Contact Invariants and Reeb Dynamics

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Classical mechanics

The **phase space** $\mathbb{R}^{2n}$ of a system consists of the position and momentum of a particle.

Lagrange: The equations of motion minimize action
$\rightsquigarrow$ n second order differential equations.

Hamilton-Jacobi: The n Euler-Lagrange equations
$\rightsquigarrow$ a Hamiltonian system of $2n$ equations.

Motion is governed by conservation of *energy*, a Hamiltonian H.

- Flow lines of $X_H = -J_0 \nabla H$ are solutions.
- Phase space is (secretly) a symplectic manifold.
- Certain time dependent H give rise to **contact manifolds**.
- Flow lines of the **Reeb vector field** are solutions.

Contact geometry shows up in...

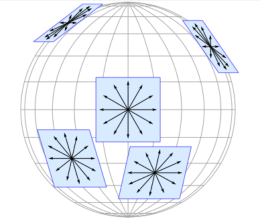
Restricted three body problems, Low energy space travel
Geodesic flow, Liquid crystals ....

Hyperplane fields

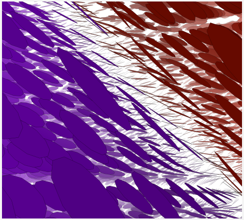
A **hyperplane field** ξ on M^n is the kernel of a 1-form α.
It is a smooth choice of an $\mathbb{R}^{n-1}$ subspace in T_pM at each point p.

Definition

ξ is **integrable** if locally there is a submanifold S with $T_pS = \xi_p$.



Nice and integrable.



Not so much.

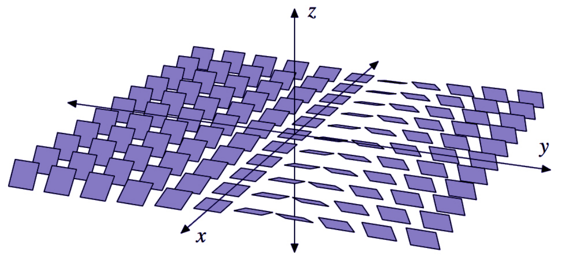
Definition

A **contact structure** is a maximally nonintegrable hyperplane field.

Contact forms

The kernel of a 1-form α on M^{2n+1} is a contact structure whenever

- $\alpha \wedge (d\alpha)^n$ is a volume form $\Leftrightarrow$ $d\alpha|_\xi$ is nondegenerate.



$\alpha = dz - ydx$

$\xi = \ker \alpha = \text{Span}\left\{ \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\}$

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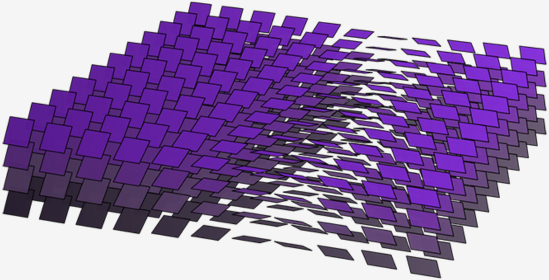


$$\alpha = dz - ydx \qquad \xi = \ker \alpha = \text{Span}\left\{ \frac{\partial}{\partial y}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \right\}$$

$$d\alpha = -dy \wedge dx = dx \wedge dy$$

$$\Rightarrow \quad \alpha \wedge d\alpha \quad = \quad dx \wedge dy \wedge dz$$

Localized equality

Theorem (Darboux's theorem)

Let α be a contact form on M^{2n+1} and $p \in M$. Then there are coordinates $(x_1, y_1, ..., x_n, y_n, z)$ on $U_p \subset M$ such that

$$\alpha|_{U_p} = dz - \sum_{i=1}^{n} y_i dx_i.$$

Thus locally all contact structures (and contact forms) look the same!

$\rightsquigarrow$ no local invariants like curvature for us to study.

Moreover, Gray's Stability Theorem tells us that compact deformations do not produce new contact structures.

Reeb vector fields and flow

Definition

The Reeb vector field R_α on (M, α) is uniquely determined by

- $\alpha(R_\alpha) = 1$,
- $d\alpha(R_\alpha, \cdot) = 0$.

The **Reeb flow**, $\varphi_t : M \to M$ is defined by $\dot{\varphi}_t(x) = R_\alpha(\varphi_t(x))$.

A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \to M, \quad \dot{\gamma}(t) = R_\alpha(\gamma(t)), \tag{0.1}$$

and is **embedded** whenever (0.1) is injective.

The linearized flow along γ defines a symplectic linear map of $(\xi, d\alpha)$. If 1 is not an eigenvalue of the linearized return map then γ is **nondegenerate.**

Reeb orbits on S^3

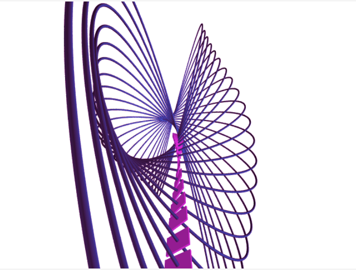
$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \alpha = \frac{i}{2}(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$

The orbits of the Reeb vector field form the Hopf fibration!

