Contact encounters in the third dimension

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Classical mechanics

The **phase space** $\mathbb{R}^{2n}$ of a system consists of the position and momentum of a particle.

Lagrange: The equations of motion minimize action
$\rightsquigarrow$ n second order differential equations.

Hamilton-Jacobi: The n Euler-Lagrange equations
$\rightsquigarrow$ a Hamiltonian system of $2n$ equations.

Motion is governed by conservation of *energy*, a Hamiltonian H.

- Flow lines of $X_H = -J_0 \nabla H$ are solutions.
- Phase space is (secretly) a symplectic manifold.
- Certain time dependent H give rise to **contact manifolds**.
- Flow lines of the **Reeb vector field** are solutions.

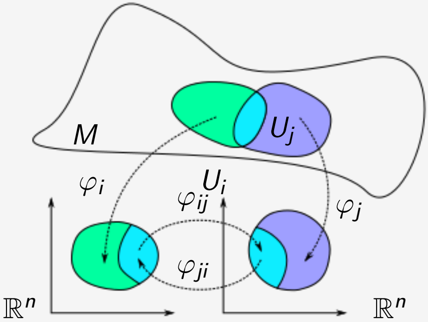
Contact geometry shows up in...

Restricted three body problems, Low energy space travel
Geodesic flow, Liquid crystals ....

Manifolds

Definition

A **smooth** n**-manifold** is a topological space that looks locally like $\mathbb{R}^n$ and admits a global differentiable structure.



A **smooth atlas** on M has

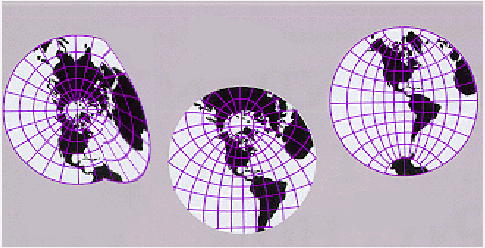
- **Charts** (U_i, φ_i) for which the U_i cover M.
- The $\varphi_i \colon U_i \to \mathbb{R}^n$ are diffeomorphisms onto an open subset of $\mathbb{R}^n$.

The **transition maps** are given by

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

A non-example

Here is a nondifferentiable atlas of charts for the globe.



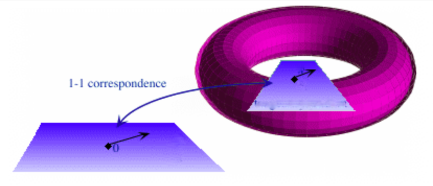
In the middle chart the Tropic of Cancer is a smooth curve, whereas in the first it has a sharp corner.

But an atlas of a differentiable manifold necessitates that the transition maps between charts be smooth.

Tangent spaces and Differential forms

Definition

The *tangent space* of M^n, denoted T_pM, is a vector space "at" a point p of the manifold diffeomorphic to $\mathbb{R}^n$.



Definition

A *1-form* is a linear function: $T_pM \to \mathbb{R}$

Differential forms are a coordinate independent approach to calculus.

They're great for defining integrals over curves, surfaces, and manifolds!

Hyperplane fields

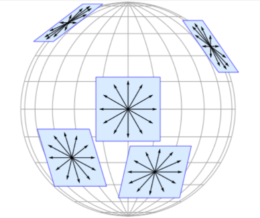
A **2-plane field** ξ on M^3 is the kernel of a 1-form α.

$$\xi = \ker \alpha_p := \{v \in T_pM \mid \alpha(v) = 0\}$$

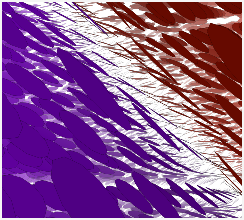
It is a smooth choice of an $\mathbb{R}^2$ subspace in T_pM at each point p.

Definition

ξ is **integrable** if at each point p there is a small open chunk of a surface S in M containing p for which $T_pS = \xi_p$.



Nice and integrable.

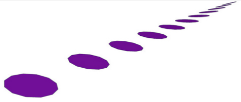


Not so much.

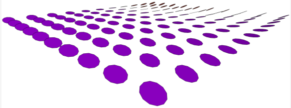
A 2-plane field ξ is a **contact structure** if it is nowhere integrable.

Visualizing a contact structure in $\mathbb{R}^3$.

