eta_t \text{ in } \mathcal{R}^\infty. \qquad (7.3)$$

To obtain a lower bound, we consider the density process Y of some $Q \in \mathscr{M}$.

(i) Using (4.4) we obtain

$$L^*_t(p) \ge \int_t^T E\big[D_s^\beta (Y_s/Y_t)^q\big|\mathcal{F}_t\big]\, \mu^\circ(ds).$$

Clearly $D_s^\beta \to D_s$ in $\mathcal{R}^\infty$ and $(Y_s/Y_t)^q \to 1$ P-a.s. for $q \to 0$. We can argue as in Proposition 6.1: For $s \ge t$ fixed, $0 \le (Y_s/Y_t)^q \le 1 + Y_s/Y_t \in L^1(P)$ yields $E\big[D_s^\beta (Y_s/Y_t)^q\big|\mathcal{F}_t\big] \to E[D_s|\mathcal{F}_t]$ P-a.s. Since Y^q is a supermartingale, $0 \le E\big[D_s^\beta (Y_s/Y_t)^q\big|\mathcal{F}_t\big] \le 1 \vee k_2$, and we conclude for each t that

$$\int_t^T E\big[D_s^\beta (Y_s/Y_t)^q\big|\mathcal{F}_t\big]\, \mu^\circ(ds) \to \int_t^T E\big[D_s\big|\mathcal{F}_t\big]\, \mu^\circ(ds) = \eta_t \;\; P\text{-a.s.}$$

Hence $L^*_t(p) \to \eta_t$ P-a.s. and the convergence in $L^r(P)$ follows by the bound (7.3).

(ii) Assume that $\mathbb{F}$ is continuous. Our argument will be similar to Proposition 6.1, but the source of monotonicity is different. Fix $(s,\omega) \in [0,T] \times \Omega$ and consider

$$g_q(t) := E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big]^{\frac{1}{1-q}}(\omega), \quad t \in [0,s].$$

Then $g_q(t)$ is continuous in t and decreasing in q by virtue of Lemma 5.5. Dini's lemma yields $g_q \to 1$ uniformly on $[0,s]$, hence $E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big] \to 1$

uniformly in t. We deduce that $E\big[D_s^\beta (Y_s/Y_t)^q \big| \mathcal{F}_t\big](\omega) \to E[D_s|\mathcal{F}_t](\omega)$ uniformly in t since

$$\begin{aligned}
&\Big| E\big[D_s^\beta (Y_s/Y_t)^q \big| \mathcal{F}_t\big] - E[D_s|\mathcal{F}_t] \Big| \\
&\le E\big[|D_s^\beta - D_s|(Y_s/Y_t)^q \big| \mathcal{F}_t\big] + \Big| E\big[D_s\{(Y_s/Y_t)^q - 1\}\big|\mathcal{F}_t\big]\Big| \\
&\le \|D_s^\beta - D_s\|_{L^\infty(P)} E\big[(Y_s/Y_t)^q \big| \mathcal{F}_t\big] + \|D_s\|_{L^\infty(P)} \Big| E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big] - 1\Big| \\
&\le \|D_s^\beta - D_s\|_{L^\infty(P)} + k_2 \Big| E\big[(Y_s/Y_t)^q\big|\mathcal{F}_t\big] - 1\Big|.
\end{aligned}$$

The rest of the argument is like the end of the proof of Proposition 6.1.

(iii) Let $u_{p_0}(x_0) < \infty$ for some $p_0 \in (0,1)$. Then we can take a different approach via Proposition 5.1, which shows that

$$L_t^*(p) \ \ge\ E\Big[\int_t^T D_s^\beta \, \mu^\circ(ds) \Big| \mathcal{F}_t\Big]^{1-q/q_0} \Big(k_1^{\beta-\beta_0} L_t^*(p_0)\Big)^{q/q_0}$$

for all $p < 0$, where we note that $q_0 < 0$. Using that almost every path of $L^*(p_0)$ is bounded and bounded away from zero (Lemma 4.1), the right hand side P-a.s. tends to $\eta_t = E[\int_t^T D_s \, \mu^\circ(ds)|\mathcal{F}_t]$ uniformly in t as $q \to 0$. Since $L^*(p_0)$ is prelocally bounded, the prelocal convergence in $\mathcal{R}^\infty$ follows in the same way. □

Remark 7.2. One can ask when the convergence in Proposition 7.1 holds even in $\mathcal{R}^\infty$. The following statements remain valid if L^* replaced by L.

(i) Assume again that $u_{p_0}(x_0) < \infty$ for some $p_0 \in (0,1)$, and in addition that $L^*(p_0)$ is (locally) bounded. Then the argument for Proposition 7.1(iii) shows $L^*(p) \to \eta$ in $\mathcal{R}^\infty$ ($\mathcal{R}^\infty_{loc}$).

(ii) Conversely, $L^*(p) \to \eta$ in $\mathcal{R}^\infty$ ($\mathcal{R}^\infty_{loc}$) implies that $L^*(p)$ is (locally) bounded away from zero for all $p < 0$ close to zero, because $\eta \ge k_1 > 0$.

As we turn to the convergence of the martingale part $M^{L(p)}$, a suitable localization will again be crucial.

Lemma 7.3. *Let $p_1 < 0$. There exists a localizing sequence (σ_n) such that*

$$\big(L(p)\big)_-^{\sigma_n} > 1/n \ \text{ simultaneously for all } p \in [p_1, 0).$$

Proof. This follows from Proposition 5.4 and Lemma 4.1. □

Next, we state a basic result (i) for the convergence of $M^{L(p)}$ in $\mathcal{H}^2_{loc}$ and stronger convergences under additional assumptions (ii) and (iii), for which Remark 7.2(i) gives sufficient conditions.

Proposition 7.4. *Assume that S is continuous. As $p \to 0-$,*

(i) $M^{L(p)} \to M^\eta$ in $\mathcal{H}^2_{loc}$.

(ii) if $L(p) \to \eta$ in $\mathcal{R}^\infty_{loc}$, then $M^{L(p)} \to M^\eta$ in BMO_{loc}.

(iii) if $L(p) \to \eta$ in $\mathcal{R}^\infty$, then $M^{L(p)} \to M^\eta$ in BMO.

