THE END CURVE THEOREM FOR NORMAL COMPLEX SURFACE SINGULARITIES

WALTER D. NEUMANN AND JONATHAN WAHL

Abstract. We prove the "End Curve Theorem," which states that a normal surface singularity (X, o) with rational homology sphere link Σ is a splice-quotient singularity if and only if it has an end curve function for each leaf of a good resolution tree.

An "end-curve function" is an analytic function $(X, o) \to (\mathbb{C}, 0)$ whose zero set intersects Σ in the knot given by a meridian curve of the exceptional curve corresponding to the given leaf.

A "splice-quotient singularity" (X, o) is described by giving an explicit set of equations describing its universal abelian cover as a complete intersection in $\mathbb{C}^t$, where t is the number of leaves in the resolution graph for (X, o), together with an explicit description of the covering transformation group.

Among the immediate consequences of the End Curve Theorem are the previously known results: (X, o) is a splice quotient if it is weighted homogeneous (Neumann 1981), or rational or minimally elliptic (Okuma 2005).

We consider normal surface singularities whose links are rational homology spheres ($\mathbb{Q}$HS for short). The $\mathbb{Q}$HS condition is equivalent to the condition that the resolution graph Γ of a minimal good resolution be a *rational tree*, i.e., Γ is a tree and all exceptional curves are genus zero.

Among singularities with $\mathbb{Q}$HS links, splice-quotient singularities are a broad generalization of weighted homogeneous singularities. We recall their definition briefly here and in more detail in Section 1. Full details can be found in [20].

Recall first that the topology of a normal complex surface singularity (X, o) is determined by and determines the minimal resolution graph Γ. Let t be the number of leaves of Γ. For $i = 1, \dots, t$, we associate the coordinate function x_i of $\mathbb{C}^t$ to the i–th leaf. This leads to a natural action of the "discriminant group" $D = H_1(\Sigma)$ by diagonal matrices on $\mathbb{C}^t$ (see Section 1).

Under two (weak) conditions on Γ, called the "semigroup" and "congruence" conditions, one can write down an explicit set of $t-2$ equations in the variables x_i, which

---

2000 *Mathematics Subject Classification.* 32S50, 14B05, 57M25, 57N10.

Key words and phrases. surface singularity, splice quotient singularity, rational homology sphere, complete intersection singularity, abelian cover, numerical semigroup, monomial curve, linking pairing.

Research supported under NSF grant no. DMS-0456227 and NSA grant no. H98230-06-1-011.

Research supported under NSA grant no. FA9550-06-1-0063.

defines an isolated complete intersection singularity $(V, 0)$ and which is invariant under the action of D. Moreover the resulting action of D on V is free away from 0, and $(X, o) = (V, 0)/D$ is a normal surface singularity whose minimal good resolution graph is Γ. This (X, o) is what we call a *splice quotient singularity*. Since the covering transformation group for the covering map $V \to X$ is $D = H_1(X - \{o\}) = \pi_1(X - \{o\})^{ab}$, the covering $(V, 0) \to (X, o)$ (branched only at the singular points) may be called the *universal abelian cover* of (X, o). In particular, for a splice quotient singularity, one can write down explicit equations for the universal abelian cover just from the resolution graph, i.e., from the topology of the link.

The link Σ of the singularity (X, o) can be expressed as the boundary of a plumbed regular neighborhood N of the exceptional divisor $E = E_1 \cup \cdots \cup E_n$ in the minimal good resolution $\tilde{X}$ of (X, o). Then each meridian curve of an E_i gives a knot K_i in Σ. A "meridian curve" means the boundary of a small transverse disk to the exceptional divisor E_i. If E_i is the exceptional curve corresponding to a leaf of Γ we call K_i an *end knot*. A (germ of a) smooth complex curve on $\tilde{X}$ which intersects E transversally on such a leaf curve (and hence which cuts out an end-knot on Σ) is called an *end curve*; we also use this name for the image curve in X.

If (X, o) is a splice-quotient singularity as described above, then some power $z_i = x_i^d$ of the coordinate function x_i on V is well defined on $X = V/D$. The zero set in Σ resp. X of z_i is the end knot resp. end curve corresponding to the i–th leaf of Γ (the degree of vanishing may be > 1). We say that the end knot or end curve is *cut out* by the function z_i and that z_i is an *end curve function*.