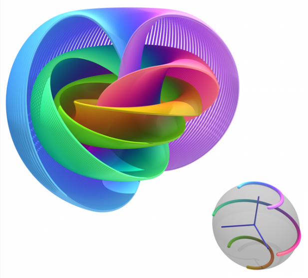
Why?

$$R_\alpha = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.

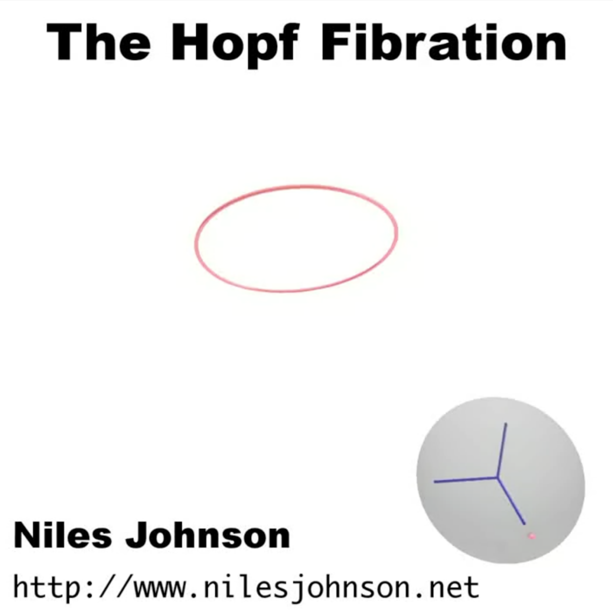


Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$

A video of the Hopf fibration



Existence of periodic orbits

The Weinstein Conjecture (1978)

Let M be a closed oriented odd-dimensional manifold with a contact form α. Then the associated Reeb vector field R_α has a closed orbit.

- Weinstein (convex hypersurfaces)
- Rabinowitz (star shaped hypersurfaces)
- Star shaped is secretly contact!
- Viterbo, Hofer, Floer, Zehnder (‘80’s fun)
- Hofer (S^3)
- Taubes (dimension 3)

Tools $>$ 1985: **Floer Theory** and **Gromov's** pseudoholomorphic curves.

A new era of contact topology

Helmut Hofer on turning 60:

> *Why did I come into symplectic and contact geometry? I had the flu, and the only thing to read was a copy of Rabinowitz's paper where he proves the existence of periodic orbits on star-shaped energy surfaces. It turned out to contain a fundamental new idea, which was to study a different action functional for loops in the phase space rather than for Lagrangians in the configuration space. Which actually if we look back, led to the variational approach in symplectic and contact topology, which is reincarnated in infinite dimensions in Floer theory and has appeared in every other subsequent approach. The flu turned out to be really good.*

Morse theory

Let $f : M \to \mathbb{R}$ be a smooth "nondegenerate" function.
Let g be a "reasonable" metric.
Then the pair (f, g) is Morse-Smale.

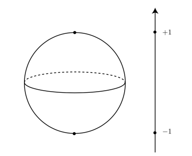
Ingredients:
$C_* = \mathbb{Z}\langle \text{Crit}(f) \rangle$.
$* = \#\{\text{negative eigenvalues Hess}(f)\}$
∂ counts flow lines of $-\nabla f$ betwen critical points

Theorem

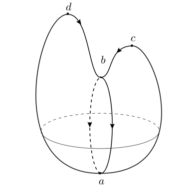
Morse $H_*(M, (f, g)) \cong$ *Singular* $H_*(M)$.
Floer $H_*(M, \omega, H) \cong$ *Morse* $H_*(M, (H, \omega(\cdot, J\cdot)))$

Necessities:
Transversality (so the implicit function theorem holds)
Compactness (so $\partial^2 = 0$ and invariance holds)

More thoughts on spheres



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 & * = 0,\ 2 \\ 0 & \text{else} \end{cases} \qquad \partial = 0$$



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & * = 2 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z}_2 & * = 0 \end{cases} \qquad \begin{aligned} \partial c &= \partial d = b \\ \partial b &= 2a = 0 \end{aligned}$$

Theorem (Reeb)

If there exists a Morse function on M with only two critical points then M is homeomorphic to a sphere.

Theorem (Hutchings-Taubes 2008)

A closed contact 3-manifold admits ≥ 2 embedded Reeb orbits and if there are exactly two then M is diffeomorphic to S^3 or a lens space.

A new hope for a chain complex

Let $(M, \xi = \ker\alpha)$ be a closed nondegenerate contact manifold.

