Embedded Contact Homology of Prequantization Bundles

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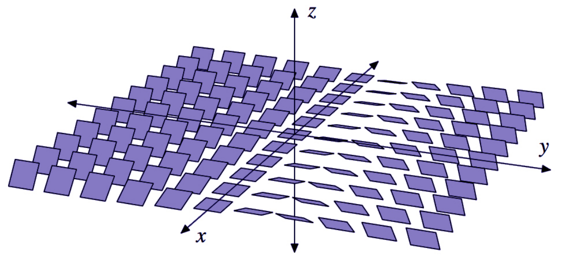
https://math.rice.edu/~jkn3/WHVSS-slides.pdf

Contact structures

Definition

A **contact structure** is a maximally nonintegrable hyperplane field.



$$\xi = \ker(dz - y dx)$$

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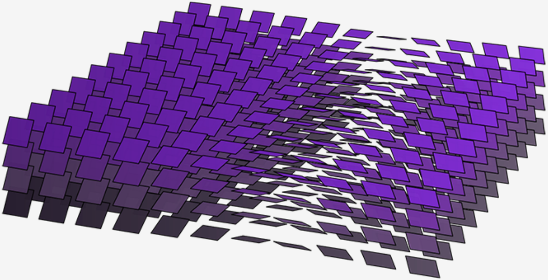


$$\xi = \ker(dz - y\,dx)$$

The kernel of a 1-form λ on Y^{2n-1} is a contact structure whenever

- $\lambda \wedge (d\lambda)^{n-1}$ is a volume form $\Leftrightarrow$ $d\lambda|_{\xi}$ is nondegenerate.

Reeb vector fields

Definition

The Reeb vector field R on (Y, λ) is uniquely determined by

- $\lambda(R) = 1$,
- $d\lambda(R, \cdot) = 0$.

The **Reeb flow**, $\varphi_t : Y \to Y$ is defined by $\frac{d}{dt}\varphi_t(x) = R(\varphi_t(x))$.

A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \tag{0.1}$$

and is **embedded** whenever (**??**) is injective.

Reeb orbits

Given an embedded **Reeb orbit** $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$,
the linearized flow along γ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$ is called the **linearized return map**.

If 1 is not an eigenvalue of $d\varphi_T$ then γ is **nondegenerate**.

Nondegenerate orbits are either **elliptic** or **hyperbolic** according to whether $d\varphi_T$ has eigenvalues on S^1 or real eigenvalues.

λ is **nondegenerate** if all Reeb orbits associated to λ are nondegenerate.

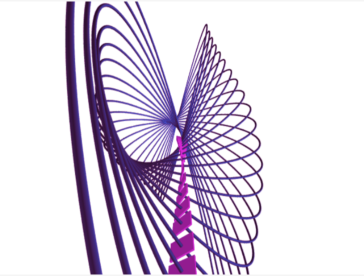
Reeb orbits on S^3

$$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \lambda = \frac{i}{2}(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$$

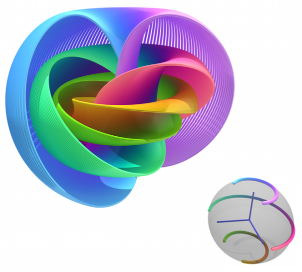
The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.

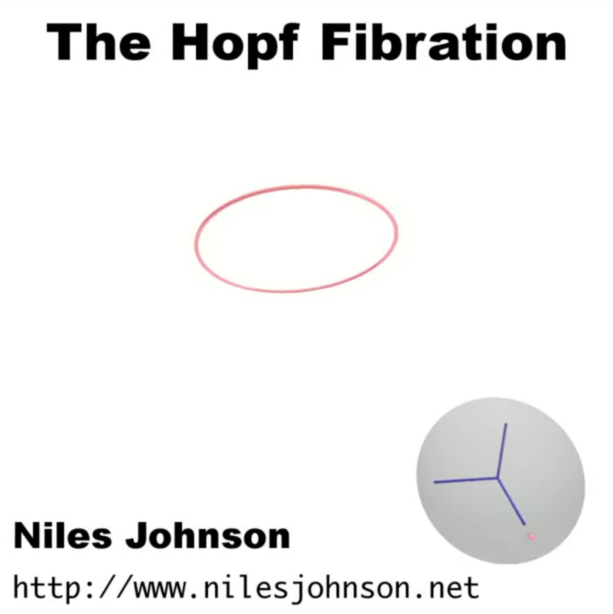


Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$

A video of the Hopf fibration



Prequantization bundles

Theorem (Boothby-Wang construction '58)

