Symplectic Embeddings on the Virtual BeECH

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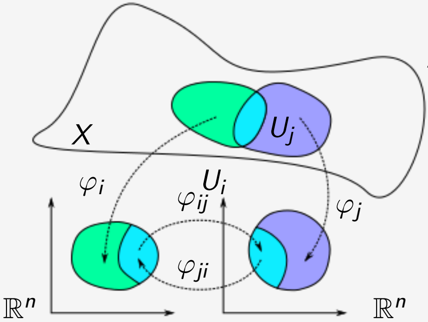
Definition

A **smooth n-manifold** X is a topological space that looks locally like $\mathbb{R}^n$ and admits a global differentiable structure.



A **smooth atlas** on X has

- **Charts** (U_i, φ_i) for which the U_i cover X.
- The $\varphi_i \colon U_i \to \mathbb{R}^n$ are diffeomorphisms onto an open subset of $\mathbb{R}^n$.

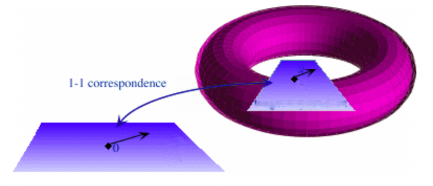
The **transition maps** are given by

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

We need 4 dimensions in order to be embedded!

Definition

The *tangent space* of X^n, denoted T_pX, is a vector space "at" a point p of the manifold diffeomorphic to $\mathbb{R}^n$.



Definition

A *1-form* is a linear function: $T_pX \to \mathbb{R}$.

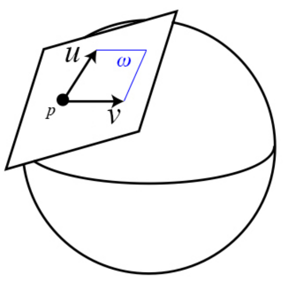
Directional derivatives $D_v f(p) = \nabla f(p) \cdot \frac{v}{|v|}$ and *fds* from $\oint_C fds$.

Differential forms are a coordinate independent approach to calculus. Great for defining integrals over curves, surfaces, and manifolds!

What is a symplectic manifold?

Definition

A 2-form ω on a manifold X is a smooth choice of anti-symmetric bilinear functions $\omega_p : T_pX \times T_pX \to \mathbb{R}$ for each $p \in X$.



At the infinitesimal level, ω measures oriented area spanned by vectors u and v at a point p.

Definition

A symplectic manifold is a pair (X^{2n}, ω) such that ω is a smooth 2-form satisfying

- Closedness: $d\omega = 0$
- Nondegeneracy: ω^n is nonvanishing, i.e. a volume form.

Examples

- $dx \wedge dy$ on $\mathbb{R}^2$
- $\sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$

Given $\mathbb{C}^n$ with $\omega_0 = \frac{\sqrt{-1}}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j$,

consider the symplectic manifolds with boundary

- Ball: $B^{2n}(r) := \{z \in \mathbb{C}^n \mid \pi|z_1^2| + \ldots + \pi|z_n^2| \leq r\}$
- Cylinder: $Z^{2n}(R) := B^2(R) \times \mathbb{C}^{n-1}$
- Ellipsoid: $E(a,b) := \left\{z \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1^2|}{a} + \frac{\pi|z_2^2|}{b} \leq 1\right\}$
- Polydisc: $P(a,b) := \{z \in \mathbb{C}^2 \mid \pi|z_1^2| \leq a,\ \pi|z_2^2| \leq b\}$

A **diffeomorphism** is a smooth bijective map with smooth inverse.

Definition

Two symplectic manifolds (X, ω) and (X', ω') are **symplectomorphic** if there exists a diffeomorphism $f : (X, \omega) \to (X', \omega')$ s.t. $f^*\omega' = \omega$.