Our main result is

End Curve Theorem. *Let (X, o) be a normal surface singularity with $\mathbb{Q}$HS link Σ. Suppose that for each leaf of the resolution diagram Γ there exists a corresponding end curve function $z_i\colon (V, o) \to (\mathbb{C}, 0)$ which cuts out an end knot $K_i \subset \Sigma$ (or end curve) for that leaf. Then (X, o) is a splice quotient singularity and a choice of a suitable root x_i of z_i for each i gives coordinates for the splice quotient description.*

An immediate corollary (conjectured in [20] and first proved by Okuma [23]) is that rational singularities and most minimally elliptic singularities (the few with non–$\mathbb{Q}$HS link must be excluded) are splice-quotients. Another direct corollary is the result of [16], that a weighted homogeneous singularity with $\mathbb{Q}$HS link has universal abelian cover a Brieskorn complete intersection. The special case of the End Curve Theorem, when the link is an *integral* homology sphere (so that D is trivial), was proved in our earlier paper [21].

We first proved the End Curve Theorem in summer of 2005, but it has taken a while to write up in what we hope is an understandable form. In the meantime, Okuma resp. Némethi and Okuma in [24, 13, 14] (see also Braun and Némethi [2]) have used this to compute the geometric genus p_g of any splice-quotient, and to prove for splice-quotients the Casson invariant conjecture [17] for singularities with $\mathbb{Z}$HS links (in which case $D = \{1\}$ so $V = X$), as well as the Némethi-Nicolaescu extension [11] of the Casson Invariant Conjecture to singularities with $\mathbb{Q}$HS links.

These results of Némethi and Okuma give topological interpretations of analytic invariants; this is analogous to the fact that for rational singularities some of the important analytic invariants are topologically determined. As happens for rational singularities, the set of resolution graphs that belong to splice quotient singularities is closed under the operations of taking subgraphs and of decreasing the intersection weight at any vertex [14]. It is worth noting, however, that rational singularities did not have explicit analytic descriptions before splice quotients were discovered; even the fact that their universal abelian covers are complete intersections was unexpected until it was conjectured in [20] (see also [22]).

Of course, unlike rationality, the property of being a splice-quotient is not topologically determined—for example, splice quotients, as quotients of Gorenstein singularities, are necessarily $\mathbb{Q}$–Gorenstein, which is generally a very special property within a topological type. Even more, "equisingular deformations" of very simple splice quotients need not be of this type (see Example 10.4).

We once over-optimistically conjectured that $\mathbb{Q}$–Gorenstein singularities with $\mathbb{Q}$HS links would have complete intersection universal abelian covers, and although this is false in general [12], we see it is true for a large class of singularities. There is a natural arithmetic analog. A standard "dictionary" that developed out of proposals of Mazur and others pairs 3–manifolds with number fields, knots with primes, and so on. A natural analog of universal abelian covers of QHS links belonging to complete intersections would be that the ring of integers of the Hilbert class field of a number field K be a complete intersection over $\mathbb{Z}$. This is true, proved by de Smit and Lenstra [5]. The analogy between splice singularities and Hilbert class fields is enticing, since it is a significant open problem to compute Hilbert class fields, while the explicit splice singularity description is easily computed from the resolution diagram.

We summarise the proof of the End Curve Theorem in Section 2 after first recalling the theory of splice quotient singularities in Section 1. We complete the proof in Section 8. Some applications and examples are discussed in the final section 10.

Some of the ingredients in our proof could be of independent interest. We need an extension to the equivariant reducible case of the theory of numerical semigroups and monomial curves developed by Delorme, Herzog, Kunz, Watanabe and the authors [4, 8, 9, 21, 28]. The necessary parts of this theory are developed in sections 3–7. Some topological results about knots in $\mathbb{Q}$–homology spheres and their linking numbers and Milnor numbers are collated in Section 9.

1. Splice quotient singularities

We recall here the detailed construction of splice-quotient singularities. For full details see [20].

Let $(\bar{X}, o) \subset (\mathbb{C}^N, o)$ be a normal surface singularity whose link $\Sigma = \bar{X} \cap S_\epsilon^{2N-1}$ is a $\mathbb{Q}$HS. Equivalently the minimal good resolution resolves the singularity by a tree of rational curves. Let Γ be the resolution graph. In some cases we can construct directly from Γ singularities which have the same link as $\bar{X}$ (but might well be analytically distinct).