Let (Σ_g, ω) be a Riemann surface and e a negative class in $H_2(\Sigma_g; \mathbb{Z})$. Let $\mathfrak{p} : Y \to \Sigma_g$ be the principal S^1-bundle with Euler class e. Then there is a connection 1-form λ on Y whose Reeb vector field R is tangent to the S^1-action.

- (Y, λ) is the **prequantization bundle** over (Σ_g, ω).
- The Reeb orbits of R are the S^1-fibers of this bundle.
- The Reeb orbits of R are degenerate.
- $d\lambda = \mathfrak{p}^*\omega$
- $\mathfrak{p}_*\xi = T\Sigma_g$

Perturbed Reeb dynamics of prequantization bundles

Use a Morse-Smale $H : \Sigma \to \mathbb{R}$, $|H|_{C^2} < 1$ to perturb λ:

$$\lambda_\varepsilon := (1 + \varepsilon \mathfrak{p}^* H)\lambda$$

The perturbed Reeb vector field is

$$R_\varepsilon = \frac{R}{1 + \varepsilon \mathfrak{p}^* H} + \frac{\varepsilon \tilde{X}_H}{(1 + \varepsilon \mathfrak{p}^* H)^2}$$

where $\tilde{X}_H$ is the horizontal lift of X_H to ξ. If $p \in \mathrm{Crit}(H)$ then $X_H(p) = 0$.

The **action** of a closed orbit γ is $\mathcal{A}(\gamma) := \int_\gamma \lambda_\varepsilon$.

Fix $L > 0$. $\exists\ \varepsilon > 0$ such that if γ is an orbit of R_ϵ and

- if $\mathcal{A}(\gamma) < L$ then γ is nondegenerate and projects to $p \in \mathrm{Crit}(H)$;
- if $\mathcal{A}(\gamma) > L$ then γ loops around the tori above the orbits of X_H, or is a larger iterate of a fiber above $p \in \mathrm{Crit}(H)$.

Fiber orbits of prequantization bundles

Recall

$$R_\varepsilon = \frac{R}{1+\varepsilon \mathfrak{p}^* H} + \frac{\varepsilon \tilde{X}_H}{(1+\varepsilon \mathfrak{p}^* H)^2}$$

Denote the k-fold cover projecting to $p \in \mathrm{Crit}(H)$ by γ_p^k.

We have

$$CZ_\tau(\gamma_p^k) = RS_\tau(\mathrm{fiber}^k) - \frac{\dim(\Sigma)}{2} + \mathrm{ind}_p(H).$$

Using the constant trivialization of $\xi = \mathfrak{p}^* T\Sigma$, $RS_\tau(\mathrm{fiber}^k) = 0$.

Thus

$$CZ_\tau(\gamma_p^k) = \mathrm{ind}_p(H) - 1.$$

Fiber orbits of prequantization bundles

Recall

$$CZ_\tau(\gamma_p^k) = \mathrm{ind}_p(H) - 1$$

Only positive hyperbolic orbits have even CZ.

If $\mathrm{ind}_p(H) = 1$ then γ_p is positive hyperbolic.

Since $\mathfrak{p}$ is a bundle, all linearized return maps are close to Id.

Hence no negative hyperbolic orbits.

If $\mathrm{ind}_p(H) = 0, 2$ then γ_p is elliptic.

Assume H is perfect. Denote

- the index zero elliptic orbit by e_-
- the index two elliptic orbit by e_+,
- the hyperbolic orbits by $h_1, \ldots, h_{2g}$.