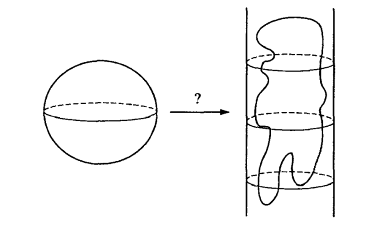
Here $f^*\omega'(\cdot, \cdot) = \omega'(df\cdot, df\cdot)$

Definition

A manifold (X, ω) is said to **symplectically embed** into (X', ω') if there exists an injective smooth map $f : X \overset{s}{\hookrightarrow} X'$ s.t. f is a symplectomorphism onto its image.

Symplectomorphisms are volume-preserving. Are all volume preserving maps are symplectomorphisms?

Are symplectic embeddings restricted by more than volume?



Theorem (Gromov Nonsqueezing, 1985)

$B^{2n}(r)$ symplectically embeds into $Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$ if and only if $r \leq R$

Definition

The **Gromov width** of (X,ω) of dimension $2n$ is the supremum over real numbers r such that $B^{2n}(r)$ embeds symplectically into X.

Symplectic capacity $\rightsquigarrow$ obstructions of symplectic embeddings: If $c(X,\omega_1) > c(X',\omega')$ then $\nexists\ (X,\omega) \overset{s}{\hookrightarrow} (X',\omega')$.

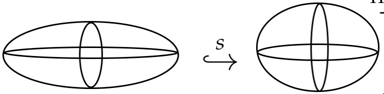
Definition

A **symplectic capacity**, $c : \left\{ \begin{array}{c} \text{symplectic} \\ \text{manifolds} \end{array} \right\} \to \mathbb{R}_{\geq 0}$, satisfies:

- Monotonicity: If $(X,\omega) \overset{s}{\hookrightarrow} (X',\omega')$ then $c(X) \leq c(X')$.
- Conformality/Scaling: for $a \in \mathbb{R} \setminus 0$, $c(X, a\omega) = |a| c(X,\omega)$
- Weak Normalization: $0 < c(B^{2n}(1)) \leq c(Z^{2n}(1)) < \infty$

$$E(a,b) := \left\{ \frac{\pi|z_1^2|}{a} + \frac{\pi|z_2^2|}{b} \leq 1 \right\}$$

Theorem (McDuff, 2011)

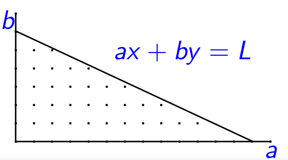


$$\overset{s}{\hookrightarrow} \quad \underset{\text{McDuff}}{\overset{\text{Hutchings}}{\Longleftrightarrow}} N_k(a,b) \leq N_k(c,d) \quad \forall k$$

$N_k(a,b)$ is k^{th} smallest entry in $(am+bn)_{m,n\in\mathbb{Z}_{\geq 0}}$ with repetitions.

$$N(1,4) = 0\ 1\ 2\ 3\ 4\ 4\ 5\ 5\ 5\ 5$$
$$N(2,2) = 0\ 2\ 2\ 4\ 4\ 4\ 6\ 6\ 6\ 6$$

Thus $E(1,4) \overset{s}{\hookrightarrow} E(2,2) = B(2)$!



- $N_k(a,b)$ are the **ECH capacities** of $E(a,b)$.
- $c_k(E(a,b))$ is the smallest L such that $k+1$ lattice points are contained in the region of $\mathbb{R}^2$ bounded by $ax+by=L$ and the x- and y-axes.

Definition

Given a symplectic 4-manifold (X,ω), its **ECH capacities** are a sequence of real numbers

$$0 = c_0(X,\omega) < c_1(X,\omega) \leq c_2(X,\omega) \leq ... \leq \infty$$

such that

$$(X,\omega) \overset{s}{\hookrightarrow} (X',\omega') \Rightarrow c_k(X,\omega) \leq c_k(X',\omega') \, \forall \, k.$$

Some properties of ECH capacities:

- ECH capacities *obstruct* low-dimensional symplectic embeddings:

$$(X,\omega) \overset{s}{\not\hookrightarrow} (X',\omega') \Leftarrow \exists\, k\; c_k(X,\omega) > c_k(X',\omega')$$