We denote by $A(\Gamma)$ the intersection matrix of the exceptional divisor (we say "intersection matrix of Γ"); this is the negative definite matrix whose diagonal entries are the weights of the vertices of Γ and whose off-diagonal entries are 1 or 0 according as corresponding vertices of Γ are connected by an edge or not. The *discriminant group* $D(\Gamma)$ is the cokernel of $A(\Gamma)\colon \mathbb{Z}^n \to \mathbb{Z}^n$. There is a canonical isomorphism $D(\Gamma) \cong H_1(\Sigma; \mathbb{Z})$ (if Σ were not a $\mathbb{Q}$HS one would have $D(\Gamma) \cong \operatorname{Tor} H_1(\Sigma; \mathbb{Z})$). The order $|D| = \det(-A(\Gamma))$ of $D(\Gamma)$ is called the *discriminant* of Γ.

1.1. **Splice diagram.** We shall denote by Δ the splice diagram corresponding to Γ. We recall its construction. If one removes from Γ a vertex v and its adjacent edges then Γ breaks into subgraphs Γ_{vj}, $j = 1, \dots, \delta_v$, where δ_v is the valency of v. We weight each outgoing edge at v by the discriminant of the corresponding subgraph; these are the "splice diagram weights" (the reader may wish to refer to the illustrative example in subsection 1.6). The graph Γ with all splice diagram weights added and with the self-intersection weights deleted is called the *maximal splice diagram*. One can still recover Γ from it. If one now drops the splice diagram weights around vertices of valency ≤ 2 and then suppresses all valency 2 vertices to get a diagram with only leaves (valency 1) and nodes (valency ≥ 3) one gets the *splice diagram* Δ. The splice diagram Δ no longer determines Γ in general.

For the purpose of this paper it is convenient to have a version of the splice diagram in which we do not discard the splice diagram weights at leaves. We call this the *splice diagram with leaf weights* and denote it also by Δ.

Definition 1.1. For two vertices v and w of Γ the *linking number*. ℓ_{vw} is the product of splice diagram weights adjacent to but not on the shortest path from v to w in Γ. If $v = w$ this means the product of splice diagram weights at v. (The name comes from the fact that ℓ_{vw} is $|D|$ times the linking number of the knots in Σ corresponding to v and w, see Proposition 9.1.)

The matrix (ℓ_{vw}) is the adjoint of $-A(\Gamma)$ ([7] Lemma 20.2):

$$(\ell_{vw}) = \operatorname{Adj}(-A(\Gamma)) = -|D| A(\Gamma)^{-1}$$

Note that for vertices v and w of Δ, ℓ_{vw} can be computed using only weights of Δ, except that leaf weights are also needed if $v = w$ is a leaf.

1.2. **Action of the discriminant group on $\mathbb{C}^t$.** Let v_i, $i = 1, \dots, t$ be the leaves of Γ or Δ and associate a coordinate Y_i of $\mathbb{C}^t$ with each leaf. Since $D = \mathbb{Z}^n / A(\Gamma)\mathbb{Z}^n$ with $\mathbb{Z}^n = \mathbb{Z}^{\mathrm{vert}(\Gamma)}$, each vertex v of Γ determines an element $e_v \in D$. There is a non-degenerate $\mathbb{Q}/\mathbb{Z}$–valued bilinear form on D satisfying

$$e_v \cdot e_w = -\ell_{vw}/|D|\,,$$

the vw–entry of $A(\Gamma)^{-1}$.

We get an action of D on $\mathbb{C}^t$ by letting the element $e \in D$ act via the diagonal matrix

$$\operatorname{diag}(e^{2\pi i\, e\cdot e_{v_1}}, \dots, e^{2\pi i\, e\cdot e_{v_t}})\,.$$

The elements e_{v_i}, $i = 1, \dots, t$ generate D (in fact any $t-1$ of them do, see [20] Proposition 5.1). We thus only need the splice diagram with leaf weights Δ to determine this action.

1.3. **Splice equations.** In contrast to the action of D on $\mathbb{C}^t$, only the splice diagram Δ and not leaf weights are needed to discuss splice equations. We will write down $t-2$ equations in the variables $Y_1, \dots, Y_t$, grouped into $\delta_v - 2$ equations for each node v of Δ. These $\delta_v - 2$ equations are weighted homogeneous with respect to weights determined by v. We first describe these weights.