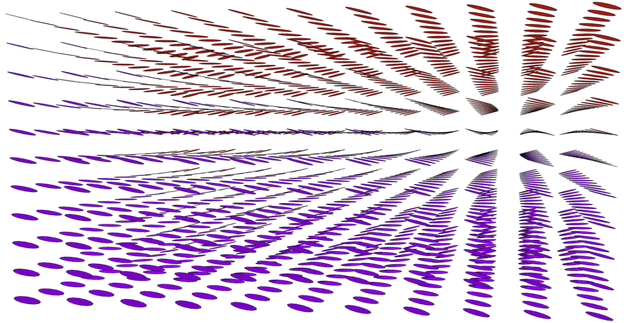
A 2-plane field ξ is a **contact structure** if it is nowhere integrable.

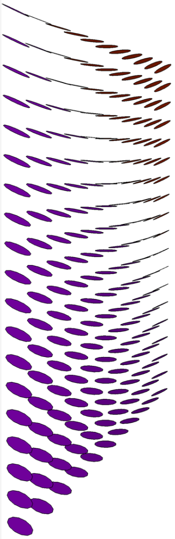


Take a line of planes rotating from $+\infty$ to $-\infty$.



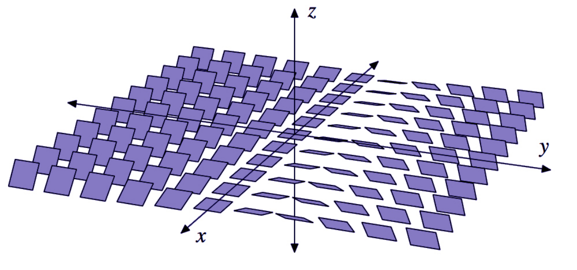
Sweep these lines left-right and up-down.





The kernel of a 1-form α on M^{2n+1} is a contact structure whenever

- $\alpha \wedge (d\alpha)^n$ is a volume form $\Leftrightarrow$ $d\alpha|_\xi$ is nondegenerate.



$\alpha = dz - ydx$

$\xi = \ker \alpha = \text{Span}\left\{ \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\}$

Contact forms

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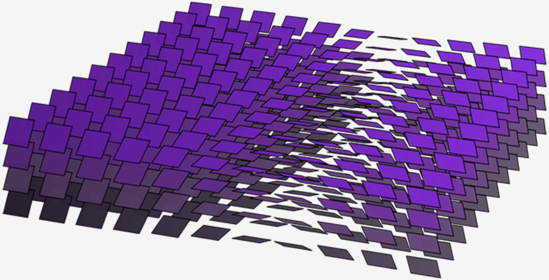
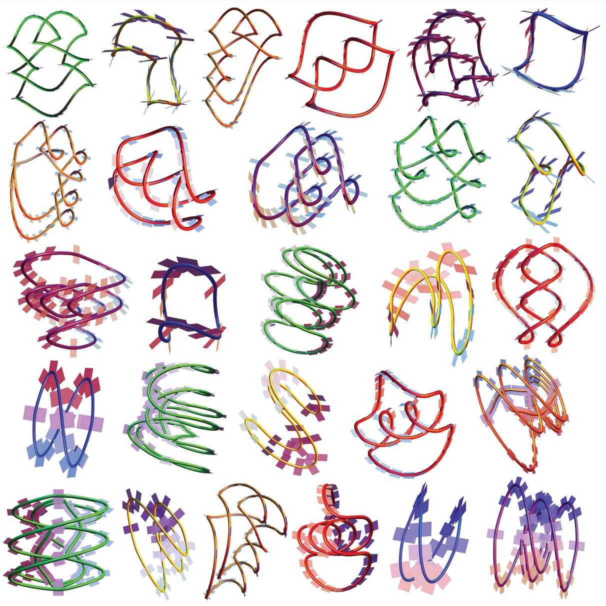


$$\alpha = dz - y dx \qquad \xi = \ker \alpha = \text{Span}\left\{ \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\}$$

$$d\alpha = -dy \wedge dx = dx \wedge dy$$

$$\Rightarrow \quad \alpha \wedge d\alpha \quad = \quad dx \wedge dy \wedge dz$$

Legendrian knots (Lenny Ng)



Localized equality

Theorem (Darboux's theorem)

Let α be a contact form on M^{2n+1} and $p \in M$. Then there are coordinates $(x_1, y_1, ..., x_n, y_n, z)$ on $U_p \subset M$ such that

$$\alpha|_{U_p} = dz - \sum_{i=1}^{n} y_i dx_i.$$

Thus locally all contact structures (and contact forms) look the same!
$\rightsquigarrow$ no local invariants like curvature for us to study.