- $c_1(B^4(r)) = r$ and $c_1(Z^4(R)) = R \Rightarrow$ 4D Gromov nonsqueezing.
- $\lim_{k\to\infty} \frac{c_k(X,\omega)^2}{k} = 4\int_X \omega\wedge\omega$, relating ECH capacities to volume.
- $c_k(X,\omega)$ measures the ω-area of a surface in X solving a "J-holomorphic curve" PDE with fixed boundary on ∂X; we have

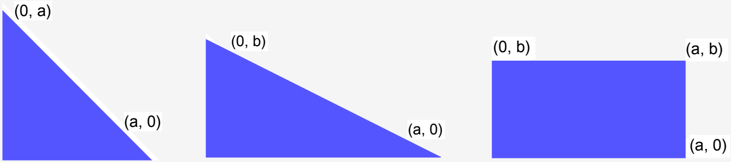
$$(X,\omega) \overset{s}{\hookrightarrow} (X',\omega') \Rightarrow c_k(X,\omega) \le c_k(X',\omega')$$

  because properties of ECH force the surface in X' with fixed boundary to agree in X with the surface for X.
  - The same reasoning implies the "Action Inequality" later in the talk.

Symplectic toric domains are defined by a polytope $\Omega \subset \mathbb{R}^2_{\geq 0}$

$$X_\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\}.$$

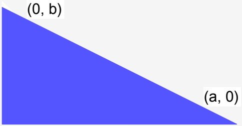
- $B^4(a) := \{\pi|z_1|^2 + \pi|z_2|^2 \leq a\}$
  isosceles right triangle with side length a.
- $E(a, b) := \{\frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\}$
  right triangle with lengths a, b.
- $P(a, b) := \{\pi|z_1|^2 \leq a, \pi|z_2|^2 \leq b\}$
  rectangle of sides a, b.



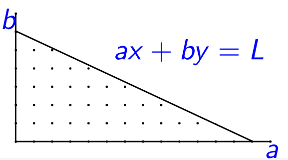
(a) *Ball* $B(a)$ (b) *Ellipsoid* $E(a, b)$ (c) *Polydisk* $P(a, b)$

The combinatorics of the polytopes Ω tell us about embeddings of their toric domains X_Ω!

- The area of Ω equals the volume $\int_{X_\Omega} \omega_0 \wedge \omega_0$ of X_Ω.
- In some cases we can compute $c_k(X_\Omega)$ from the geometry of Ω. When $X_\Omega = E(a,b)$, we have



(d) *Ellipsoid* $E(a,b)$



(e) $c_k(E(a,b))$ $P(a,b)$

  And we can also compute $c_k(X_\Omega)$ from Ω in more complex ways for more general Ω.
- Soon you'll see even more ways to obstruct embeddings of X_Ω using combinatorics of Ω.

Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq A,\ 0 \leq y \leq f(x)\}$, $f \geq 0$ nonincreasing.

Definition

If f is concave, then X_Ω is a **convex** toric domain. If f is convex, then X_Ω is a **concave** toric domain.

Theorem (Cristofaro-Gardiner '19, generalizing McDuff '11)

If X_Ω is concave and $X_{\Omega'}$ is convex, then

$$X_\Omega \overset{s}{\hookrightarrow} X_{\Omega'} \Leftrightarrow c_k(X_\Omega) \leq c_k(X_{\Omega'})\,\forall\, k, \tag{0.1}$$

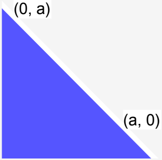
However, if X_Ω is convex (e.g. a polydisk), then (0.1) is *not* an equivalence, only $\Rightarrow$!

So we need other means to **obstruct** $X_\Omega \overset{s}{\hookrightarrow} X_{\Omega'}$.

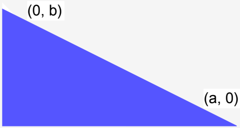
Definition

If f is concave, then X_Ω is a **convex** toric domain. If f is convex, then X_Ω is a **concave** toric domain.

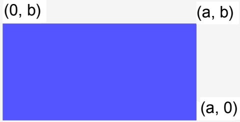
Polydisks are convex, not concave!