Proof. Set $X = X(p) = \eta - L(p)$. Then X is bounded uniformly in p by Lemma 4.1 and our aim is to prove $M^{X(p)} \to 0$. Lemma 5.9 applied to $\|\eta\|_\infty - \eta$ shows that $M^\eta \in BMO$. We may restrict our attention to p in some interval $[p_1, 0)$ and Lemma 5.11 shows that $\sup_{p\in[p_1,0)} \|M^{L(p)}\|_{BMO} < \infty$. Due to the orthogonality of the sum $M^L = Z^L \bullet M + N^L$, we have in particular that

$$\sup_{p\in[p_1,0)} \|Z^L(p) \bullet M\|_{BMO} < \infty. \tag{7.4}$$

Under the condition of (iii), $L(p)$ is bounded away from zero for all p close to zero since $\eta \geq k_1 > 0$; moreover, $\lambda \bullet M \in BMO$ by Corollary 5.12. For (i) and (ii) we may assume by a localization as in Lemma 7.3 that $L_-(p)$ is bounded away from zero uniformly in p. Since M is continuous, we may also assume that $\lambda \bullet M \in BMO$, by another localization.

Using the formula (4.7) for A^L and the decomposition (7.1) of η, the finite variation part A^X is continuous and

$$\begin{aligned} 2\,dA^X = {} & 2\Big\{(1-p)D^\beta L^q_- - D\Big\}\,d\mu \\ & - q\Big\{L_-\lambda^\top\, d\langle M\rangle\,\lambda + 2\lambda^\top\, d\langle M\rangle\, Z^L + L_-^{-1}\big(Z^L\big)^\top\, d\langle M\rangle\, Z^L\Big\}. \end{aligned} \tag{7.5}$$

In particular, we note that

$$[M^X] = [X] - X_0^2 = X^2 - X_0^2 - 2\int X_-\,dX. \tag{7.6}$$

For case (i) we have $X_0^2 \to 0$ and $E[X_T^2] \to 0$ by Proposition 7.1 (Remark 6.2 applies). In case (iii) we have $X \to 0$ in $\mathcal{R}^\infty$ by assumption and under (ii) the same holds after a localization. If we denote $o^1_t := E\big[X_T^2 - X_t^2\big|\mathcal{F}_t\big]$, we therefore have that $o^1_0 \to 0$ in case (i) and $o^1 \to 0$ in $\mathcal{R}^\infty$ in cases (ii) and (iii). Denote also $o^2_t := 2E\big[\int_t^T X_-\{(1-p)D^\beta L^q_- - D\}\,d\mu\big|\mathcal{F}_t\big]$. Recalling that $p \to 0-$ implies $q \to 0+$ and $\beta \to 1-$, we have $(1-p)D^\beta L^q_- - D \to 0$ in $\mathcal{R}^\infty$ and since X_- is bounded uniformly in p, it follows that $o^2 \to 0$ in $\mathcal{R}^\infty$. As $M^X \in BMO$ and X_- is bounded, $\int X_-\,dM^X$ is a martingale and (7.6) yields

$$E\big[[M^X]_T - [M^X]_t\big|\mathcal{F}_t\big] = E\big[X_T^2 - X_t^2\big|\mathcal{F}_t\big] - 2E\Big[\int_t^T X_-\,dA^X\Big|\mathcal{F}_t\Big].$$

Using (7.5) and the definitions of o^1 and o^2, we can rewrite this as

$$
\begin{aligned}
&E\big[[M^X]_T - [M^X]_t \big| \mathcal{F}_t\big] - o_t^1 + o_t^2 \\
&= qE\Big[\int_t^T X_- \big\{ L_- \lambda^\top \, d\langle M\rangle \, \lambda + 2\lambda^\top \, d\langle M\rangle \, Z^L + L_-^{-1} \big(Z^L\big)^\top \, d\langle M\rangle \, Z^L \big\} \Big| \mathcal{F}_t \Big].
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and using that X_-, L_-, L_-^{-1} are bounded uniformly in p, it follows that

$$
\begin{aligned}
E\big[[M^X]_T &- [M^X]_t \big| \mathcal{F}_t\big] - o_t^1 + o_t^2 \\
&\le qE\Big[\int_t^T X_-(1 + L_-)\lambda^\top \, d\langle M\rangle \, \lambda \Big| \mathcal{F}_t\Big] \\
&\quad + qE\Big[\int_t^T X_-(1 + L_-^{-1}) \big(Z^L\big)^\top \, d\langle M\rangle \, Z^L \Big| \mathcal{F}_t\Big] \\
&\le qC\big(\|\lambda \bullet M\|_{BMO} + \|Z^{L(p)} \bullet M\|_{BMO}\big),
\end{aligned}
$$

where $C > 0$ is a constant independent of p and t. In view of (7.4), the right hand side is bounded by qC' with a constant $C' > 0$ and we have

$$E\big[[M^X]_T - [M^X]_t \big| \mathcal{F}_t\big] \le qC' + o_t^1 - o_t^2.$$

For (i) we only have to prove the convergence to zero of the left hand side for $t = 0$ and so this ends the proof. For (ii) and (iii) we observe that $[M^X]_t = [M^X]_{t-} + (\Delta M_t^X)^2$ and $|\Delta M^X| = |\Delta X| \le 2\|X\|_{\mathcal{R}^\infty}$ to obtain

$$\sup_{t \le T} E\big[[M^X]_T - [M^X]_{t-} \big| \mathcal{F}_t\big] \ \le \ qC' + \|o^1\|_{\mathcal{R}^\infty} + \|o^2\|_{\mathcal{R}^\infty} + 4\|X\|_{\mathcal{R}^\infty}^2$$

and we have seen that the right hand side tends to 0 as $p \to 0-$. $\square$

V.7.2 The Limit $p \to 0+$

We notice that the limit of $L(p)$ for $p \to 0+$ is meaningless without supposing that $u_{p_0}(x_0) < \infty$ for some $p_0 \in (0, 1)$, so we make this a *standing assumption* for the entire Section V.7.2. We begin with a result on the integrability of the tail of the sequence.

