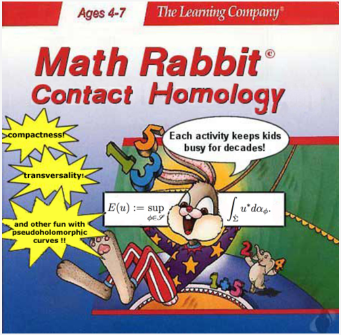
Reflections on cylindrical contact homology

Jo Nelson (Rice)

Symplectic Zoominar, May 2020

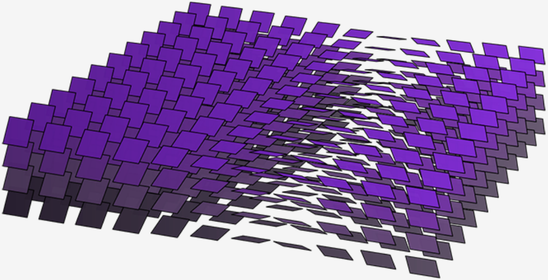
https://math.rice.edu/~jkn3/Zoominar-slides.pdf



Contact structures

Definition

A **contact structure** is a maximally nonintegrable hyperplane field.



$$\xi = \ker(dz - ydx)$$

The kernel of a 1-form λ on Y^{2n-1} is a contact structure whenever

- $\lambda \wedge (d\lambda)^{n-1}$ is a volume form $\Leftrightarrow$ $d\lambda|_\xi$ is nondegenerate.

Reeb vector fields

Definition

The Reeb vector field R on (Y, λ) is uniquely determined by

- $\lambda(R) = 1$,
- $d\lambda(R, \cdot) = 0$.

The **Reeb flow**, $\varphi_t : Y \to Y$ is defined by $\frac{d}{dt}\varphi_t(x) = R(\varphi_t(x))$.

A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \tag{0.1}$$

and is **embedded** whenever (0.1) is injective.

Reeb orbits

Given an embedded **Reeb orbit** $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$,
the linearized flow along γ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$ is called the **linearized return map**.

If 1 is not an eigenvalue of $d\varphi_T$ then γ is **nondegenerate**.

λ is **nondegenerate** if all Reeb orbits associated to λ are nondegenerate.

In dim 3, nondegenerate orbits are either **elliptic** or **hyperbolic** according to whether $d\varphi_T$ has eigenvalues on S^1 or real eigenvalues.

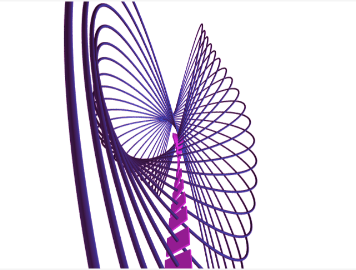
Reeb orbits on S^3

$$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \lambda = \frac{i}{2}(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$$

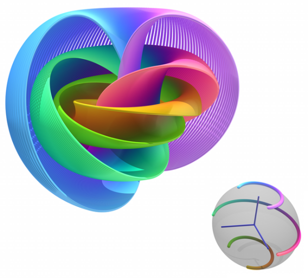
The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.

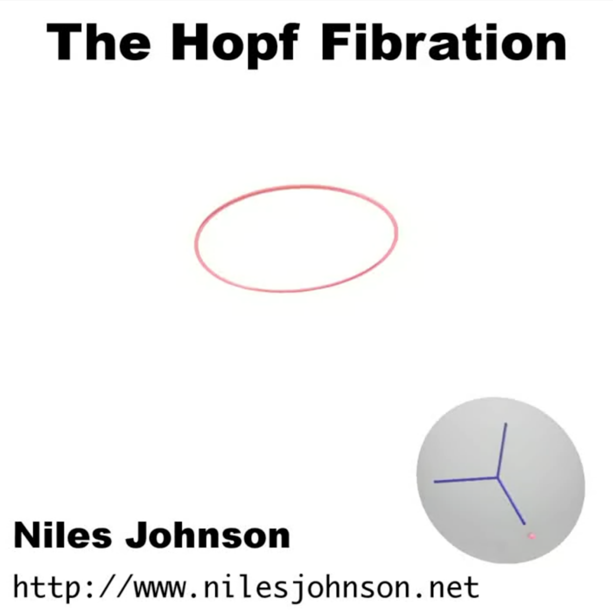


Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$

A video of the Hopf fibration



A new era of contact geometry

Helmut Hofer on the origins of the field:

> *So why did I come into symplectic and contact geometry? So it turned out I had the flu and the only thing to read was a copy of Rabinowitz's paper where he proves the existence of periodic orbits on star-shaped energy surfaces. It turned out to contain a fundamental new idea, which was to study a different action functional for loops in the phase space rather than for Lagrangians in the configuration space. Which actually if we look back, led to the variational approach in symplectic and contact topology, which is reincarnated in infinite dimensions in Floer theory and has appeared in every other subsequent approach. ...Ja, the flu turned out to be really good.*

Existence of periodic orbits

The Weinstein Conjecture (1978)

Let Y be a closed oriented odd-dimensional manifold with a contact form λ. Then the associated Reeb vector field R admits a closed orbit.