Fix a node v of Δ. The *v-weight* of Y_i is ℓ_{vv_i}. We will write down equations of total weight ℓ_{vv}. Number the outgoing edges at v by $j = 1, \dots, \delta_v$. For each j, a monomial M_{vj} of total weight ℓ_{vv}, using only the variables Y_j that are beyond the outgoing edge j from v, is called an *admissible monomial*. The existence of admissible monomials for every edge at every node is the *semigroup condition* of [20]. Assuming this condition, choose one admissible monomial M_{vj} for each outgoing edge at v. Then *splice diagram equations* for the node v consist of equations of the form

$$\sum_{j=1}^{\delta_v} a_{vij} M_{vj} + H_{vi} = 0 \quad i = 1, \dots, \delta_v - 2,$$

where

- all maximal minors of the $(\delta_v - 2) \times \delta_v$ matrix (a_{vij}) have full rank;
- H_{vi} is an optional extra summand in terms of monomials of v–weight $> \ell_{vv}$ (most generally, a convergent power series in such monomials).

Choosing splice diagram equations for each node gives exactly $t-2$ equations, called a *system of splice diagram equations*. In Theorem 2.6 of [20], it is shown that they determine a 2-dimensional complete intersection V with isolated singularity at the origin.

Claim. *The link of this singularity has the topology of the universal abelian cover of the singularity link determined by Γ. In particular, this topology is determined by the splice diagram Δ alone.*

For Γ that admit splice-quotient singularities (i.e., the equations can be chosen D–equivariantly), this is in [20]. The second sentence has been proved in full generality by Helge Pedersen [25], but a complete proof of the first sentence for general Γ has not yet been written up, so it should be considered conjectural.

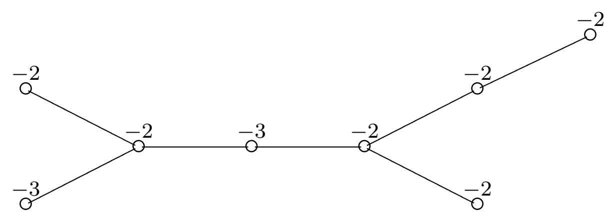
1.4. **Splice-quotient singularities.** Suppose now that we can choose a system of splice diagram equations as above, which are additionally equivariant with respect to the action of D; this is a combinatorial condition on Γ, called the *congruence condition* ([18, 20]). Then

Theorem ([20]). *D acts freely on $V - \{0\}$ and the quotient $(X, o) := (V, 0)/D$ is a normal surface singularity whose resolution graph is Γ; moreover, $(V, 0) \to (X, o)$ is the universal abelian cover. We call (X, o) a* splice-quotient singularity.

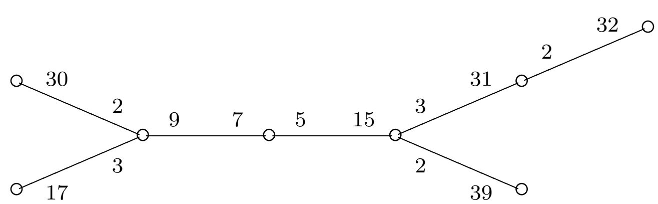
Theorem 10.1 of [20] says that the class of splice-quotient singularities is natural, in the sense that it does not depend on the choice of which admissible monomials M_{vj} one chooses to use (so long as they are chosen equivariantly for the action of D). A change in choice can be absorbed in the extra higher order summands of the splice equations.

1.5. **Reduced weights.** The v–weights of the Y_i used to define splice equations may have a common factor, so in practice one should use reduced v-weights that divide out this common factor. Precisely, if v is a node the *reduced v–weight* of the variable Y_j is the v-weight ℓ_{vv_j} of Y_j divided by the GCD of the v–weights of all the Y_i's.

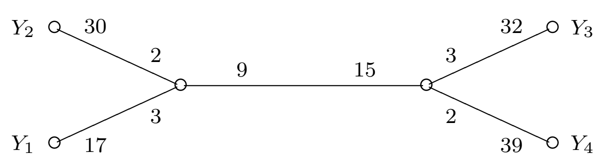
1.6. **Example.** Consider the resolution graph

$\Gamma =$ 

Its maximal splice diagram is



and its splice diagram with leaf weights is

$\Delta =$ 

We have also shown the association of $\mathbb{C}^4$–coordinates $Y_1, \dots, Y_4$ with leaves of Δ.