Lemma 7.5. *Let $1 \le r < \infty$. There exists a localizing sequence (σ_n) such that*

$$\operatorname*{ess\,sup}_{t \in [0,T],\, p \in (0, p_0/r]} L_{t \wedge \sigma_n}(p) \quad \text{is in } L^r(P) \text{ for all } n.$$

Proof. Let $p_1 = p_0/r$ and $\sigma_n = \inf\{t > 0 : L_t(p_1) > n\} \wedge T$, then by Corollary 5.2(ii), $\sup_t L_{t \wedge \sigma_n}(p_1) \le n + \Delta L_{\sigma_n}(p_1) \in L^r(P)$. But $L(p) \le CL(p_1)$ by Corollary 5.2(i), so (σ_n) already satisfies the requirement. $\square$

Proposition 7.6. *As $p \to 0+$,*

$$L^*(p) \to \eta,$$

uniformly in t, P-a.s.; in $\mathcal{R}^r_{loc}$ for $r \in [1,\infty)$; and prelocally in $\mathcal{R}^\infty$. Moreover, the convergence takes place in $\mathcal{R}^\infty$ (in $\mathcal{R}^\infty_{loc}$) if and only if $L(p_1)$ is (locally) bounded for some $p_1 \in (0,p_0)$. The same assertions hold for L^* replaced by L.

Proof. We consider only $p \in (0,p_0)$ in this proof and recall that $p \to 0+$ implies $q \to 0-$ and $\beta \to 1-$. Since $L = (L^*)^{1/\beta}$, it suffices to prove the claims for L^*. Using Lemma 4.1,

$$L^*_t(p) \geq \mu^\circ[t,T]^{-\beta p}\, E\Big[\int_t^T D_s\, \mu^\circ(ds)\Big|\mathcal{F}_t\Big]^\beta \to \eta_t \text{ in } \mathcal{R}^\infty. \tag{7.7}$$

Conversely, by Proposition 5.1,

$$L^*_t(p) \;\leq\; E\Big[\int_t^T D^\beta_s\, \mu^\circ(ds)\Big|\mathcal{F}_t\Big]^{1-q/q_0} \Big(k_1^{\beta-\beta_0} L^*_t(p_0)\Big)^{q/q_0}. \tag{7.8}$$

Since almost every path of $L^*(p_0)$ is bounded, the right hand side P-a.s. tends to η_t uniformly in t as $q \to 0-$. By localizing $L^*(p_0)$ to be prelocally bounded, the same argument shows the prelocal convergence in $\mathcal{R}^\infty$.

We have proved that $L^*(p) \to \eta$ uniformly in t, P-a.s. In view of Lemma 7.5, the convergence in $\mathcal{R}^r_{loc}$ follows by dominated convergence.

For the second claim, note that the "if" statement is shown exactly like the prelocal $\mathcal{R}^\infty$ convergence and the converse holds by boundedness of η. Of course, if $L(p_1)$ is (locally) bounded for some $p_1 \in (0,p_0)$, then in fact $L(p)$ has this property for all $p \in (0,p_1]$, by Corollary 5.2(i). □

We turn to the convergence of the martingale part. The major difficulty will be that $L(p)$ may have unbounded jumps; i.e., we have to prove the convergence of quadratic BSDEs whose solutions are not locally bounded.

Proposition 7.7. *Assume that S is continuous. As $p \to 0+$,*

(i) $M^{L(p)} \to M^\eta$ *in* $\mathcal{H}^2_{loc}$.

(ii) if there exists $p_1 \in (0,p_0]$ such that $L(p_1)$ is locally bounded, then $M^{L(p)} \to M^\eta$ in BMO_{loc}.

(iii) if there exists $p_1 \in (0,p_0]$ such that $L(p_1)$ is bounded, then $M^{L(p)} \to M^\eta$ in BMO.

The following terminology will be useful in the proof. We say that real numbers (x_ε) converge to x *linearly* as $\varepsilon \to 0$ if

$$\limsup_{\varepsilon\to 0+} \tfrac{1}{\varepsilon}|x_\varepsilon - x| < \infty.$$

Lemma 7.8. *Let $x_\varepsilon \to x$ linearly and $y_\varepsilon \to y$ linearly. Then*

(i) $\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon| < \infty$ *if* $x = y$,

(ii) $x_\varepsilon y_\varepsilon \to xy$ *linearly,*

(iii) if $x > 0$ *and* φ *is a real function with* $\varphi(0) = 1$ *and differentiable at* 0, *then* $(x_\varepsilon)^{\varphi(\varepsilon)} \to x$ *linearly.*

Proof. (i) This is immediate from the triangle inequality.

(ii) This follows from $|x_\varepsilon y_\varepsilon - xy| \leq |x_\varepsilon||y_\varepsilon - y| + |y||x_\varepsilon - x|$ because convergent sequences are bounded.

(iii) Here we use

$$|(x_\varepsilon)^{\varphi(\varepsilon)} - x| \leq |x_\varepsilon||(x_\varepsilon)^{\varphi(\varepsilon)-1} - 1| + |x_\varepsilon - x|;$$

as $\{x_\varepsilon\}$ is bounded and $x_\varepsilon \to x$ linearly, the question is reduced to the boundedness of $\varepsilon^{-1}|(x_\varepsilon)^{\varphi(\varepsilon)-1} - 1|$. Fix $0 < \delta_1 < x < \delta_2$ and $\varrho(\delta,\varepsilon) := |\delta^{\varphi(\varepsilon)-1} - 1|$. For ε small enough, $x_\varepsilon \in [\delta_1, \delta_2]$ and then

$$\varrho(\delta_1,\varepsilon) \wedge \varrho(\delta_2,\varepsilon) \;\leq\; |(x_\varepsilon)^{\varphi(\varepsilon)-1} - 1| \;\leq\; \varrho(\delta_1,\varepsilon) \vee \varrho(\delta_2,\varepsilon).$$

For $\delta > 0$ we have $\lim_\varepsilon \varepsilon^{-1}|\varrho(\delta,\varepsilon)| = \big|\frac{d}{d\varepsilon}\delta^{\varphi(\varepsilon)}|_{\varepsilon=0}\big| = |\log(\delta)\varphi'(0)| < \infty$. Hence the upper and the lower bound above converge to 0 linearly. □

Proof of Proposition 7.7. We first prove (ii) and (iii), i.e, we assume that $L(p_1)$ is locally bounded (resp. bounded). Recall $L(p) \geq k_1$ from Lemma 4.1. By Corollary 5.2(i) there exists a constant $C > 0$ independent of p such that $L(p) \leq CL(p_1)$ for all $p \in (0, p_1]$. Hence $L(p)$ is bounded uniformly in $p \in (0, p_1]$ in the case (iii) and for (ii) this holds after a localization. Now Lemma 5.11(ii) implies $\sup_{p\in(0,p_1]} \|M^{L(p)}\|_{BMO} < \infty$ and we can proceed exactly as in the proof of items (ii) and (iii) of Proposition 7.4.