- Weinstein (convex hypersurfaces)
- Rabinowitz (star shaped hypersurfaces)
- Star shaped is secretly contact!
- Viterbo, Hofer, Floer, Zehnder ('80's fun)
- Hofer (S^3)
- Taubes (dimension 3)

Tools $>$ 1985: **Floer Theory** and **Gromov's** pseudoholomorphic curves.

The machinery that was invented

Let $(Y^{2n-1}, \xi = \ker \lambda)$ be a closed nondegenerate contact manifold.

Floerify Morse theory on

$$\begin{aligned} \mathcal{A}: \quad C^\infty(S^1, Y) &\to \mathbb{R}, \\ \gamma &\mapsto \int_\gamma \lambda. \end{aligned}$$

Proposition

$\gamma \in \mathrm{Crit}(\mathcal{A}) \Leftrightarrow \gamma$ *is a closed Reeb orbit.*

- Grading: $|\gamma| = CZ(\gamma) + n - 3$,
- $C_*^{EGH}(Y, \lambda, J) = \mathbb{Q}\langle \{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\} \rangle$
- **3-D**: Even covers of embedded negative hyperbolic orbits are bad.

The letter J is for pseudoholomorphic

A λ-**compatible almost complex structure** is a J on $T(\mathbb{R} \times Y)$:

- J is $\mathbb{R}$-invariant
- $J\xi = \xi$, positively with respect to $d\lambda$
- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate

Gradient flow lines are a no go; instead count **pseudoholomorphic cylinders** $u \in \mathcal{M}^J(\gamma_+, \gamma_-)/\mathbb{R}$.

$$u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times Y, J)$$
$$\bar{\partial}_J u := du + J \circ du \circ j \equiv 0$$

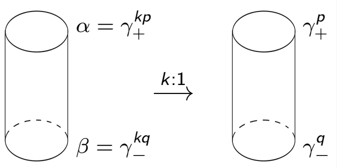
$$\lim_{s \to \pm\infty} \pi_{\mathbb{R}}\ u(s,t) = \pm\infty$$
$$\lim_{s \to \pm\infty} \pi_M\ u(s,t) = \gamma_\pm$$

up to reparametrization.

Note: J is S^1-INDEPENDENT

Cylindrical contact homology

- $C_*(Y, \lambda, J) = \mathbb{Q}\langle\{\text{closed}\} \setminus \{\text{bad}\}\rangle$
- $|\gamma| = CZ(\gamma) + n - 3$
- $\displaystyle \langle \partial^{EGH}\alpha, \beta\rangle = \sum_{\substack{u \in \mathcal{M}^J(\alpha,\beta)/\mathbb{R}, \\ |\alpha| - |\beta| = 1}} \frac{m(\alpha)}{m(u)} \epsilon(u)$
- $\displaystyle \langle \partial^{\text{HƆƎ}}\alpha, \beta\rangle = \sum_{\substack{u \in \mathcal{M}^J(\alpha,\beta)/\mathbb{R}, \\ |\alpha| - |\beta| = 1}} \frac{m(\beta)}{m(u)} \epsilon(u)$



$\gamma_\pm$ embedded, $\gcd(p, q) = 1$

Conjecture (Eliashberg-Givental-Hofer '00)

If there are no contractible Reeb orbits with $|\gamma| = -1, 0, 1$ then (C_*, ∂) is a chain complex and $CH_*^{EGH}(Y, \ker\lambda; \mathbb{Q}) = H(C_*(Y, \lambda, J), \partial)$ is an invariant of $\xi = \ker\lambda$.

Progress...

Definition

- (Y^{2n+1}, λ) is **hypertight** if there are no contractible Reeb orbits.
- (Y^3, λ) is **dynamically convex** whenever
  $c_1(\xi)|_{\pi_2(Y)} = 0$ and every contractible γ satisfies $CZ(\gamma) \geq 3$.

For us $\{\text{hypertight}\} \subset \{\text{dynamically convex}\}$.