Embedded contact homology (ECH) is a Floer theory for closed (Y^3, λ) and $\Gamma \in H_1(Y;\mathbb{Z})$.

For nondegenerate λ, the chain complex $ECC_*(Y, \lambda, \Gamma, J)$ is generated as a $\mathbb{Z}_2$ vector space by **orbit sets** $\alpha = \{(\alpha_i, m_i)\}$, which are finite sets for which:

- α_i is an embedded Reeb orbit
- $m_i \in \mathbb{Z}_{>0}$
- $\sum_i m_i[\alpha_i] = \Gamma$
- If α_i is hyperbolic, $m_i = 1$.

The **grading** $*$ comes from the relative **ECH index** $I(\alpha, \beta)$, a combination of $c_1(\ker \lambda)$, $CZ(\alpha_i^k)$, $CZ(\beta_j^k)$, and the relative self-intersection.

Almost complex structures and ∂^{ECH}

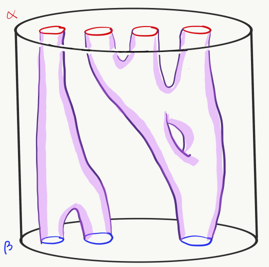
A λ-**compatible almost complex structure** is a complex structure J on $T(\mathbb{R} \times Y)$, for which:

- J is $\mathbb{R}$-invariant
- $J\xi = \xi$, positively with respect to $d\lambda$
- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate

$\langle \partial^{ECH} \alpha, \beta \rangle$ counts **currents**, disjoint unions of J-**holomorphic curves**

$$u : (\dot{\Sigma}, j) \to (\mathbb{R} \times Y, J), \quad du \circ j = J \circ du$$

which are asymptotically cylindrical to orbit sets α and β at $\pm\infty$.



For generic J,
ECH index one yields
somewhere injective.

-Hutchings' Haiku

Embedded contact homology differential ∂^{ECH}

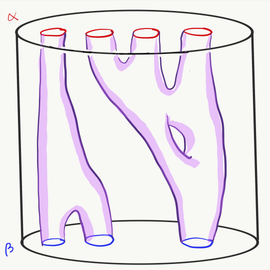
Theorem (Hutchings-Taubes '09)

$\left(\partial^{ECH}\right)^2 = 0$, *so* $(ECC_*(Y, \lambda, \Gamma, J), \partial^{ECH})$ *is a chain complex.*

Theorem (Taubes, Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda)

The homology depends only on $(Y, \ker \lambda, \Gamma)$.

We denote the homology by $ECH_*(Y, \ker \lambda, \Gamma)$.



Dee squared is zero;
obstruction bundle gluing
is complicated.

-Hutchings-Taubes' Haiku

ECH from H_*

Theorem (Nelson-Weiler, 90%)

Let $(Y, \xi = \ker\lambda)$ *be a prequantization bundle over* (Σ_g, ω). *Then*

$$\bigoplus_{\Gamma \in H_1(Y;\mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$$

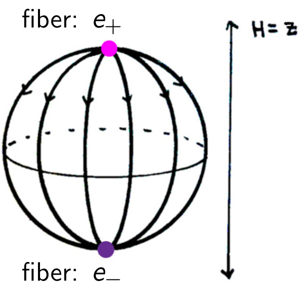
Inspired by the 2011 PhD thesis of Farris.

1. The critical points of a perfect H form a basis for $H_*(\Sigma_g; \mathbb{Z}_2)$. The generators of ECH are of the form $e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$ where $m_i = 0, 1$, so correspond to a basis for $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$.
2. We will prove ∂^{ECH} only counts cylinders corresponding to Morse flows on Σ_g, therefore $\partial^{ECH}(e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+})$ is a sum over all ways to apply ∂^{Morse} to h_i or e_+.

Our favorite fibration on S^3

Example (S^3, λ)

The ECH of S^3 is the $\mathbb{Z}_2$-vector space generated by terms $e_-^{m_-} e_+^{m_+}$, where $|e_-| = 2, |e_+| = 4$. Note that $*$ is not the grading on $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$, since $|e_-^2| = 6$.