(i) This case is more difficult because we have to use prelocal bounds and Lemma 5.11(ii) does not apply. Again, we want to imitate the proof of Proposition 7.4(i), or more precisely, the arguments after (7.6). We note that for the claimed $\mathcal{H}^2_{loc}$-convergence those estimates are required only at $t = 0$ and so the BMO-norms can be replaced by $\mathcal{H}^2$-norms. Inspecting that proof in detail, we see that we can proceed in the same way once we establish:

- There exists a localizing sequence (σ_n) and constants C_n such that for all n,
  (a) $(H1_{[0,\sigma_n]}) \bullet M^{L(p)}$ is a martingale for all H predictable and bounded, and all $p \in (0, p_0)$,
  (b) $\sup_{p\in(0,p_0]} \big(L_-(p) + L_-^{-1}(p)\big) \leq C_n$ on $[0, \sigma_n]$,
  (c) $\limsup_{p\to 0+} \|Z^{L(p)} 1_{[0,\sigma_n]}\|_{L^2(M)} \leq C_n$.

We may assume by localization that $\lambda \bullet M \in \mathcal{H}^2$. We now prove (a)-(c); instead of indicating (σ_n) explicitly, we write "by localization..." as usual.

(a) Fix $p \in (0, p_0)$. By Lemma 4.1 and Lemma 5.2(ii), $L = L(p)$ is a supermartingale of class (D). Hence its Doob-Meyer decomposition $L = L_0 + M^L + A^L$ is such that A^L is decreasing and nonpositive, and M^L is a true martingale. Thus

$$0 \leq E[-A_T^L] = E[L_0 - L_T] < \infty.$$

After localizing as in Lemma 7.5 (with $r = 1$), we have $\sup_t L_t \in L^1(P)$. Hence $\sup_t |M_t^L| \leq \sup_t L_t - A_T^L \in L^1(P)$. Now (a) follows by the BDG inequalities exactly as in the proof of Lemma 5.9.

(b) We have $L_-(p) \geq k_1$ by Lemma 4.1. Conversely, by Corollary 5.2(i), $L_-(p) \leq C L_-(p_0)$ for $p \in (0, p_0]$ with some universal constant $C > 0$, and $L_-(p_0)$ is locally bounded by left-continuity.

(c) We shall use the rate of convergence obtained for $L(p)$ and the information about Z^L contained in A^L via the Bellman BSDE. We may assume by localization that (a) and (b) hold with σ_n replaced by T. Thus it suffices to show that

$$\limsup_{p \to 0+} \bigg\| \sqrt{L_-(p)}\lambda + \frac{Z^{L(p)}}{\sqrt{L_-(p)}} \bigg\|_{L^2(M)} < \infty.$$

Suppressing again p in the notation, (a) and the formula (4.7) for A^L imply

$$\begin{aligned} &E[L_0 - L_T] = E[-A_T^L] \\ &= E\Big[(1-p)\int_0^T D^\beta L_-^q \, d\mu\Big] - \frac{q}{2} E\Big[\int_0^T L_-\Big(\lambda + \frac{Z^L}{L_-}\Big)^\top d\langle M\rangle \Big(\lambda + \frac{Z^L}{L_-}\Big)\Big]. \end{aligned}$$

Recalling that $L_T = D_T$, this yields

$$\begin{aligned} \tfrac{1}{2}\Big\| \sqrt{L_-}\lambda + \frac{Z^L}{\sqrt{L_-}} \Big\|_{L^2(M)} &= \tfrac{1}{2} E\Big[\int_0^T L_-\Big(\lambda + \frac{Z^L}{L_-}\Big)^\top d\langle M\rangle \Big(\lambda + \frac{Z^L}{L_-}\Big)\Big] \\ &= \tfrac{1}{|q|}\bigg(E[L_0 - L_T] - E\Big[(1-p)\int_0^T D^\beta L_-^q \, d\mu\Big]\bigg) \\ &= \tfrac{1}{|q|}\bigg(L_0 - E\Big[D_T + (1-p)\int_0^T D^\beta L_-^q \, d\mu\Big]\bigg) \\ &= \tfrac{1}{|q|}(L_0 - \Gamma_0), \end{aligned}$$

where we have set $\Gamma_0 = \Gamma_0(p) = E[D_T + (1-p)\int_0^T D^\beta L_-^q \, d\mu]$. We know that both L_0 and Γ_0 converge to $\eta_0 = E\big[\int_0^T D_s \, \mu^\circ(ds)\big]$ as $p \to 0+$ (and hence $q \to 0-$). However, we are asking for the stronger result

$$\limsup_{p \to 0+} \tfrac{1}{|q|} |L_0(p) - \Gamma_0(p)| < \infty.$$

By Lemma 7.8(i), it suffices to show that $L_0(p) \to \eta_0$ linearly and $\Gamma_0(p) \to \eta_0$ linearly. Using $L^* = L^\beta$, inequalities (7.7) and (7.8) evaluated at $t = 0$ read

$$\mu^\circ[0,T]^{-p}\eta_0 \leq L_0(p) \leq E\Big[\int_0^T D_s^\beta\,\mu^\circ(ds)\Big]^{1/\beta+p/q_0}\Big(k_1^{1-\beta_0/\beta}L_0(p_0)\Big)^{q/q_0}.$$

Recalling the bound (2.4) for D, items (ii) and (iii) of Lemma 7.8 yield that $L_0(p) \to \eta_0$ linearly. The second claim, that $\Gamma_0(p) \to \eta_0$ linearly, follows from the definitions of $\Gamma_0(p)$ and η_0 using again (2.4) and the uniform bounds for L_- from (b). This ends the proof. □

V.7.3 Proof of Theorem 3.4 and Other Consequences

Lemma 7.9. *Assume that S is continuous and that there exists $p_0 > 0$ such that $u_{p_0}(x_0) < \infty$. As $p \to 0$,*

$$\frac{Z^{L(p)}}{L_-(p)} \to \frac{Z^\eta}{\eta_-} \text{ in } L^2_{loc}(M) \quad \text{and} \quad \frac{1}{L_-(p)} \bullet N(p) \to \frac{1}{\eta_-} \bullet N^\eta \text{ in } \mathcal{H}^2_{loc}. \tag{7.9}$$

For a sequence $p \to 0-$ the convergence $\frac{Z^{L(p)}}{L_-(p)} \to \frac{Z^\eta}{\eta_-}$ in $L^2_{loc}(M)$ holds also without the assumption on p_0.