A convex hypersurface transverse to the radial vector field Y in $(\mathbb{R}^4, \omega_0)$ admits a dynamically convex contact form $\lambda_0 := \omega_0(Y, \cdot)$.

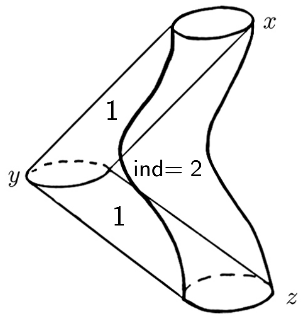
Theorem (Hutchings-N. '14 (JSG 2016))

If (Y^3, λ) is dynamically convex, J generic, and every contractible Reeb orbit γ has $CZ(\gamma) = 3$ only if γ is embedded then $\partial^2 = 0$.

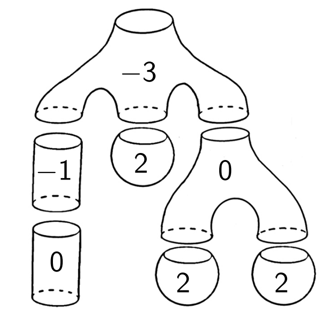
- Intersection theory is key to our proof that $\partial^2 = 0$.
- Can allow contractible $CZ(\gamma) = 3$ for prime covers of embedded Reeb orbits
  (Cristofaro-Gardiner - Hutchings - Zhang)
- 3D hypertight: invariance via obg for chain homotopy (Bao - Honda '14)
- Any dim hypertight: $\partial^2 = 0$ and invariance via Kuranishi atlases (Pardon '15)

The pseudoholomorphic menace

- Transversality for multiply covered curves is hard.
- Is $\mathcal{M}^J(\gamma_+;\gamma_-)$ more than a set?
- $\mathcal{M}^J(\gamma_+;\gamma_-)$ can have **nonpositive** virtual dimension...
- Compactness issues are "severe".



Desired compactification when $CZ(x) - CZ(z) = 2$.



Adding to 2 becomes hard

The return of regularity (domain dependent $\mathbb{J}$)

- S^1-independent J cylinders in $\mathbb{R} \times Y^3$ are reasonable
- All hope is lost in cobordisms, and no obvious chain maps.
- Invariance of $CH_*^{EGH}(Y, \lambda, J)$ requires S^1-dependent $\mathbb{J} := \{J_t\}_{t \in S^1}$.
- But breaking S^1-symmetry invalidates $\left(\partial^{EGH}\right)^2 = 0$.
- We define a Morse-Bott non-equivariant chain complex

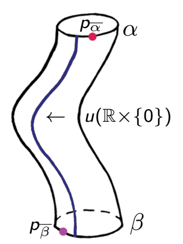
$$NCC_* := \bigoplus_{\text{all Reeb orbits } \gamma} \mathbb{Z}\langle \check{\gamma}, \hat{\gamma} \rangle, \quad \partial^{NCH} := \begin{pmatrix} \check{\partial} & \partial^+ \\ \partial^- + \ \mathbf{obg} & \hat{\partial} \end{pmatrix}$$

- If sufficient regularity exists to use J, then between good orbits,

$$\check{\partial} = \partial^{EGH}, \quad \hat{\partial} = -\partial^{HGE}, \quad \partial^+ = 0$$

- Compactness issues require obstruction bundle gluing, producing a novel correction term.
- The nonequivariant theory NCH_* is a contact invariant, which we relate to CH_*^{EGH} via family Floer methods.

Enter the point constraints



Given a generic λ-compatible family $\mathbb{J} := \{J_t\}_{t\in S^1}$,

$$
\begin{array}{rccc}
e_\pm : & \mathcal{M}^{\mathbb{J}}(\gamma_+,\gamma_-) & \to & \mathrm{im}(\gamma_\pm) = \overline{\gamma_\pm} \\
 & u & \mapsto & \lim_{s\to\pm\infty} \pi_Y u(s,0)
\end{array}
$$

Can use to specify a generic **base point** $p_{\overline{\gamma}}$ on each embedded $\overline{\gamma}$: $\quad e_+(u) = p_{\overline{\alpha}}, \quad e_-(u) = p_{\overline{\beta}}$.