(f) *Ball* $B(a)$



(g) *Ellipsoid* $E(a, b)$



(h) *Polydisk* $P(a, b)$

Our results use the combinatorics of Ω “beyond” the ECH capacities of X_Ω to obstruct

$$P(a, 1) \overset{s}{\hookrightarrow} E(bc, c)$$

based on the relationships between a, b, and c.

Theorem (Hutchings, 2016)

Let $1 \leq a \leq 2$ and $b \in \mathbb{Z}_{>0}$. Then $P(a,1) \overset{s}{\hookrightarrow} E(bc,c)$ if and only if $a+b \leq bc$.

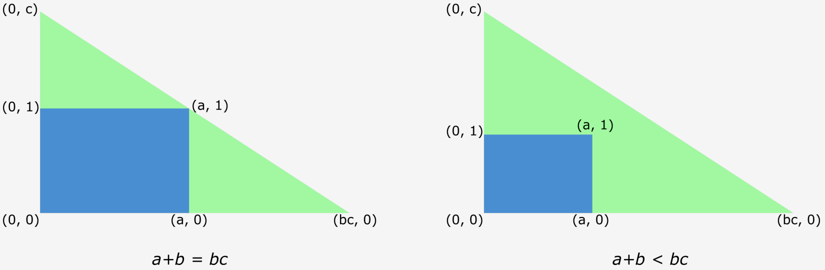


Figure: $a+b \leq bc$ means that the polydisk $P(a,1)$ can be directly included into the ellipsoid $E(bc,c)$.

Theorem (Ning-Yang, 2020)

Let $d_0 \geq 3$ be a prime number. Let $1 \leq a \leq (2d_0 - 1)/d_0$, $c > 0$ and $b = p/2$ for some odd integer $p \geq 4d_0 + 1$. Then $P(a, 1) \overset{s}{\hookrightarrow} E(bc, c)$ if and only if $a + b \leq bc$.

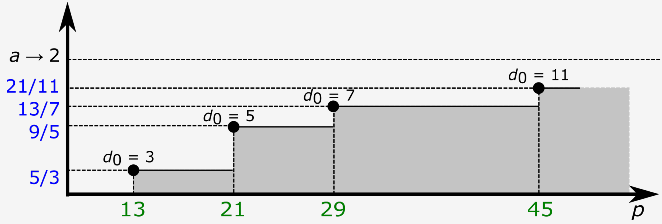


Figure: Each dot represents some prime $d_0 \geq 3$ and the shaded regions show the applicability of our theorem. With an increasing restriction on p, the theorem works for more $a \geq 1$ approaching $a = 2$.

For $p \leq 13$, we can provide sharp obstructions for $1 \leq a < 2$.

Theorem (Ning-Yang, 2020)

Let $1 \leq a \leq 4/3$, $c > 0$ and $b = p/2$ for some odd integer $p > 2$. Then $P(a,1) \overset{s}{\hookrightarrow} E(bc,c)$ if and only if $a + b \leq bc$.

Theorem (Ning-Yang, 2020)

Let $1 \leq a \leq 3/2$, $c > 0$ and $b = p/2$ for some odd integer $p \geq 7$. Then $P(a,1) \overset{s}{\hookrightarrow} E(bc,c)$ if and only if $a + b \leq bc$.

Definition

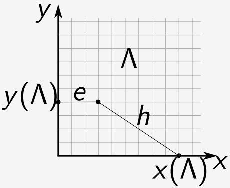
A *convex generator* is a convex integral path Λ such that:

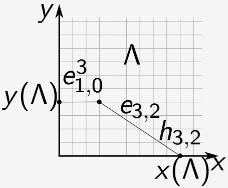
- Each edge of Λ is labeled 'e' or 'h'.
- Horizontal and vertical edges can only be labeled 'e'.

Definition

If Λ be a convex generator, then its *ECH index* is defined to be

$$I(\Lambda) = 2(L(\Lambda) - 1) - h(\Lambda).$$





The **symplectic action** of a convex generator Λ wrt $P(a,1)$ is

$$A_{P(a,1)}(\Lambda) = x(\Lambda) + ay(\Lambda).$$

The **symplectic action** of Λ with respect to $E(bc,c)$ is

$$A_{E(bc,c)}(\Lambda) = r, \text{ where } cx + bcy = r \text{ is tangent to } \Lambda.$$

A convex generator Λ with $I(\Lambda) = 2k$ is **minimal** for $E(bc,c)$ if:

- All edges of Λ are labeled 'e'.
- Λ uniquely minimizes $A_{E(bc,c)}$ among convex generators with $I = 2k$ and all edges labeled 'e'.