The discriminant group D is cyclic of order 33. Its action on $\mathbb{C}^4$ can be read off from the splice diagram and leaf weights, and is generated by the four diagonal matrices corresponding to the four leaves ($\zeta = e^{-2\pi i/33}$):

$$\begin{aligned} e_1 &= \langle \zeta^{17}, \zeta^9, \zeta^4, \zeta^6 \rangle \\ e_2 &= \langle \zeta^{9}, \zeta^{30}, \zeta^6, \zeta^9 \rangle \\ e_3 &= \langle \zeta^{4}, \zeta^6, \zeta^{32}, \zeta^{15} \rangle \\ e_4 &= \langle \zeta^{6}, \zeta^9, \zeta^{15}, \zeta^{39} \rangle \,. \end{aligned}$$

In this case e_1 clearly suffices to generate the group.

Calling the left node v, the v–weights of the variables Y_1, Y_2, Y_3, Y_4 are read off from the splice diagram as 18, 27, 12, and 18. The GCD is 3, so the reduced weights are 6, 9, 4, 6. The total reduced weight for that node is $54/3 = 18$, so admissible monomials for the two edges departing v to the left must be Y_1^3 and Y_2^2. For the edge going right there are two monomials of the desired weight: $Y_3^3 Y_4$ and Y_4^3. One checks that all these monomials transform with the same character under the D–action (e_1 acts on each by ζ^{18}), so we can choose either one of $Y_3^3 Y_4$ and Y_4^3; we choose Y_4^3. The number of equations to write down for this node is $\delta_v - 2 = 1$. We choose "general coefficients" $1, 1, 1$ and write down

$$Y_1^3 + Y_2^2 + Y_4^3 = 0 \,.$$

For the right node v' the reduced v'–weights of Y_1, Y_2, Y_3, Y_4 are 4, 6, 10, 15, and total reduced weight is 30. Hence admissible monomials are Y_3^3 and Y_4^2 for the edges going right, and a choice of Y_2^5, $Y_1^3 Y_2^3$, or $Y_1^6 Y_2$ for the leftward edge; again all choices are D–equivariant (e_1 acts by ζ^{12} on all). We make our choices and write a second equation

$$Y_2^5 + Y_3^3 + Y_4^2 = 0 \,.$$

The results of [20] tell us that the two equations define a normal complete intersection singularity $(V, 0)$, that the action of $D = \mathbb{Z}/33$ on it is free off the singular point, and the quotient is a normal singularity (X, o) with resolution graph Γ. A mental calculation shows that our coefficients are in fact general; any other choice can be reduced to these by diagonal coordinate transformation of $\mathbb{C}^4$.

The End Curve Theorem tells us that if a singularity has this resolution graph and has end curve functions for its four leaves, then it is a higher weight deformation of the above example, i.e., (possibly) deformed by adding higher weight terms equivariantly in the two equations.

2. Overview of the proof

The End Curve Theorem was proved in [21] when the link Σ is a $\mathbb{Z}$HS (so there is no group action). We first outline the proof in this case. We thus assume we have an end curve function z_i on X associated with each leaf of Γ. By replacing z_i by a suitable root if necessary one can assume its zero-set is not only irreducible but also reduced (this uses that Σ is a $\mathbb{Z}$HS–cf. Section 8). The claim then is that these functions z_i generate the maximal ideal of the local ring of X at o and that X is a complete intersection given by splice equations in these generators. The main step is to show that each curve $C_i = \{z_i = 0\} \subset X$ is a complete intersection curve.

2.1. Consider the curve C_1 given by $z_1 = 0$. For $j > 1$ the function z_j, restricted to C_1, has degree of vanishing ℓ_{1j} at o, so the numbers ℓ_{1j}, $j = 2, \dots, t$, generate a numerical semigroup $S \subset \mathbb{N}$ which is a sub-semigroup of the full value semigroup $V(C_1)$ (the semigroup generated by all degrees of vanishing at o of functions on C_1). The δ–invariant $\delta(S)$ and the δ–invariant $\delta(C_1) := \delta(V(C_1))$ therefore satisfy

$$2\delta(C_1) \leq 2\delta(S) \tag{1}$$

(the δ-invariant counts the number of gaps in the semigroup, i.e., the size of $\mathbb{N} - S$).