The base level **cascade Morse-Bott moduli spaces**, $\mathcal{M}^{\mathbb{J}}(\cdot,\cdot)_1$:

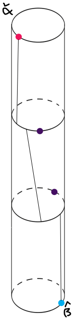
$$
\begin{array}{rcl}
\mathcal{M}^{\mathbb{J}}\left(\hat{\alpha},\check{\beta}\right)_1 & := & \mathcal{M}^{\mathbb{J}}(\alpha,\beta) \\
\mathcal{M}^{\mathbb{J}}\left(\check{\alpha},\check{\beta}\right)_1 & := & \left\{u \in \mathcal{M}^{\mathbb{J}}(\alpha,\beta) \mid e_+(u) = p_{\overline{\alpha}}\right\} \\
\mathcal{M}^{\mathbb{J}}\left(\hat{\alpha},\hat{\beta}\right)_1 & := & \left\{u \in \mathcal{M}^{\mathbb{J}}(\alpha,\beta) \mid e_-(u) = p_{\overline{\beta}}\right\} \\
\mathcal{M}^{\mathbb{J}}\left(\check{\alpha},\hat{\beta}\right)_1 & := & \left\{u \in \mathcal{M}^{\mathbb{J}}(\alpha,\beta) \mid e_+(u) = p_{\overline{\alpha}},\ e_-(u) = p_{\overline{\beta}}\right\}
\end{array}
$$

Higher levels consist of certain tuples $(u_1, .., u_\ell)$ of broken cylinders.

As a set, each of these spaces is a disjoint union of subsets $\mathcal{M}^{\mathbb{J}}(\cdot,\cdot)_\ell$.

Enter the cascade moduli spaces

Definition



$\mathcal{M}_3^{\mathbb{J}}(\check{\alpha}, \hat{\beta})$

Assuming $\alpha = \gamma_0, \gamma_1, \dots, \gamma_{\ell-1}, \gamma_\ell = \beta$ are all distinct, the **higher levels** $\mathcal{M}^{\mathbb{J}}_{|\alpha|-|\beta|}\left(\tilde{\alpha}, \tilde{\beta}\right)_\ell$, are the set of tuples

$$(u_1, .., u_\ell) \in \prod_{i=1}^{\ell} \mathcal{M}^{\mathbb{J}}(\gamma_{i-1}, \gamma_i) \quad \text{such that}$$

- if $\tilde{\alpha} = \check{\alpha}$ then $e_+(u_0) = p_{\overline{\alpha}}$;
- if $\tilde{\beta} = \hat{\beta}$ then $e_-(u_\ell) = p_{\overline{\beta}}$;
- $e_-(u_{i-1})$, $e_+(u_i)$, $p_{\overline{\gamma_i}}$, are cyclically ordered wrt Reeb flow.

When $\alpha = \beta$, define $\mathcal{M}^{\mathbb{J}}(\check{\alpha}; \check{\alpha}) = \mathcal{M}^{\mathbb{J}}(\check{\alpha}; \hat{\alpha}) = \mathcal{M}^{\mathbb{J}}(\hat{\alpha}; \hat{\alpha}) = \emptyset$,

$$\mathcal{M}^{\mathbb{J}}(\hat{\alpha}; \check{\alpha}) := \left\{ \begin{array}{cc} -2\{\text{pt}\} & \text{if } \alpha \text{ is bad;} \\ \emptyset & \text{if } \alpha \text{ is good.} \end{array} \right\}$$

A new hope for a chain complex

$$NCC_* := \bigoplus_{\text{all Reeb orbits } \gamma} \mathbb{Z}\langle \check{\gamma}, \hat{\gamma} \rangle, \quad \partial^{NCH} := \begin{pmatrix} \check{\partial} & \partial^+ \\ \partial^- + \textbf{obg} & \hat{\partial} \end{pmatrix}$$

$$\check{\partial}: \quad \widecheck{CC}_* \to \widecheck{CC}_{*-1}$$

$$\check{\alpha} \mapsto \sum_{\substack{\check{\beta},\ |\alpha|-|\beta|=1 \\ u \in \mathcal{M}^{\mathbb{J}}(\check{\alpha}, \check{\beta})}} \epsilon(u) \check{\beta}$$

$$\partial^+: \quad \widehat{CC}_* \to \widecheck{CC}_*$$

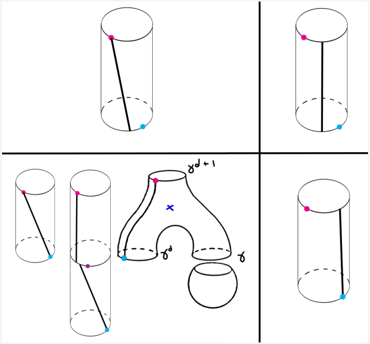
$$\hat{\alpha} \mapsto \sum_{\substack{\check{\beta},\ |\alpha|-|\beta|=0 \\ u \in \mathcal{M}^{\mathbb{J}}(\hat{\alpha}, \check{\beta})}} \epsilon(u) \check{\beta}$$