Proof. By localization we may assume that $L_-(p)$ is bounded away from zero and infinity, uniformly in p (Lemma 7.3 and Lemma 4.1 and the preceding proof); we also recall (7.2). We have

$$\Big|\frac{Z^{L(p)}}{L_-(p)} - \frac{Z^\eta}{\eta_-}\Big| \leq \Big|\frac{1}{L_-(p)}\big(Z^{L(p)} - Z^\eta\big)\Big| + \Big|\big(\eta_- - L_-(p)\big)\frac{Z^\eta}{L_-(p)\eta_-}\Big|.$$

Let $u_{p_0}(x_0) < \infty$. The first part of (7.9) follows from the $L^2_{loc}(M)$ and prelocal $\mathcal{R}^\infty$ convergences obtained in Propositions 7.4, 7.7 and Propositions 7.1, 7.6, respectively. The proof of the second part of (7.9) is analogous.

Now drop the assumption that $u_{p_0}(x_0) < \infty$ and consider a sequence $p_n \to 0-$. Then Proposition 7.1 only yields $L_t(p_n) \to \eta_t$ P-a.s. for each t, rather than the convergence of $L_{t-}(p_n)$ to η_{t-}. Consider the optional set $\Lambda := \bigcap_n\{L_-(p_n) = L(p_n)\} \cap \{\eta = \eta_-\}$. Because $L(p_n)$ and η are càdlàg, $\{t : (\omega,t) \in \Lambda^c\} \subset [0,T]$ is countable P-a.s. and as M is continuous is follows that $\int_0^T 1_{\Lambda^c}\,d\langle M\rangle = 0$ P-a.s. Now dominated convergence for stochastic integrals yields that $\{(\eta_- - L_-(p_n))Z^\eta\} \bullet M = \{(\eta - L(p_n))1_\Lambda Z^\eta\} \bullet M \to 0$ in $\mathcal{H}^2_{loc}$ and the rest is as before. □

Proof of Theorem 3.4 and Remark 3.5. The convergence of the optimal consumption is contained in Propositions 7.1 and 7.6 by the formula (4.2). The convergence of the portfolios follows from Lemma 7.9 in view of (4.8).

For $p \in (0, p_0]$ we have the uniform bound $\hat\kappa(p) \leq (k_2/k_1)^{\beta_0}$ by Lemma 4.1 and (4.2); while for $p \in [p_1, 0)$, $\hat\kappa(p)$ is prelocally uniformly bounded by Lemma 7.3 and (4.2). Hence the convergence of the wealth processes follows from Corollary 8.4(i). □

We complement the convergence in the primal problem by a result for the solution $\widehat{Y}(p)$ of the dual problem (4.3).

Proposition 7.10. *Assume that S is continuous and that there exists $p_0 > 0$ such that $u_{p_0}(x_0) < \infty$ holds. Moreover, assume that there exists $p_1 \in (0, p_0]$ such that $L(p_1)$ is locally bounded. As $p \to 0$,*

$$\widehat{Y}(p) \to \frac{\eta_0}{x_0}\mathcal{E}\Big(-\lambda \bullet M + \frac{1}{\eta_-} \bullet N^\eta \Big) \quad \text{in } \mathcal{H}^r_{loc} \text{ for all } r \in [1, \infty).$$

If η and $L(p)$ are continuous for $p < 0$, the convergence for a limit $p \to 0-$ holds in the semimartingale topology without the assumptions on p_0 and p_1.

Proof. (i) If $L(p_1)$ is locally bounded, then $L(p) \to \eta$ in $\mathcal{R}^\infty_{loc}$ by Remark 7.2 and Proposition 7.6. Moreover, $M^{L(p)} \to M^\eta$ in BMO_{loc} by Propositions 7.4 and 7.7. This implies $N^{L(p)} \to N^\eta$ in BMO_{loc} by orthogonality of the KW decompositions. It follows that

$$-\lambda \bullet M + \frac{1}{L_-(p)} \bullet N^{L(p)} \to -\lambda \bullet M + \frac{1}{\eta_-} \bullet N^\eta \quad \text{in } BMO_{loc}.$$

This implies that the corresponding stochastic exponentials converge in $\mathcal{H}^r_{loc}$ for $r \in [1, \infty)$ (see Theorem 3.4 and Remark 3.7(2) in Protter [63]). In view of the formula (4.9) for $\widehat{Y}(p)$, this ends the proof of the first claim.

(ii) Using Lemma 7.9 and Lemma 8.2(ii), the proof of the second claim is similar. □

Note that in the standard case $D \equiv 1$ the normalized limit in Proposition 7.10 is $\mathcal{E}(-\lambda \bullet M)$, i.e., the "minimal martingale density" (cf. [71]). We conclude by an additional statement concerning the convergence of the wealth processes in Theorem 3.4.

Proposition 7.11. *Let the conditions of Theorem 3.4(ii) hold and assume in addition that there exists $p_1 \in (0, p_0]$ such that $L(p_1)$ is locally bounded. Then the convergence of the wealth processes in Theorem 3.4(ii) takes place in $\mathcal{H}^r_{loc}$ for all $r \in [1, \infty)$.*

Proof. Under the additional assumption, the results of this section yield the convergence of $\hat{\kappa}(p)$ in $\mathcal{R}^\infty_{loc}$ and the convergence of $\hat{\pi}(p) \bullet M$ in BMO_{loc} (and hence in $\mathcal{H}^\omega_{loc}$) by the same formulas as before. Corollary 8.4(ii) yields the claim. □

V.8 Appendix A: Convergence of Stochastic Exponentials

This appendix provides some continuity results for stochastic exponentials of continuous semimartingales in an elementary and self-contained way. They are required for the main results of Section V.3 because our wealth processes are exponentials. We also use a result from the (much deeper) theory of $\mathcal{H}^\omega$-differentials; but this is applied only for refinements of the main results.