Key: $e_{p,2}^{d_0}$ is **minimal** for $E(pc/2,c)$ for any $c > 0$, $d_0 \geq 1$.

Remark

If $I(\Lambda) = 2k$ and Λ is minimal for X_Ω then $A_\Omega(\Lambda) = c_k(X_\Omega)$.

Definition (Hutchings, 2016)

Let Λ, Λ' be convex generators s.t. all edges of Λ' are labeled 'e'. We write $\Lambda \leq_{P(a,1),E(bc,c)} \Lambda'$ whenever:

1. (Index requirement)
   $I(\Lambda) = I(\Lambda')$;
2. (Action inequality)
   $A_{P(a,1)}(\Lambda) \leq A_{E(bc,c)}(\Lambda')$;
3. (J-holomorphic curve genus inequality)
   $x(\Lambda) + y(\Lambda) - h(\Lambda)/2 \geq x(\Lambda') + y(\Lambda') + m(\Lambda') - 1$.

We abbreviate '$\leq$' for '$\leq_{P(a,1),E(bc,c)}$' between generators when when a, b and c are specified without ambiguity.

Theorem (The Hutchings criterion, 2016)

Let X_Ω and $X_{\Omega'}$ be convex toric domains. Suppose $X_\Omega \overset{s}{\hookrightarrow} X_{\Omega'}$. Let Λ' be a convex generator which is minimal for $X_{\Omega'}$. Then there exists a convex generator Λ with $I(\Lambda) = I(\Lambda')$, a nonnegative integer n, and product decompositions

$$\Lambda = \Lambda_1 \cdots \Lambda_n \quad \text{and} \quad \Lambda' = \Lambda'_1 \cdots \Lambda'_n,$$

such that

1. $\Lambda_i \leq_{\Omega,\Omega'} \Lambda'_i$ *for each* $i = 1, \dots, n$.
2. *Given* $i, j \in \{1, \dots, n\}$, *if* $\Lambda_i \neq \Lambda_j$ *or* $\Lambda'_i \neq \Lambda'_j$, *then* Λ_i *and* Λ_j *have no elliptic orbit in common.*
3. *If* S *is any subset of* $\{1, \dots, n\}$, *then* $I\left(\prod_{i \in S} \Lambda_i\right) = I\left(\prod_{i \in S} \Lambda'_i\right)$.

In our case, $X_\Omega = P(a, 1)$ and $X_{\Omega'} = E(bc, c)$.

Theorem (Ning-Yang, 2020)

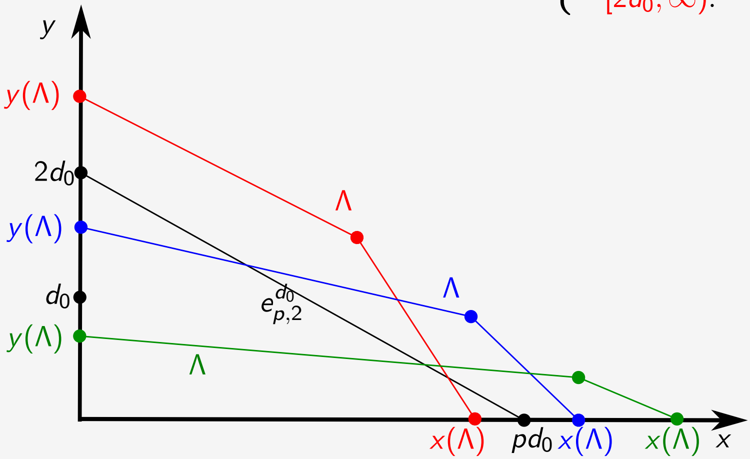
Let $d_0 \geq 3$ be a prime number. Let $1 \leq a \leq (2d_0 - 1)/d_0$, $c > 0$ and $b = p/2$ for some odd integer $p \geq 4d_0 + 1$. Then $P(a,1) \overset{s}{\hookrightarrow} E(bc, c)$ if and only if $a + b \leq bc$.