The fibers above the critical points of the height function on S^2 represent $e_\pm$.

We have $\partial^{ECH} = 0$ because $\partial^{Morse} = 0$.

Lens spaces $L(k,1)$

$L(k,1)$ is the total space of the prequantization bundle with Euler number $-k$ on S^2.

Corollary (Nelson-Weiler, 95%)

With its prequantization contact structure ξ_k,

$$ECH_*(L(k,1), \xi_k, \Gamma) \cong \begin{cases} \mathbb{Z}_2 & \text{if } * \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{else} \end{cases}$$

for all $\Gamma \in H_1(L(k,1); \mathbb{Z})$.

Finer points of the isomorphism

Fix a negative Euler class e. For $\Gamma \in \{0, \ldots, -e-1\}$,

$$ECH_*(Y, \xi, \Gamma) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Lambda^{\Gamma - ne}(H_*(\Sigma_g; \mathbb{Z}_2))$$

Proposition (Nelson-Weiler)

Let $\alpha = e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$ and let $\beta = e_-^{n_-} h_1^{n_1} \cdots h_{2g}^{n_{2g}} e_+^{n_+}$. Let $N = n_- + n_+ + \sum_j n_j$ and $m = \frac{(m_- + m_+ + \sum_i m_i) - N}{-e}$. Then

$$I(\alpha, \beta) = (2-2g)m - m^2 e + 2mN + m_+ - m_- - n_+ + n_-$$

Using this formula, we obtain

$$I(e_+^{N+e}, e_-^N) = 2g - 2$$

$ECH_*(Y, \xi, 0)$ for $g = 2, e = -1$

Recall $I(e_+^{N+e}, e_-^N) = 2g - 2$. Set $*(\alpha) = I(\alpha, \emptyset)$.

	$* = -2$	$* = -1$	$* = 0$	$* = 1$	$* = 2$	$* = 3$	$* = 4$	
Λ^0			$\emptyset$					
Λ^1	e_-	h_i	e_+					
Λ^2	e_-^2	$e_- h_i$	$e_- e_+$	$h_i e_+$	e_+^2			
			$h_i h_j$					
Λ^3			e_-^3	$e_-^2 h_i$	$e_-^2 e_+$	$e_- h_i e_+$	$e_- e_+^2$	$\cdots$
					$e_- h_i h_j$	$h_i h_j h_k$	$h_i h_j e_+$	
Λ^4							e_-^4	$\cdots$

Outline of proof

Theorem (Nelson-Weiler, 90%)

Let $(Y, \xi = \ker\lambda)$ *be a prequantization bundle over* (Σ_g, ω). *Then*

$$\bigoplus_{\Gamma \in H_1(Y;\mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$$

1. There exists $\varepsilon > 0$ so that the generators of $ECC_*^L(Y, \lambda_\varepsilon, J)$ consist solely of orbits which are fibers over critical points.
2. Prove that $\partial^{ECH,L}$ only counts cylinders which are the union of fibers over Morse flow lines in Σ.
3. Finish with a direct limit argument, sending $\varepsilon \to 0$ and $L \to \infty$, in addition to the isomorphism with Seiberg-Witten.

Pseudoholomorphic Cylinders

- Pseudoholomorphic cylinders correspond to Floer trajectories on Σ_g (*Moreno, Siefring*)
- Floer trajectories on Σ_g correspond to Morse flows (*Floer, Salamon-Zehnder*)
- Cylinder counts permit use of fiberwise S^1-invariant J, even for multiply covered curves, by automatic transversality (*Wendl*)

Theorem (N. 2017)

The cylindrical contact homology chain complex of a prequantization bundle over Σ_g is generated by infinitely many copies of the Morse complex of Σ_g, and on each copy the cylindrical differential agrees with the Morse differential.

Higher genus curve counting difficulties (Farris)

- Can count cylinders using the complex structure $J_{\Sigma_g} = \mathfrak{p}^* j_{\Sigma_g}$, the S^1-invariant lift of j_{Σ_g}.

  (YAY! Automatic transversality!)