$$\partial^-: \quad \widecheck{CC}_* \to \widehat{CC}_{*-2}$$

$$\check{\alpha} \mapsto \sum_{\substack{\hat{\beta},\ |\alpha|-|\beta|=2 \\ u \in \mathcal{M}^{\mathbb{J}}(\check{\alpha}, \hat{\beta})}} \epsilon(u) \hat{\beta}$$

$$\hat{\partial}: \quad \widehat{CC}_* \to \widehat{CC}_{*-1}$$

$$\hat{\alpha} \mapsto \sum_{\substack{\hat{\beta},\ |\alpha|-|\beta|=1 \\ u \in \mathcal{M}^{\mathbb{J}}(\hat{\alpha}, \hat{\beta})}} \epsilon(u) \hat{\beta}$$

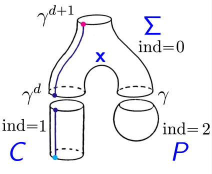


Theorem (Hutchings-N '19 (obg details in progress))

If (Y^{2n-1}, λ) is hypertight or (Y^3, λ) is dynamically convex, then for a generic family $\mathbb{J}$, ∂^{NCH} is well-defined, $\left(\partial^{NCH}\right)^2 = 0$, and $NCH_*(Y, \ker \lambda)$ is independent of the choice of λ and $\mathbb{J}$.

Rise of the obstruction bundles $(\partial^-+\mathbf{obg})\check{\partial}+\hat{\partial}(\partial^-+\mathbf{obg})=0$

Given $\Sigma_R \to \mathbb{R}\times S^1$ we can form a preglued curve. Next try to perturb to an honest pseudoholomorphic curve.



- Near **x** of Σ only perturb in directions normal to $\mathbb{R}\times\gamma$
- Obtain a unique curve **iff** the gluing obstruction $\mathfrak{s}(\Sigma)=0$, where $\mathfrak{s}$ is a section of the obstruction bundle $\mathcal{O}\to\mathcal{M}_R$.
- Count of gluings is related to count of 0's of obstruction section $\mathfrak{s}$.
- Fiber = coker$(D_\Sigma)^*$, Rank = dim$\mathcal{M}_R$.
- [*HT*]: branch points varied but objects being glued are fixed.
- [*HN*]: **x** is fixed but the glued object P varies in its moduli space.

Theorem (Hutchings-N, in progress)

Let (Y^3,λ) be dynamically convex and J generic. If γ is an embedded elliptic contractible Reeb orbit then

$$\langle(\mathbf{obg})\widecheck{\gamma^d},\widehat{\gamma^{d-1}}\rangle = n(\gamma),$$

given by the leading coefficient of the asymptotic op associated to P:

$$n(\gamma)=\deg\left(\mathcal{M}^J(\text{index 2 planes asymptotic to }\gamma)/\mathbb{R}\to S^1\right),$$

Revenge of the obstruction bundles

Essentialness of **obg** for the ellipsoid $E(a,b)=\left\{(u,v)\in\mathbb{C}^2 \,\middle|\, \frac{|u|^2}{a}+\frac{|v|^2}{b}=1\right\}$

Let α and β be orbits for the ellipsoid with $|\alpha| - |\beta| = 2$.

The differential coefficient from $\hat{\alpha} \to \check{\beta}$ has to be ± 1 or else the homology comes out wrong.

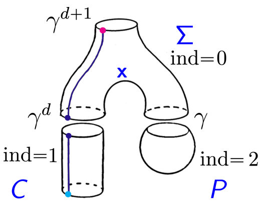
The **obg** arises when $\alpha = \gamma^{k+1}$, $\beta = \gamma^k$ with γ the short orbit.

HWZ: *the holomorphic planes bounded by γ give a foliation of $E(a,b)$.*

It follows directly from this that the obstruction bundle term is ± 1, assuming you know what you are doing (stay tuned).

An intersection theory argument shows there are no non-**obg** contributions to the differential coefficient from $\hat{\alpha}$ to $\check{\beta}$.

Obstruction bundle gluing setup



γ is embedded, C and P are immersed
Σ is a branched cover of $\gamma \times \mathbb{R}$
Fix $\mathbb{R}$-coor of **x** (akin to gluing parameter)
$\dim(\mathrm{Coker} D_\Sigma)) = 2$
Fix point constraint at bottom of Σ
Fix translation of C (another gluing parameter)

After fixing the $\mathbb{R}$-coor of C, Σ, P we have three degrees of freedom:

1. S^1-coordinate of the branch point **x**.
2. Choice of P, a point in the moduli space of planes.

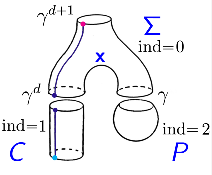
We have three constraints:

1. Conformal constraint corresponding to the point constraint.
2. The gluing obstruction.

$\exists$! **choice of S^1-coor agreeing with the conformal constraint.**

Next, the gluing obstruction...