Floerify Morse theory on

$$\begin{aligned} \mathcal{A}: \quad C^\infty(S^1, M) &\to \mathbb{R}, \\ \gamma &\mapsto \int_\gamma \alpha. \end{aligned}$$

Proposition

$\gamma \in \text{Crit}(\mathcal{A}) \Leftrightarrow \gamma$ *is a closed Reeb orbit.*

- Grading on orbits given by Conley-Zehnder index,
- $C_*(M,\alpha) = \mathbb{Q}\langle\{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}\rangle$

The letter J is for pseudoholomorphic

$(\xi, d\alpha)$ symplectic vector bundle $\rightsquigarrow \overline{J}$ almost complex structure

Define J on $T(\mathbb{R} \times M) = \mathbb{R} \oplus \mathbb{R}\langle R_\alpha \rangle \oplus \xi$

$$J|_\xi = \overline{J}$$
$$J\frac{\partial}{\partial \tau} = R_\alpha$$

Gradient flow lines are a no go; instead count **pseudoholomorphic cylinders** $u \in \mathcal{M}_J(\gamma_+; \gamma_-)$.

$$u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)$$
$$\bar{\partial}_{j,J}\ u := du + J \circ du \circ j \equiv 0$$

Note: J is S^1-INDEPENDENT

$$\lim_{s \to \pm\infty} \pi_{\mathbb{R}}\ u(s,t) = \pm\infty$$
$$\lim_{s \to \pm\infty} \pi_M\ u(s,t) = \gamma_\pm$$

up to reparametrization.

Cylindrical contact homology

- $\partial : C_* \to C_{*-1}$ is a weighted count of cylinders.
- Hope this is finite count.
- Hope the resulting homology is independent of our choices.

Conjecture (Eliashberg-Givental-Hofer '00)

Under minimal assumptions (C_*, ∂) *is a chain complex and* $CH_*^{EGH}(M, \alpha, J; \mathbb{Q}) = H(C_*, \partial)$ *is an invariant of* $\xi = \ker \alpha$.

Theorem (Hutchings-N. 2014)

Under essentially the same minimal assumptions in dimension three, ∂ *is well-defined and* $\partial^2 = 0$.

Invariance required the construction of two "new" contact homology theories.

Cylindrical contact homology

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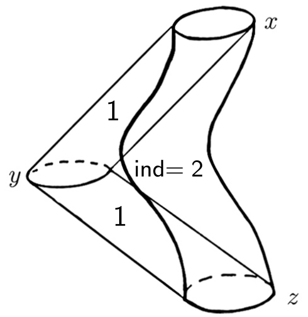
which we can actually compute :

Theorem (N. 2017)

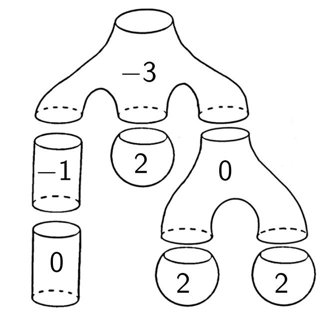
The chain complex of nontrivial S^1 bundles over a closed surface Σ_g is generated by infinitely many copies of the Morse complex of Σ_g and on each copy ∂ agrees with the Morse differential.

The pseudoholomorphic menace

- Transversality for multiply covered curves is hard.
- Is $\mathcal{M}_J(\gamma_+;\gamma_-)$ more than a set?
- $\mathcal{M}_J(\gamma_+;\gamma_-)$ can have **nonpositive** virtual dimension...
- Compactness issues are severe



Desired compactification when $CZ(x) - CZ(z) = 2$.



Adding to 2 becomes hard

Technical considerations in obtaining invariance

- S^1-independent J work in $\mathbb{R} \times M$
- But not in cobordisms, so no chain maps.
- Invariance of $CH_*^{EGH}(M, \alpha, J)$ requires S^1-dependent J.

- Breaking S^1 symmetry invalidates $\partial^2 = 0$.

- We define a "new" Morse-Bott non-equivariant chain complex.
- Compactness issues require obstruction bundle gluing, producing a correction term.

- The nonequivariant theory NCH_* is a well-defined contact invariant but what about CH_*^{EGH}??

Full circle

- We S^1-equivariantize the nonequivariant theory NCH_* algebraically, yielding an integral lift of contact homology,

$$CH_*^{S^1} = H_*(\mathbb{Z}\langle\check{\gamma}, \hat{\gamma}\rangle \otimes \mathbb{Z}[[u]], \partial^{S^1}), \ \deg(u) = 2.$$

- $CH_*^{S^1}$ rescues the bad orbits, which contribute torsion
- Expect isomorphisms with flavors of symplectic homology

Theorem (Hutchings-N '19; via formalism of Hutchings-N '17)

If $(M^{2n+1}, \ker\alpha)$ admits no contractible Reeb orbits or is a dynamically convex 3-manifold, NCH_* and $CH_*^{S^1}$ are defined with coefficients in $\mathbb{Z}$ and are contact invariants. In dimension 3,

$$CH_*^{S^1}(M, \alpha, J) \otimes \mathbb{Q} \cong CH_*^{EGH}(M, \alpha, J),$$

thus CH_*^{EGH} is also a contact invariant.

Thanks!