2.2. Classical theory of numerical semigroups, developed further in Theorem 3.1 of [21], shows via an induction over subdiagrams of Γ that

$$2\delta(S) \leq 1 + \sum_{v \neq 1} (\delta_v - 2)\ell_{1v}\,, \tag{2}$$

with equality if and only if the semigroup condition holds for Δ at every vertex and edge pointing away from 1. Moreover, if equality holds, the monomial curve for this semigroup is a complete intersection, with maximal ideal generated by $z_2, \dots, z_n$.

2.3. The δ-invariant of a curve is determined by Milnor's μ invariant, which in our case is a topological invariant, computable in terms of the splice diagram (Sect. 11 of [7], see also Lemma 9.4) as

$$2\delta(C_1) = 1 + \sum_{v \neq 1} (\delta_v - 2)\ell_{1v}\,. \tag{3}$$

2.4. Comparing (1), (2), and (3), we must have equality in (1) and (2). It follows that $S = V(C_1)$. Moreover, by step 2.2, the monomial curve for S is a complete intersection with maximal ideal generated by $z_2, \dots, z_t$. Since C_1 is a positive weight deformation of this monomial curve, C_1 is also a complete intersection, with maximal ideal generated by $z_2, \dots, z_t$. It follows that (X, o) is a complete intersection with maximal ideal generated by $z_1, \dots, z_t$.

2.5. Repeating the above for all leaves $i = 1, \dots, t$ shows that the semigroup conditions hold. One can thus choose admissible monomials for every node, and it is then not hard to deduce that equations of splice type hold. Finally, one deduces that (X, o) is defined by these equations, completing the proof.

2.6. **General case.** We must now describe how the above proof is modified when Σ is not a $\mathbb{Z}$HS, so D is non-trivial. It is not hard to show that the functions x_i (appropriate roots of the end curve functions z_i) are defined on the universal abelian cover V of X. But the curve $C_1 = \{x_1 = 0\}$ is no longer an irreducible curve, so the theory of numerical semigroups of 2.2 above cannot be used.

We extend the theory of value semigroups and the appropriate results concerning them to reducible curves that have an action of a group D that is transitive on components; the value semigroup is now a subsemigroup of $\mathbb{N} \times \hat{D}$, where $\hat{D}$ is the character group of D. The inductive argument of step 2.2 must now deal with subsemigroups of a semigroup $\mathbb{N} \times \hat{D}$, where $\hat{D}$ changes at each step of the induction. Moreover, we must show in the end that this extension allows us to deduce, as before, that V is a complete intersection with maximal ideal generated by the x_i, that the semigroup conditions hold, which guarantee that admissible monomials exist, but also that the congruence conditions hold, allowing us to choose the monomials D–equivariantly. Once this is done, one again deduces that equations of splice type hold on V. Finally, using the main theorem of [20] one deduces that V is defined by these equations and that $X = V/D$, thus completing the proof.

The necessary theory of reducible curves and their value semigroups is developed in sections 3–7 and the proof is completed in section 8. Some needed topological computations are collected together in section 9.

3. D–curves

A *D–curve* is a reduced curve germ (C, o) on which a finite abelian group D acts effectively (i.e., $D \to \mathrm{Aut}(C)$ is injective) and transitively on the set of branches. Denote the branches (C_i, o), $i = 1, \dots, r$. If $H \subset D$ is the subgroup stabilizing (any) one branch, then $F := D/H$ acts simply transitively on the set of branches of C (recall that any effective transitive action of an abelian group is simply transitive). D also acts on the normalization $\tilde{C}$ of C, a disjoint union of r smooth curves $\tilde{C}_i$. On the level of analytic local rings, D acts on $(R, \mathfrak{m})$ (the local ring of C), on the direct sum of its branches $R/\mathcal{P}_i$, and on its normalization $\tilde{R} = \oplus_{i=1}^{r} \tilde{R}_i$, where each $\tilde{R}_i$ is a convergent power series ring $\mathbb{C}\{\{y_i\}\}$. One may assume that the parameters y_i form one D–orbit (up to multiplication by scalars). H then acts on each y_i via the same character, independent of i.