The gluing obstruction



– Given nonzero $\psi \in \text{coker}(D_\Sigma)=\ker(D_\Sigma^*)$, consider its asymptotic eigenfunctions $\{\psi_i\}$. ψ is a 1-D $\mathbb{C}$-vector space and hence a section.
– Let ψ_d, ψ_1 be in the leading eigenspace of the asymoptotic operators L_{γ^d}, L_γ. (espace for L_γ pulls back to espace for L_{γ^d}).
– Take ψ_C, ψ_P to be the associated asymptotic eigenfunctions for Cylinder and Plane.
– If J is generic then $\psi_d, \psi_1, \psi_C, \psi_P$ are all nonvanishing.

- The gluing obstruction comes from a count of zeros.
- Many pages of math permit use of the following approximation given by the coefficients of the leading order term

$$\mathfrak{s}(\Sigma) \approx \langle \psi_d, \psi_C \rangle + \langle \psi_1, \psi_P \rangle,$$

- Everything is fixed except for ψ_P as P can move in its moduli space.
- Since we fixed **x** for Σ, the number of ways to glue is given by the choices of P such that $\langle \psi_1, \psi_P \rangle = -\langle \psi_d, \psi_C \rangle$
- It suffices to find the zeros of the linearized section, given by the coefficients of the leading order terms: $\mathfrak{s}_0(\Sigma) = a_C\psi_C + a_P\psi_P$
- $\exists$ unique $a_P \in \mathbb{C} \setminus \{0\}$ because ψ is a 1-D $\mathbb{C}$-vector space $\cong \mathbb{C}$.
- Translation in $\mathbb{R}$ corresponds to multiplication by e^s

OBSTRUCTION BUNDLE *ZOOMING*

"It's the best paper you've never read!" –Ko Honda, 2020

Brought to you by:

Jo Nelson (Rice) and Jacob Rooney (UCLA $\to$ Simons Center)

With contributions by

- Dan Cristofaro-Gardiner (Santa Cruz $\xrightarrow{\text{IAS}}$ Maryland) *tbc
- Chris Gerig (Harvard)
- Ko Honda (UCLA) *tbc
- Michael Hutchings (UC Berkeley)

Join us weekly in September, someday after 10am PDT for the fun!

Family Floer (cf. Bourgeois-Oancea IMRN '17)

$$CC_*^{S^1}(Y,\lambda) = NCC_* \otimes \mathbb{Z}[U], \ \deg(U) = 2, \quad \partial^{S^1} = \partial^{NCH} \otimes 1 + \ldots + \partial_k \otimes U^{-k} + \ldots$$

- Let $\mathfrak{J}$ be an S^1-equivariant $S^1 \times ES^1$ family.
- Fix a perfect Morse f on $BS^1 = S^\infty$.
- Given $\gamma_\pm$ and $x_\pm \in \mathrm{Crit}(f)$, consider pairs (η, u) of grad flow

  $\eta : \mathbb{R} \to ES^1 = \mathbb{CP}^\infty$ asymptotic to points in $\pi^{-1}(x_\pm)$ and

  $u : \mathbb{R} \times S^1 \to \mathbb{R} \times Y, \quad \partial_s u + \mathfrak{J}_{t,\eta(s)} u = 0$, asymptotic to $\gamma_\pm$.
- Let $\mathcal{M}^{\mathfrak{J}}((x_+,\gamma_+),(x_-,\gamma_-))$ be the quotient of this solution set.
- Have evaluation maps and can run *Axiomatic S^1 Morse-Bott framework* (HN '17).

Theorem (Hutchings-N '19 (obg details in progress))

If (Y^{2n-1},λ) is hypertight or (Y^3,λ) is dynamically convex, then for a generic family $\mathfrak{J}$, $(CC_*^{S^1}(Y,\lambda,\mathfrak{J}),\partial^{S^1})$ is a chain complex and $CH_*^{S^1}(Y,\ker\lambda)$ is independent of the choice of λ and $\mathfrak{J}$.