Key: use Hutchings' criterion to show the non-existence of $P(a,1) \overset{s}{\hookrightarrow} E(bc,c)$ when $a + b > bc$.

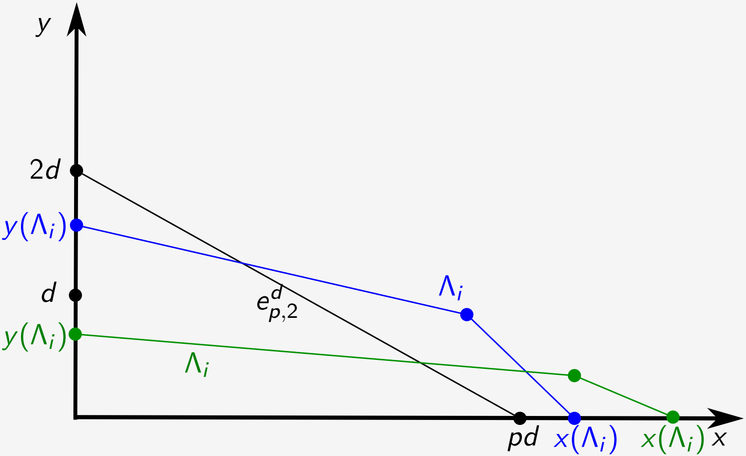
Take $\Lambda' = e_{p,2}^{d_0}$, we need to show the non-existence of Λ such that

- (Trivial factorization) $\Lambda \leq \Lambda'$.
- (Full factorization) $\Lambda = \prod_i \Lambda_i$ where $\Lambda_i \leq e_{p,2}$ for $1 \leq i \leq d_0$.
- Λ factors into $2 \leq k \leq d_0 - 1$ factors.

Split $\Lambda \leq \Lambda' = e_{p,2}^{d_0}$ into three cases: $y(\Lambda) \in \begin{cases} [0, d_0), \\ [d_0, 2d_0), \\ [2d_0, \infty). \end{cases}$

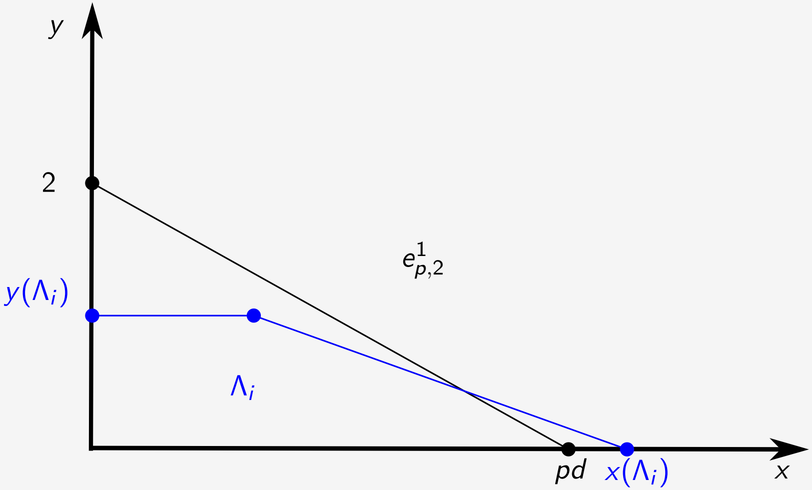


Previous argument can also show that for $d \leq d_0 - 1$, $\Lambda_i \leq e_{p,2}^d$ is only possible if $y(\Lambda_i) = d$.



Such Λ_i must contain an $e_{1,0}$ factor! Now use primality of d_0.

In this case, we only need to consider $y(\Lambda_i) < 2$. Explicit index computations finishes the proof.



"Optimal" restrictions

Theorem (Ning-Yang)

Let $d_0 \geq 3$ be a prime number. Let $1 \leq a \leq (2d_0 - 1)/d_0$, $c > 0$ and $b = p/2$ for some odd integer $p \geq 4d_0 + 1$. Then $P(a, 1) \overset{s}{\hookrightarrow} E(bc, c)$ if and only if $a + b \leq bc$.