Gray's Stability Theorem tells us that compact deformations do not produce new contact structures.

Definition

(M, ξ_1) and (N, ξ_2) are **contactomorphic** whenever there exists a diffeomorphism $f : M \to N$ such that $df(\xi_1) = \xi_2$.

Reeb vector fields and flow

Definition

The Reeb vector field R_α on (M, α) is uniquely determined by

- $\alpha(R_\alpha) = 1$,
- $d\alpha(R_\alpha, \cdot) = 0$.

The **Reeb flow**, $\varphi_t : M \to M$ is defined by $\dot{\varphi}_t(x) = R_\alpha(\varphi_t(x))$.

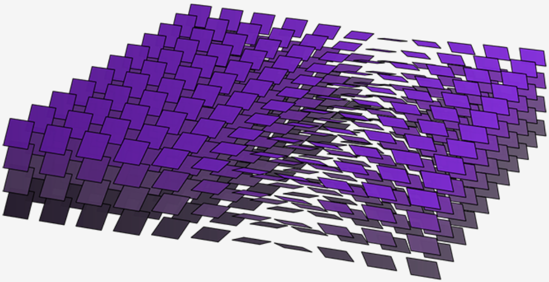
A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \to M, \quad \dot{\gamma}(t) = R_\alpha(\gamma(t)), \tag{0.1}$$

and is **embedded** whenever (0.1) is injective.

The Reeb vector field on $(\mathbb{R}^3, \ker \alpha)$.

R_α satisfies $\alpha(R_\alpha) = 1$, $d\alpha(R_\alpha, \cdot) = 0$.
R_α is never parallel to ξ.



Let $\alpha = dz - ydx$, $\; d\alpha = dx \wedge dy$

$$R_\alpha = \frac{\partial}{\partial z}, \quad \varphi_t(x, y, z) = (x, y, z + t)$$

Reeb orbits on S^3

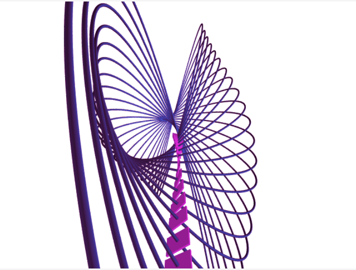
$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \alpha = \frac{i}{2}(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$

The orbits of the Reeb vector field form the Hopf fibration!

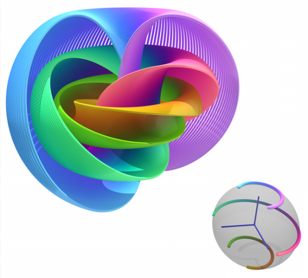
Why?

$$R_\alpha = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.



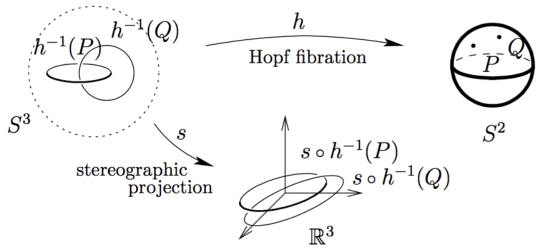
Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$

The Hopf fibration [Heinz Hopf 1931]

- Describes the 3-sphere in terms of S^1 and S^2, an example of an *fiber bundle*.
- The 3-sphere is composed of fibers which are all S^1's, and these fibers get all twisted up, e.g. $S^3 \neq S^1 \times S^2$.
- $h : S^3 \to S^2$ is a many to 1 map such that each distinct point of S^2 comes from a distinct circle of the 3-sphere
- Each circle links with every other circle exactly once.

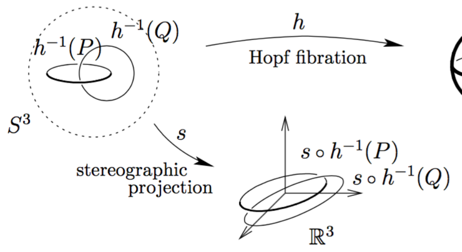


Explicit construction

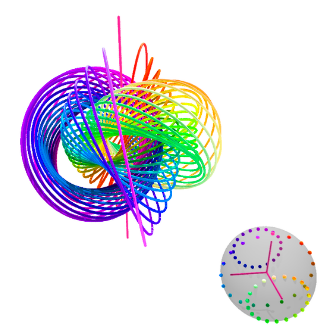
The Hopf fibration is defined by $h(z_0, z_1) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$.
Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$:

$$\begin{aligned} (x_1, x_2, x_3, x_4) \quad &\leftrightarrow (z_0, z_1) &= \quad &(x_1 + ix_2, x_3 + ix_4) \\ (x_1, x_2, x_3) \quad &\leftrightarrow (z, x) &= \quad &(x_1 + ix_2, x_3). \end{aligned}$$