Autonomous simplification (Hutchings-N '19)

Suppose J is λ-compatible on $\mathbb{R} \times Y$ which satisfies the necessary transversality conditions to define ∂^{EGH} and show $\left(\partial^{EGH}\right)^2 = 0$. Then we can use the "autonomous" family $\mathfrak{J} = \{J\}$:

$$\partial^{S^1} = \partial^{NCH} \otimes 1 + \partial_1 \otimes U^{-1},$$

where the "BV operator" ∂_1 is given by

$$\partial_1 \widehat{\alpha} = 0, \quad \partial_1 \check{\alpha} = \begin{cases} d(\alpha)\widehat{\alpha}, & \alpha \text{ good}, \\ 0, & \alpha \text{ bad}. \end{cases}$$

If α and β are good Reeb orbits, then

$$\left\langle \partial^{NCH} \check{\alpha}, \check{\beta} \right\rangle = \left\langle \partial^{EGH} \alpha, \beta \right\rangle, \quad \left\langle \partial^{NCH} \widehat{\alpha}, \widehat{\beta} \right\rangle = \left\langle -\partial^{\reflectbox{EGH}} \alpha, \beta \right\rangle.$$

If α is a bad Reeb orbit, then $\left\langle \partial^{NCH} \check{\alpha}, \check{\beta} \right\rangle = 0$ for any Reeb orbit β;

If β is a bad Reeb orbit, then $\left\langle \partial^{NCH} \widehat{\alpha}, \widehat{\beta} \right\rangle = 0$ for any Reeb orbit α.

$\left\langle \partial^{NCH} \widehat{\alpha}, \check{\beta} \right\rangle = 0$, except when $\alpha = \beta$ is bad, yielding a coefficient of -2.

Full circle

$$CC_*^{S^1}(Y,\lambda,J) = \bigoplus_{\alpha,k\geq 0} \mathbb{Z}\langle \check{\alpha}\otimes U^k, \hat{\alpha}\otimes U^k\rangle, \ \partial^{S^1} = \partial^{NCH}\otimes 1 + \partial_1 \otimes U^{-1}$$

Theorem (Hutchings-N. '19)

When there exists a regular pair (λ, J), meaning ∂^{EGH} is well-defined and $(\partial^{EGH})^2 = 0$, e.g. $\dim(Y) = 3$, λ dynamically convex, J generic, then

$$H_*\left(CC_*^{S^1}(Y,\lambda,J), \partial^{S^1}\right)\otimes\mathbb{Q} = CH_*^{EGH}(Y,\lambda,J).$$

1. Let C'_* be the submodule missing generators of the form $\check{\beta}\otimes 1$ where β is good. Then C'_* is a subcomplex of $CC_*^{S^1}(Y,\lambda)$.
2. $H_*\left(C'\otimes\mathbb{Q}, \partial^{S^1}\otimes 1\right) = 0.$
3. $CH_*^{S^1}(Y,\xi)\otimes\mathbb{Q} = H_*\left(\left(CC_*^{S^1}(Y,\lambda)/C'_*\right)\otimes\mathbb{Q}, \partial^{S^1}\otimes 1\right).$
   A basis for this quotient complex is given by $\check{\alpha}\otimes 1$, for α good. The differential is induced by $\check{\partial}$ & after tensoring w/$\mathbb{Q}$ agrees with ∂^{EGH}.

Corollary (Hutchings-N '19 (modulo obg))

$CH_*^{EGH}(Y^3, \ker\lambda)$ *does not depend on J or dynamically convex λ!*

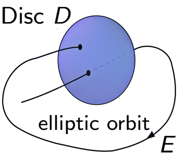
The period doubling bifurcation

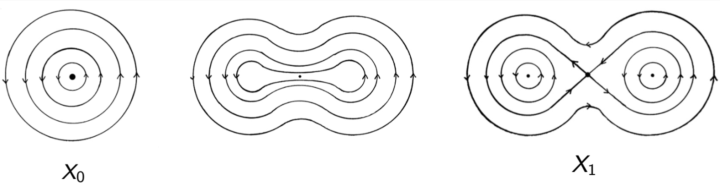
Bifurcations of Reeb orbits occur as we deform λ.

Let $\{\Phi_\tau\}_{\tau\in[0,1]} : D \to D$, be a partial return map

$$\Phi_\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \circ \varphi_\epsilon^{X_\tau},$$

$\varphi_\epsilon^{X_\tau}$ is the ϵ-flow of a 180°-rotation invariant X_τ.





