MATH 354: PROBLEM SET 1

RICE UNIVERSITY, SPRING 2024

Due date: Tuesday, January 16th, by 11pm in Gradescope.

Read the handout `Induction.pdf`, located in the Handouts folder under the Files tab in Canvas. It slightly expands upon the content of our first lecture. It may take a few readings to digest the contents of this handout, and I welcome any questions you might have about it. Please pose these as a discussion on Canvas.

Problem 1. Use induction to show that, for all natural numbers n,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problems 2-3. Consider the set $\mathbb{F}_3 = \{0, 1, 2\}$. Let us define an addition and a multiplication on $\mathbb{F}_3$ by following the rules in the following tables:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

You may assume, without verification, that these operations turn $\mathbb{F}_3$ into a field with additive identity 0 and multiplicative identity 1.

(2) Fill out the additive and multiplicative inverse tables below:

x	0	1	2
$-x$			

x	0	1	2
$1/x$			

Briefly explain how you filled out the tables.

(3) Compute the value of the following expression in $\mathbb{F}_3$:

$$\frac{2^3 + 1}{2}.$$

Briefly explain how you carried out the computation.

Problems 4-6. The set $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ of complex numbers, together with its usual operations of addition and multiplication (see Definition 1.1 in §1.A), is a field. This is verified by the combination of Example 1.2 in §1.A and Exercises 1.A.1, 1.A.4, 1.A.5, 1.A.6, 1.A.7, 1.A.8, 1.A.9. You are welcome to try to do the proofs. They are tedious, and we will do similar exercises in other contexts, so let's just skip these and assume them to be true.

By definition, every $z \in \mathbb{C}$ can be expressed as $a + bi$ with a, $b \in \mathbb{R}$; we refer to this as the **standard form** of the complex number z. For example, $-7 + 22i$ is the standard form of $(2 + 3i)(4 + 5i)$; see Example 1.2 in §1.A. Now, work these out:

(4) Let $z \in \mathbb{C}$ have standard form $z = a + bi$, where a, $b \in \mathbb{R}$. Carry out the computation $z^3 = (a + ib)^3$ to find the standard form of z^3. Show your work.

(5) Consider $w = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \in \mathbb{C}$. Compute w^3 and show it equals 1. Show your work.

(6) Consider the same w as in (5). Write its additive inverse and its multiplicative inverse in standard form. Confirm that these inverses are correct by explicitly verifying that $w + (-w) = 0$ and $w \cdot 1/w = 1$ for them. Show your work for this verification.

Problem 7. Exercise 1.A.10.

Problem 8. Exercise 1.A.14.

Problem 9. Exercise 1.B.1.

Problem 10. Exercise 1.B.2.

Comments and Hints

- For Problem 7, the book only says "Find." Don't just write down an answer, though; verify that your answer is correct. We will verify everything in this course unless explicitly told otherwise.

- For Problem 8, take a look at 1.14 in Axler's book (page 7). The proof there serves as a model for the kind of proof that is expected.

- For Problem 9, be careful to articulate your reasoning correctly. Do you understand the meaning of what you are trying to prove?

- For Problem 10, proceed by considering two possible cases: (i) $a = 0$, or (ii) $a \neq 0$. In one of the two cases (which?), the desired conclusion ("$a = 0$ or $\mathbf{v} = \mathbf{0}$") is essentially immediate. In the other case, you need to perform an algebraic step in order to reach the desired conclusion.