Lemma 8.1. *Let $X^n = M^n + A^n$, $n \geq 1$ be continuous semimartingales with continuous canonical decompositions and assume that $\sum_n \|X^n\|_{\mathcal{H}^2} < \infty$. Then M^n, $[M^n]$ and $\int |dA^n|$ are locally bounded uniformly in n.*

Proof. Let $\sigma_k = \inf\{t > 0 : \sup_n |M^n_t| > k\} \wedge T$. We use the notation $M^{n\star}_t = \sup_{s \leq t} |M^n_s|$, then the norms $\|M^{n\star}_T\|_{L^2}$ and $\|M^n\|_{\mathcal{H}^2}$ are equivalent by the BDG inequalities. Now

$$P\Big[\sup_n M^{n\star}_T > k\Big] \leq k^{-2} \sum_n \|M^{n\star}_T\|_{L^2}$$

shows $P[\sigma_k < T] \to 0$. Similarly, $P\big[\sup_n [M^n]_T > k\big] \leq k^{-1} \sum_n \|M^n\|_{\mathcal{H}^2}$ and $P\big[\sup_n \int_0^T |dA^n| > k\big] \leq k^{-2} \sum_n \|A^n\|_{\mathcal{H}^2}$ yield the other claims. □

We sometimes write "in $\mathcal{S}^0$" to indicate convergence in the semimartingale topology.

Lemma 8.2. *Let $X^n = M^n + A^n$, $n \geq 1$ and $X = M + A$ be continuous semimartingales with continuous canonical decompositions.*

(i) $\sum_n \|X^n - X\|_{\mathcal{H}^2} < \infty$ implies $\mathcal{E}(X^n) \to \mathcal{E}(X)$ in $\mathcal{H}^2_{loc}$.
(ii) $X^n \to X$ in $\mathcal{H}^2_{loc}$ implies $\mathcal{E}(X^n) \to \mathcal{E}(X)$ in $\mathcal{S}^0$.
(iii) $X^n \to X$ in $\mathcal{S}^0$ implies $\mathcal{E}(X^n) \to \mathcal{E}(X)$ in $\mathcal{S}^0$.

Proof. (i) By localization we may assume that M and $\int |dA|$ are bounded and, by Lemma 8.1, that $|M^n|$ and $\int |dA^n|$ are bounded by a constant C independent of n. Note that $X^n \to X$ in $\mathcal{H}^2$; we shall show $\mathcal{E}(X^n) \to \mathcal{E}(X)$ in $\mathcal{H}^2$. Since this is a metric space, no loss of generality is entailed by passing to a subsequence. Doing so, we have $M^n \to M$, $[M^n] \to [M]$, and $A^n \to A$ uniformly in time, P-a.s. In view of the uniform bound

$$Y^n := \mathcal{E}(X^n) = \exp\big(X^n - \tfrac{1}{2}[M^n]\big) \leq e^{2C}$$

we conclude that $Y^n \to Y := \mathcal{E}(X) = \exp(X - \frac{1}{2}[M])$ in $\mathcal{R}^2$. By definition of the stochastic exponential we have $Y - Y^n = Y \bullet X - Y^n \bullet X^n$, where

$$\|Y \bullet X - Y^n \bullet X^n\|_{\mathcal{H}^2} \leq \|(Y - Y^n) \bullet X\|_{\mathcal{H}^2} + \|Y^n \bullet (X - X^n)\|_{\mathcal{H}^2}.$$

The first norm tends to zero by dominated convergence for stochastic integrals and for the second we use that $|Y^n| \leq e^{2C}$ and $X^n \to X$ in $\mathcal{H}^2$.

(ii) Consider a subsequence of (X^n). After passing to another subsequence, (i) shows the convergence in $\mathcal{H}^2_{loc}$ and Proposition 2.2 yields (ii).

(iii) This follows from (ii) by using Proposition 2.2 twice. □

We return to the semimartingale R of asset returns, which is assumed to be continuous in the sequel. We recall the structure condition (2.6) and define $L^\omega(M) := \{\pi \in L(M) : \|\pi\|_{L^\omega(M)} < \infty\}$, where $\|\pi\|_{L^\omega(M)} := \|\pi \bullet M\|_{\mathcal{H}^\omega}$ and $\mathcal{H}^\omega$ was introduced at the end of Section V.2.2.

Lemma 8.3. *Let R be continuous, $r \in \{2, \omega\}$, and $\pi, \pi^n \in L^r_{loc}(M)$. Then $\pi^n \to \pi$ in $L^r_{loc}(M)$ if and only if $\pi^n \bullet R \to \pi \bullet R$ in $\mathcal{H}^r_{loc}$.*

Proof. By (2.6) we have $\pi \bullet R = \pi \bullet M + \int \pi^\top d\langle M \rangle \lambda$. Let $\chi := \int \lambda^\top d\langle M \rangle \lambda$ denote the mean-variance tradeoff process. The inequality

$$E\Big[\Big(\int_0^T |\pi^\top d\langle M \rangle \lambda|\Big)^2\Big] \leq E\Big[\Big(\int_0^T \pi^\top d\langle M \rangle \pi\Big)\Big(\int_0^T \lambda^\top d\langle M \rangle \lambda\Big)\Big]$$

implies $\|\pi \bullet M\|_{\mathcal{H}^2} \leq \|\pi \bullet R\|_{\mathcal{H}^2} \leq (1 + \|\chi_T\|_{L^\infty})\|\pi \bullet M\|_{\mathcal{H}^2}$. As χ is locally bounded due to continuity, this yields the result for $r = 2$. The proof for $r = \omega$ is analogous. □

Corollary 8.4. *Let R be continuous and $(\pi, \kappa), (\pi^n, \kappa^n) \in \mathcal{A}$.*

(i) Assume that $\pi^n \to \pi$ in $L^2_{loc}(M)$, that (κ^n) is prelocally bounded uniformly in n, and that $\kappa^n_t \to \kappa_t$ P-a.s. for each $t \in [0, T]$. Then $X(\pi^n, \kappa^n) \to X(\pi, \kappa)$ in the semimartingale topology.

(ii) Assume $\pi^n \to \pi$ in $L^\omega_{loc}(M)$ and $\kappa^n \to \kappa$ in $\mathcal{R}^\infty_{loc}$. Then $X(\pi^n, \kappa^n) \to X(\pi, \kappa)$ in $\mathcal{H}^r_{loc}$ for all $r \in [1, \infty)$.