- J_{Σ_g}-holomorphic cylinders correspond to Morse trajectories on Σ_g.

- **Cannot use** J_{Σ_g} for higher genus curves!

  J_{Σ_g} cannot be independently perturbed at the intersection points of $\pi_Y u$ with a given S^1-orbit by an S^1-invariant perturbation.

  (YIKES! J_{Σ_g} is not typically regular!)

- There will always be a regular J for moduli spaces of higher genus curves, but we cannot assume J is S^1-invariant.

  (CURVE COUNTING NO LONGER OBVIOUS...)

Domain dependent almost complex structures (Farris)

Forsake J_{Σ_g} for an S^1-invariant domain dependent perturbation,

$$\{J^z_{\Sigma_g}\}_{z\in\dot{\Sigma}}$$

- Akin to time-dependence in Hamiltonian Floer theory.
- Implicit function theorem relates counts of nearby moduli spaces

Higher genus curves and multiply covered cylinders do not contribute to ∂^{ECH}

- Transversality guarantees index 1 holomorphic curves do not exist unless they are fixed by the S^1-action.
- Otherwise the curve lives in a moduli space of dimension ≥ 2.
- But $\langle \partial^{ECH}\alpha, \beta\rangle$ only counts curves where $\pi_Y \circ u$ is isolated.
- So we only count cylinders projecting to Floer trajectories.

Remaining issue (modulo direct limits):

- Hutchings set up ECH with a domain independent J...

One parameter family of complex structures

Consider $\{\mathfrak{J}_t\}_{t\in[0,1]}$ a family of S^1-invariant domain dependent almost complex structures in $\mathbb{R} \times Y$,

$$\mathfrak{J}_0 := \{J^z_{\Sigma_g}\}_{z\in\dot{\Sigma}}$$
$$\mathfrak{J}_1 := J \in \mathcal{J}^{reg}(Y, \lambda).$$

Lemma

For generic paths, the moduli space $\mathcal{M}_t = \mathcal{M}(\alpha, \beta, \mathfrak{J}_t)$ is cut out transversely save for a discrete number of times $t_0, ..., t_\ell \in (0, 1)$. For each such t_i, ∂^{ECH} can change either by

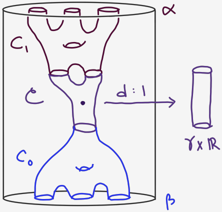
- *creation/destruction of a pair of oppositely signed curves;*
- *an "ECH handleslide."*

In either case, the homology is unaffected. PHEW!

Handleslides do not impact curve counts

At a handleslide t_i, $\{C_k \mid \text{ind}_{\text{Fred}}(C_k) = 1\}$ breaks into a building with:

- an index 1 curve C_1 at top (or bottom)
- branched covers $\mathcal{C}$ of $\gamma \times \mathbb{R}$ with $\text{ind}_{\text{Fred}}(\mathcal{C}) = 0$
- an index 0 curve C_0 at bottom (or top)



Branched covers cannot appear as the top-most or bottom-most level.
(Hutchings - N '16, Cristofaro-Gardiner - Hutchings - Zhang)
Hooray! We can invoke obstruction bundle gluing...

$$\#\mathcal{M}(\alpha, \beta, \mathfrak{J}_{t_i+\epsilon}) = \#\mathcal{M}(\alpha, \gamma, \mathfrak{J}_{t_i-\epsilon}) + \#G(C_1, C_0) \cdot \#\mathcal{M}(\gamma, \beta, \mathfrak{J}_{t_i}),$$

OBG gives a combinatorial formula for $\#G \in \mathbb{Z}$, based on
the partitions at $-\infty$ ends of C_1,
the partitions at $+\infty$ ends of C_0.
No need to explicitly compute $\#G$ as inductively $\#\mathcal{M}(\gamma, \beta, \mathfrak{J}_{t_i}) = 0$!

Filtrations and computations

There is no geometric Morse-Bott ECH.

Denote by $ECH_*^L(Y, \lambda_\varepsilon, \Gamma)$ the homology of the chain complex of ECH generators with action $\leq L$. (It's independent of J.)