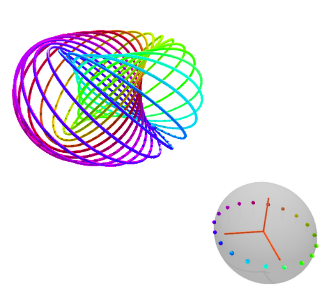
The image of $\{|z_0|^2 + |z_1|^2 = 1\}$ is the unit 2-shere in $\mathbb{C} \times \mathbb{R}$.
If $h(z_0, z_1) = h(w_0, w_1)$, then $(w_0, w_1) = (\lambda z_0, \lambda z_1)$ for $\lambda \in \mathbb{C}$, $|\lambda|^2 = 1$



Going around in even more circles...



- A collection of fibers over a circle in S^2 yields a torus, $T^2 := S^1 \times S^1$
- Each such pair of tori is linked exactly once.
- Fibers over lines of latitude form nested tori.

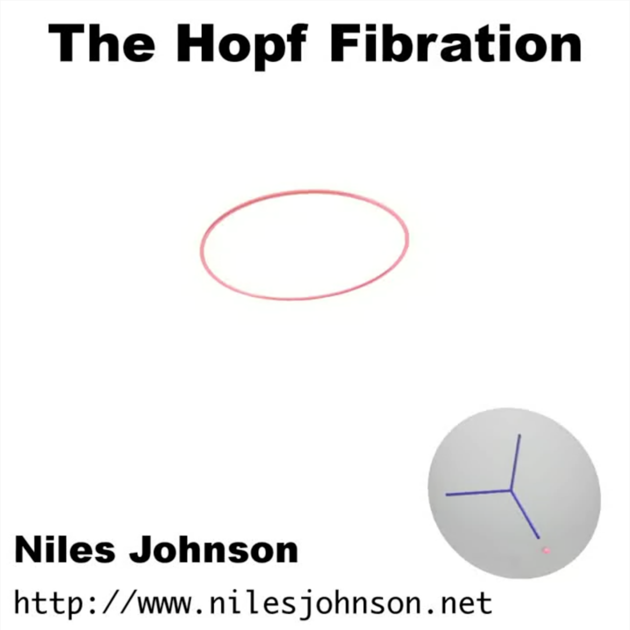


- The three-sphere is a union of two solid tori, joined along their boundary. This boundary is the torus of fibers over the "equator" on S^2.
- One solid torus is formed by the fibers over the Southern hemisphere, and the other by the fibers over the Northern hemisphere.

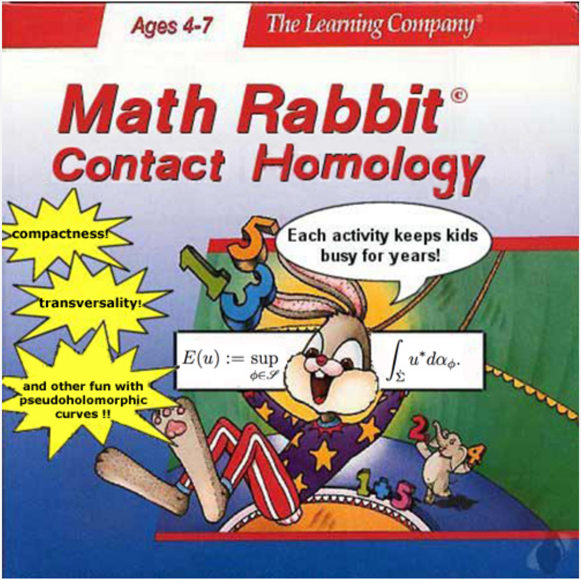
A video!

- Here we represent S^3 as a solid $3D$ ball-esque object, with the understanding that its entire boundary must be collapsed in $4D$ to just 1 point *without collapsing its interior.*
- This is similar to how we can form S^2 by taking a filled in 2-disk, blowing it into a bubble and pinching ∂D^2 to a point.
- This is the inverse procedure to a stereographic projection.

A video of the Hopf fibration



Thanks!


Ages 4-7
The Learning Company
Math Rabbit ©
Contact Homology
compactness!
transversality!
and other fun with pseudoholomorphic curves !!
Each activity keeps kids busy for years!
$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\Sigma} u^* d\alpha_\phi.$