Proof. (i) By continuity of μ, $\kappa^n_s \bullet \mu(ds)_t = \kappa^n_s \bullet \mu(ds)_{t-}$ for all t. After localization, bounded convergence yields $\int_0^T |\kappa^n_t - \kappa_t| \, \mu(dt) \to 0$ P-a.s. and in $L^2(P)$. Using Lemma 8.3, we have $\pi^n \bullet R + \kappa^n \bullet \mu(dt) \to \pi \bullet R + \kappa \bullet \mu(dt)$ in $\mathcal{H}^2_{loc}$. In view of (2.2) we conclude by Lemma 8.2(ii).

(ii) With Lemma 8.3 we obtain $\pi^n \bullet R + \kappa^n \bullet \mu(dt) \to \pi \bullet R + \kappa \bullet \mu(dt)$ in $\mathcal{H}^\omega_{loc}$. Thus the stochastic exponentials converge in $\mathcal{H}^r_{loc}$ for all $r \in [1, \infty)$ (see Theorem 3.4 and Remark 3.7(2) in [63]). □

V.9 Appendix B: Proof of Lemma 6.12

In this section we give the proof of Lemma 6.12. As mentioned above, the argument is adapted from the Brownian setting of [48, Proposition 2.4].

We use the notation introduced before Lemma 6.12, in particular, recall (6.8). We fix k throughout and let $\tau := \tau_k$. For fixed integers $m \geq n$ we abbreviate $\delta L = L^n - L^m$, moreover, δM, δZ, δN have the analogous meaning. Note that $\delta L \geq 0$ as $m \geq n$. The technique consists in applying Itô's formula to $\Phi(\delta L)$, where, with $K := 6C_k$,

$$\Phi(x) = \frac{1}{8K^2}\big(e^{4Kx} - 4Kx - 1\big).$$

On $\mathbb{R}_+$ this function satisfies

$$\Phi(0) = \Phi'(0) = 0, \quad \Phi \geq 0, \quad \Phi' \geq 0, \quad \tfrac{1}{2}\Phi'' - 2K\Phi' \equiv 1.$$

Moreover, $\Phi'' \geq 0$ and hence $h(x) := \frac{1}{2}\Phi''(x) - K\Phi'(x) = 1 + K\Phi'(x)$ is nonnegative and nondecreasing.

(i) By Itô's formula we have

$$\begin{aligned}\Phi(\delta L_0) = \Phi(\delta L_\tau) - \int_0^\tau \Phi'(\delta L_s)\big[f^n(s, L_s^n, Z_s^n) - f^m(s, L_s^m, Z_s^m)\big]\, d\langle M\rangle_s \\ - \int_0^\tau \tfrac{1}{2}\Phi''(\delta L_s)\, d\,\langle \delta M\rangle_s - \int_0^\tau \Phi'(\delta L_s)\, d\delta M_s.\end{aligned}$$

By elementary inequalities we have for all m and n that

$$|f^n(t, L^n, Z^n) - f^m(t, L^m, Z^m)|^\tau \leq \xi + K\big(|Z^n - Z^m|^2 + |Z^n - \widetilde{Z}|^2 + |\widetilde{Z}|^2\big)^\tau,$$

where the index t was omitted. Hence

$$\begin{aligned}\Phi(\delta L_0) \leq \Phi(\delta L_\tau) + \int_0^\tau \Phi'(\delta L_s)\Big[\xi_s + K\big(|\delta Z_s|^2 + |Z_s^n - \widetilde{Z}_s|^2 + |\widetilde{Z}_s|^2\big)\Big]\, d\langle M\rangle_s \\ - \int_0^\tau \tfrac{1}{2}\Phi''(\delta L_s)\, d\,\langle \delta M\rangle_s - \int_0^\tau \Phi'(\delta L_s)\, d\delta M_s.\end{aligned}$$

The expectation of the stochastic integral vanishes since δL is bounded and $\delta M \in \mathcal{H}^2$. We deduce

$$E\int_0^\tau \big[\tfrac{1}{2}\Phi''(\delta L_s) - K\Phi'(\delta L_s)\big]|\delta Z_s|^2\, d\langle M\rangle_s + E\int_0^\tau \tfrac{1}{2}\Phi''(\delta L_s)\, d\langle \delta N\rangle_s \tag{9.1}$$

$$- E\int_0^\tau K\Phi'(\delta L_s)|Z_s^n - \widetilde{Z}_s|^2\, d\langle M\rangle_s + \Phi(\delta L_0) \tag{9.2}$$

$$\leq \; E\big[\Phi(\delta L_\tau)\big] + E\int_0^\tau \Phi'(\delta L_s)\big[\xi_s + K|\widetilde{Z}_s|^2\big]\, d\langle M\rangle_s. \tag{9.3}$$

We let m tend to infinity, then $\delta L_t = L_t^n - L_t^m$ converges to $L_t^n - \widetilde{L}_t$ P-a.s. for all t and with a uniform bound, so (9.3) converges to

$$E\big[\Phi(L_\tau^n - \widetilde{L}_\tau)\big] + E\int_0^\tau \Phi'(L_s^n - \widetilde{L}_s)\big[\xi_s + K|\widetilde{Z}_s|^2\big]\, d\langle M\rangle_s;$$

while (9.2) converges to

$$-E\int_0^\tau K\Phi'(L_s^n - \widetilde{L}_s)|Z_s^n - \widetilde{Z}_s|^2\, d\langle M\rangle_s + \Phi(L_0^n - \widetilde{L}_0).$$

We turn to (9.1). The continuous function $h(x) = \frac{1}{2}\Phi''(x) - K\Phi'(x)$ occurs in the first integrand. We recall that h is nonnegative and nondecreasing and note that Φ'' has the same properties. Moreover, as L_t^m is monotone decreasing in m,

$$h(\delta L_s) = h(L_s^n - L_s^m) \uparrow h(L_s^n - \widetilde{L}_s); \quad \Phi''(\delta L_s) = \Phi''(L_s^n - L_s^m) \uparrow \Phi''(L_s^n - \widetilde{L}_s)$$

P-a.s. for all s. Hence we have for any fixed $m_0 \le m$ that

$$E\int_0^\tau h(L_s^n - L_s^m)|Z_s^n - Z_s^m|\,d\langle M\rangle_s \ge E\int_0^\tau h(L_s^n - L_s^{m_0})|Z_s^n - Z_s^m|\,d\langle M\rangle_s;$$
$$E\int_0^\tau \Phi''(L_s^n - L_s^m)\,d\langle N^n - N^m\rangle_s \ge E\int_0^\tau \Phi''(L_s^n - L_s^{m_0})\,d\langle N^n - N^m\rangle_s.$$