Hutchings-Taubes '13: Cobordism and inclusion maps give us

$$\begin{array}{ccc} ECH_*^L(Y, \lambda_\varepsilon, \Gamma) & \longrightarrow & ECH_*^L(Y, \lambda_{\varepsilon'}, \Gamma) \\ \downarrow & & \downarrow \\ ECH_*^{L'}(Y, \lambda_\varepsilon, \Gamma) & \longrightarrow & ECH_*^{L'}(Y, \lambda_{\varepsilon'}, \Gamma) \end{array}$$

which commute, for $\varepsilon' < \varepsilon, L' > L$.

We can now compute

$$\lim_{\varepsilon \to 0, L \to \infty} ECH_*^L(Y, \lambda_\varepsilon, \Gamma, J) \cong_{\mathbb{Z}_2} \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2). \qquad (0.2)$$

That the LHS of (**??**) is $ECH_*(Y, \xi, \Gamma)$ uses a similar filtration on Seiberg-Witten Floer homology from Hutchings-Taubes.

Future work: U map

There is a degree -2 map

$$U : ECC_*(Y, \xi, \Gamma) \to ECC_{*-2}(Y, \xi, \Gamma)$$

which counts J-holomorphic curves passing through a base point.

U is equivalent to the U maps on Seiberg-Witten and Heegaard Floer homologies.

In the case of prequantization bundles, we expect U to count meromorphic sections of the line bundle associated to Y.

U is Useful:

- Find index 2 holomorphic curves, since U is an invariant;
- **ECH capacities**, which obstruct symplectic embeddings;
- Proving stabilization results.

Future work: stabilization

From Seiberg-Witten and Heegaard Floer homologies we know U is an isomorphism if $*$ is large enough. Therefore:

Theorem (Nelson-Weiler, 90%)

If $e = -1$ and $g > 1$, then for $*$ large enough,

$$ECH_*(Y, \xi) \cong \mathbb{Z}_2^{2^{2g-1}}.$$

and U is an isomorphism.

We expect to prove this theorem entirely in ECH once we can characterize the U map via meromorphic sections.

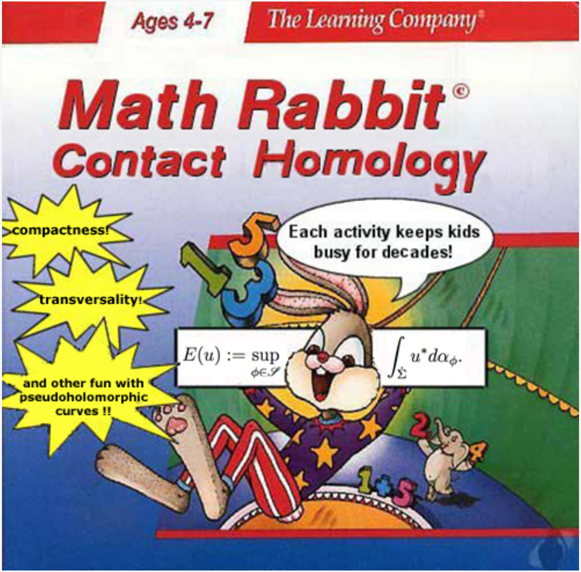
Future work: surface dynamics

Proposition (Colin-Honda '13)

If $\phi \in Mod(\Sigma_g)$ is periodic, then (Y, ξ) is supported by an open book decomposition with page Σ_g and monodromy ϕ and is a Seifert fiber space over the orbifold Σ_g/ϕ. There is a contact form for ξ whose Reeb vector field is tangent to the fibers.

We will generalize our prequantization methods to circle bundles over orbifolds to understand the dynamics of symplectomorphisms which are freely homotopic to ϕ, extending the Calabi invariant bounds in Weiler's thesis to genus 0 open books.

Thanks


Ages 4-7
The Learning Company
Math Rabbit©
Contact Homology
compactness!
transversality!
and other fun with pseudoholomorphic curves !!
Each activity keeps kids busy for decades!
$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\dot\Sigma} u^* d\alpha_\phi.$