The bound $(2d_0 - 1)/d_0 \leq 2$ on a is "optimal" in this sense:

Example

Under the same hypothesis, let $\varepsilon > 0$ and take instead

$$a = (2d_0 - 1)/d_0 + \varepsilon.$$

There always exists a convex generator $\Lambda \leq e_{p,2}^{d_0}$ for $d_0 \geq 2$, when

$$a + b - \varepsilon/2 < bc < a + b,$$

i.e. Hutchings' criterion cannot provide sharp obstructions.

The restriction $p \geq 4d_0 + 1$ is similarly “optimal”:

Example

Under the same hypothesis, consider $a = (2d_0 - 1)/d_0$ and take

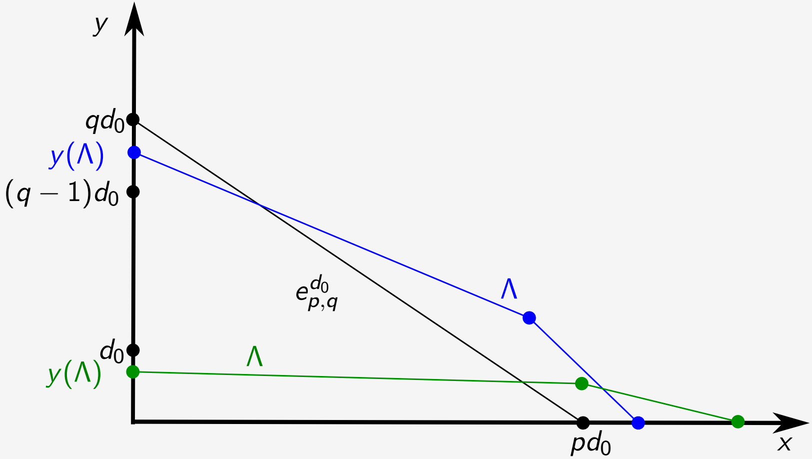
$$p = 4d_0 - 3 \leq 4d_0 + 1.$$

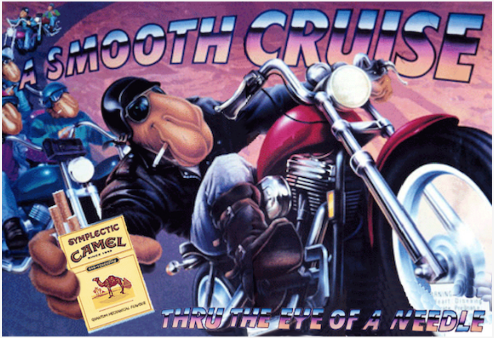
There always exists a convex generator $\Lambda \leq e_{p,2}^{d_0}$ for $d_0 \geq 2$, when

$$a + b - \frac{d_0 - 1}{2d_0^2} < bc < a + b,$$

again in which case Hutchings’ criterion cannot provide sharp obstructions.

When $b = p/q$ and $q > 2$, "Beyond ECH" tools are insufficient to disprove the existence of $\Lambda \leq e_{p,q}^{d_0}$ for $a + b > bc$.





Dusa McDuff
What is symplectic geometry?
(Survey for students)

Felix Schlenk
Symplectic embedding problems, old and new
(Detailed Survey)

Cristofaro-Gardiner, *Symplectic embeddings from concave toric domains into convex ones*, J. Diff. Geom. 112 (2019), 199-232.

Digiosia, Nelson, Ning, Weiler, Yang, *Symplectic embeddings of four-dimensional polydisks into half integer ellipsoids*, **arXiv:2010.06687**

Hutchings, *Beyond ECH Capacities*, Geom. & Topol. 20 (2016).

Hutchings, *Quantitative embedded contact homology*, J. Diff. Geom. 88 (2011), no. 2, 231–266.

McDuff, *The Hofer Conjecture on Embedding Symplectic Ellipsoids*, J. Diff. Geom., 88 (2011), no. 3, 519–532.