The right hand sides are convex lower semicontinuous functions of $Z^m \in L^2(M)$ and $N^m \in \mathcal{H}^2$, respectively, hence also weakly lower semicontinuous. We conclude from the weak convergences $Z^m \to \widetilde{Z}$ and $N^m \to \widetilde{N}$ that

$$\liminf_{m\to\infty} E\int_0^\tau h(L_s^n - L_s^m)|Z_s^n - \widetilde{Z}_s^m|\,d\langle M\rangle_s$$
$$\ge E\int_0^\tau h(L_s^n - L_s^{m_0})|Z_s^n - \widetilde{Z}_s|\,d\langle M\rangle_s;$$

$$\liminf_{m\to\infty} E\int_0^\tau \Phi''(L_s^n - L_s^m)\,d\langle N^n - N^m\rangle_s \ge E\int_0^\tau \Phi''(L_s^n - L_s^{m_0})\,d\langle N^n - \widetilde{N}\rangle_s$$

for all m_0. We can now let m_0 tend to infinity, then by monotone convergence the first right hand side tends to $E\int_0^\tau h(L_s^n - \widetilde{L}_s)|Z_s^n - \widetilde{Z}_s|\,d\langle M\rangle_s$ and the second one tends to

$$E\int_0^\tau \Phi''(L_s^n - \widetilde{L}_s)\,d\langle N^n - \widetilde{N}\rangle_s \ge 2E\int_0^\tau d\langle N^n - \widetilde{N}\rangle_s = 2E\big[\langle N^n - \widetilde{N}\rangle_\tau\big],$$

where we have used that $L^n - \widetilde{L} \ge 0$ and $\Phi''(x) = 2e^{4Kx} \ge 2$ for $x \ge 0$. Altogether, we have passed from (9.1)–(9.3) to

$$E\int_0^\tau \Big(\tfrac{1}{2}\Phi'' - 2K\Phi'\Big)(L_s^n - \widetilde{L}_s)\,|Z_s^n - \widetilde{Z}_s|^2\,d\langle M\rangle_s + E\big[\langle N^n - \widetilde{N}\rangle_\tau\big]$$
$$\le E\Phi(L_\tau^n - \widetilde{L}_\tau) - \Phi(L_0^n - \widetilde{L}_0) + E\int_0^\tau \Phi'(L_s^n - \widetilde{L}_s)\big[\xi_s + K|\widetilde{Z}_s|^2\big]\,d\langle M\rangle_s.$$

As $\frac{1}{2}\Phi'' - 2K\Phi' \equiv 1$, the first integral reduces to $E\int_0^\tau |Z_s^n - Z_s|^2\,d\langle M\rangle_s$. If we let n tend to infinity, the right hand side converges to zero by dominated convergence, so that we conclude

$$E\int_0^\tau |Z_s^n - \widetilde{Z}_s|^2\,d\langle M\rangle_s \to 0; \quad E\big[\langle N^n - \widetilde{N}\rangle_\tau\big] \to 0$$

as claimed.

(ii) For all m and n we have

$$|L^n_{t\wedge\tau} - L^m_{t\wedge\tau}| \leq |L^n_\tau - L^m_\tau| + \int_{t\wedge\tau}^{\tau} |f^n(s, L^n_s, Z^n_s) - f^m(s, L^m_s, Z^m_s)|\, d\langle M\rangle_s + \big|(M^n_\tau - M^m_\tau) - (M^n_{t\wedge\tau} - M^m_{t\wedge\tau})\big|. \tag{9.4}$$

The sequence $M^m = Z^m \bullet M + N^m$ is Cauchy in $\mathcal{H}^2_\tau$. We pick a fast subsequence, still denoted by M^m, such that $\|M^m - M^{m+1}\|_{\mathcal{H}^2_\tau} \leq 2^{-m}$. This implies that

$$M^* := \sup_m |M^m| \in \mathcal{H}^2_\tau; \quad Z^* := \sup_m |Z^m| \in L^2_\tau(M)$$

and that Z^m converges $P\otimes\langle M^\tau\rangle$-a.e. to $\widetilde{Z}$. Therefore, $\lim_n f^m(t, L^m_t, Z^m_t) = f(t, \widetilde{L}_t, \widetilde{Z}_t)$ $P\otimes\langle M^\tau\rangle$-a.e. Moreover, $|f^m(t, L^m_t, Z^m_t)^\tau| \leq \xi_t + C|Z^*_t|^2$ and this bound is in $L^1_\tau(M)$. Passing to a subsequence if necessary, we have

$$\lim_{m\to\infty} \int_0^\tau |f^n(s, L^n_s, Z^n_s) - f^m(s, L^m_s, Z^m_s)|\, d\langle M\rangle_s = \int_0^\tau |f^n(s, L^n_s, Z^n_s) - f(s, \widetilde{L}_s, \widetilde{Z}_s)|\, d\langle M\rangle_s \quad P\text{-a.s.}$$

As $M^m \to \widetilde{M}$ in $\mathcal{H}^2_\tau$, we have $E\big[\sup_{t\leq T} |M^m_{t\wedge\tau} - \widetilde{M}_{t\wedge\tau}|\big] \to 0$ and, after picking a subsequence, $\sup_{t\leq T} |M^m_{t\wedge\tau} - \widetilde{M}_{t\wedge\tau}| \to 0$ P-a.s. We can now take $m \to \infty$ in (9.4) to obtain

$$\sup_{t\leq T} |L^n_{t\wedge\tau} - \widetilde{L}_{t\wedge\tau}| \leq |L^n_\tau - \widetilde{L}_\tau| + \int_0^\tau |f^n(s, L^n_s, Z^n_s) - f(s, \widetilde{L}_s, \widetilde{L}_s)|\, d\langle M\rangle_s + \sup_{t\leq T} \big|(M^n_\tau - \widetilde{M}_\tau) - (M^n_{t\wedge\tau} - \widetilde{M}_{t\wedge\tau})\big|.$$

With exactly the same arguments, extracting another subsequence if necessary, the right hand side converges to zero P-a.s. as $n \to \infty$. We have shown that $\lim_n \sup_{t\leq T} |L^n_{t\wedge\tau} - \widetilde{L}_{t\wedge\tau}| = 0$, along a subsequence. But by monotonicity, we obtain the result for the whole sequence. □

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