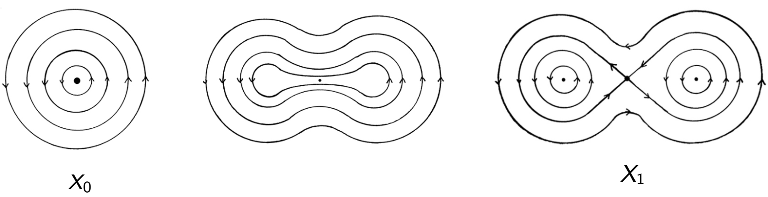
elliptic E	$\rightsquigarrow$	negative hyperbolic h_1
rotation$(E) \sim \frac{1}{2} - \epsilon$		new elliptic orbit e_2

$$\text{period}(h_1) \sim \text{period}(E), \qquad \text{rotation}(h_1) \sim \text{rotation}(E)$$

$$\text{period}(e_2) \sim 2 \cdot \text{period}(E), \qquad \text{rotation}(e_2) \sim 2 \cdot \text{rotation}(E)$$

Bifurcated E has become

$$\Phi_\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \circ \varphi_\epsilon^{X_\tau}.$$



- ($r = 0$) the critical point of X_0 is a fixed point of Φ_0, corresponding to the **elliptic** orbit E,
- ($r = 1$) the central critical point of X_1 is a fixed point of Φ_1, corresponding to the **negative hyperbolic** orbit h_1.
- ($r = 1$) the left and right critical points of X_1 are exchanged by Φ_1, giving rise to a period 2 orbit, aka the **elliptic** orbit e_2.

Compute you will

1. Fix an embedded Reeb orbit γ.
2. Locate other orbits winding k times around tubular nhood N_γ of γ.
3. Compute cylindrical contact homology in this tube.

Local $CH_*^{EGH} = H_*(\mathbb{Q}\langle \text{good orbits winding } k \text{ times around } N_\gamma\rangle, \partial|_{N_\gamma})$.

$(k = 2)$: E^2 is a generator before the bifurcation
e_2 is a generator after.

Even though h_1^2 winds twice around N_γ, it is a bad orbit,
and banished to the Sarlacc pits.

$$\text{Local } CH_*^{EGH}(\lambda_0, N_\gamma, 2) = \begin{cases} \mathbb{Q} & \text{if } * = 0 \quad \text{(generated by } E\text{)}, \\ 0 & \text{otherwise.} \end{cases}$$

Local $CH_*^{S^1}$ sees more, rescuing the bad orbits!

Local $CH_*^{S^1} = H_*(\mathbb{Z}\langle \check{\gamma}, \widehat{\gamma} \mid \gamma \text{ winds } k \text{ times around } N_\gamma \rangle \otimes \mathbb{Z}[U], \partial^{S^1}|_{N_\gamma})$

For λ_0, there is only one orbit in N_γ which winds twice around: E^2

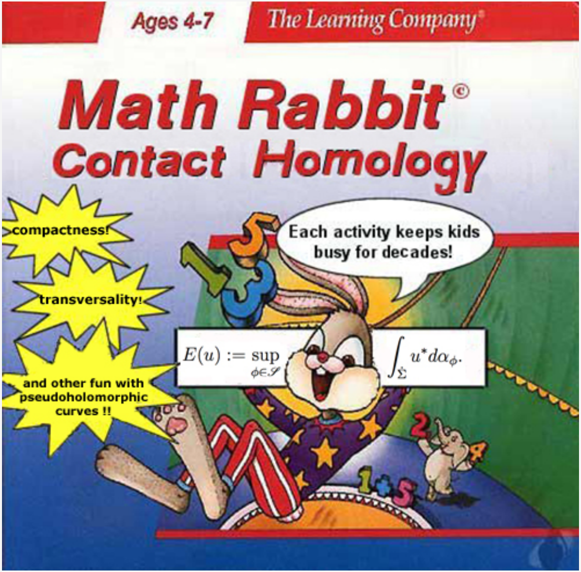
$$CH_*^{S^1}(\lambda_0, N_\gamma) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \quad (\text{generated by } \check{E}), \\ \mathbb{Z}/2 & \text{if } * = 2k+1 \quad (\text{generated by } u^k \hat{E}), \\ 0 & \text{otherwise.} \end{cases}$$

For λ_1, there are two orbits in N_γ which wind twice around: e_2 and h^2

$$CH_*^{S^1}(\lambda_1, N_\gamma) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \quad (\text{generated by } \check{e}_2), \\ \mathbb{Z}/2 & \text{if } * = 2k+1 \quad (\text{generated by } u^k \check{h}^2), \\ 0 & \text{otherwise.} \end{cases}$$

The 2-torsion before the bifurcation sees the bad Reeb orbit that can be created in the bifurcation!

Thanks!


Ages 4-7
The Learning Company
Math Rabbit
Contact Homology
compactness!
transversality!
and other fun with pseudoholomorphic curves !!
Each activity keeps kids busy for decades!
$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\dot{\Sigma}} u^* d\alpha_\phi.$