The natural valuation v_i on $\tilde{R}_i$ induces one on $\tilde{R}$ by value on the $\tilde{R}_i$–component; the induced valuation on R is given by order of vanishing of a function $f \in R$ along the branch C_i. (Of course, define $v_i(0) = \infty$.) D permutes the branches and hence the valuations, with

$$v_i(\sigma(f)) = v_{\sigma(i)}(f) \text{ for all } \sigma \in D.$$

Thus, if f is an eigenfunction for the D–action, then $v_i(f) = v_j(f)$ for all i, j; we then just write $v(f)$.

Definition 3.1. Denote the character group of D by $\hat{D}$. The *value semigroup* $\mathcal{S}(C)$ of the D–curve (C,o) is the subsemigroup of $\mathbb{N}\times\hat{D}$ consisting of all pairs $(v(f),\chi)$ with $\chi\in\hat{D}$ and $f\in R$ a D–eigenfunction with character χ.

$\tilde{R}$, and hence R, has a natural D–filtration given by the ideals

$$J_n=\{f \mid v_i(f)\geq n \text{ for all } i\}.$$

We denote by $\tilde{R}'$ and R' the associated gradeds for this filtration. R' is the graded ring of a reduced weighted homogeneous curve C' (thus a union of "monomial curves"), again with an effective action of D acting transitively on the r branches. Each $J_n\tilde{R}/J_{n+1}\tilde{R}$ is for $n\geq 0$ a vector space of dimension r and the associated graded ring $\tilde{R}'=Gr_J\tilde{R}$ is a direct sum of r polynomial rings in one variable (of degree 1).

Recall that the *delta invariant* $\delta(C)=\delta(R)$ is the length of the R–module $\tilde{R}/R$, which in this case is its dimension as a $\mathbb{C}$–vector space.

Proposition 3.2. *$\delta(R)=\delta(R')$ and $\mathcal{S}(C)=\mathcal{S}(C')$. The value semigroup $\mathcal{S}(\tilde{C})$ of the normalization has exactly r elements in each degree. The delta-invariant of R (or C) is equal to the size of the complement of $\mathcal{S}(C)$ in $\mathcal{S}(\tilde{C})$.*

Proof. The delta invariant $\delta(R)$ is finite, so for n sufficiently large, $J_nR=J_n\tilde{R}$. Computing $\delta(R)$ by summing over graded pieces of R and $\tilde{R}$, one sees that $\delta(R)=\delta(R')$ (this is a general fact about reduced curves).

$\tilde{R}$ splits (as a vector space) as a sum $\tilde{R}=\bigoplus\tilde{R}_\chi$ over the characters of D and this splitting is compatible with the filtration $\{J_n\}$ (in the sense that $J_n\,\tilde{R}_\chi=(J_n\tilde{R})_\chi$ for any D–character χ). Thus $\mathcal{S}(\tilde{R})=\mathcal{S}(\tilde{R}')$. The same argument applies to show $\mathcal{S}(R)=\mathcal{S}(R')$.

The following lemma shows that the splitting by characters splits each $J_n\tilde{R}/J_{n+1}\tilde{R}$ into r 1–dimensional summands (and trivial summands for the remaining $|D|-r$ characters), and hence also splits the subspaces $J_nR/J_{n+1}R$ into 1–dimensional summands. Since we can compute the delta invariant $\delta(R')$ by counting these summands, the proposition then follows. □

Consider the natural exact sequence of character groups

$$0\to\hat{F}\to\hat{D}\to\hat{H}\to 0.$$

Lemma 3.3. *The D–eigenfunctions of $J_n\tilde{R}/J_{n+1}\tilde{R}$ $(n>0)$ all have the same H–character. A collection of them are $\mathbb{C}$–linearly independent if and only if their D–characters are distinct. They form a basis of $J_n\tilde{R}/J_{n+1}\tilde{R}$ if and only if their D–characters form exactly one $\hat{F}$–coset of $\hat{D}$.*

Proof. Diagonalize the action of D on the r–dimensional space $J_1\tilde{R}/J_2\tilde{R}$. As mentioned above, H acts via one character for this space, so the D–characters involved form one fiber of the map $\hat{D}\to\hat{H}$. Choose any D–eigenfunction $g\in J_1/J_2$. Then multiplication by g^{n-1} maps $J_1\tilde{R}/J_2\tilde{R}$ isomorphically onto $J_n\tilde{R}/J_{n+1}\tilde{R}$, shifting the characters by $(n-1)$ times the character of g. So, r distinct D–characters appear in $J_n\tilde{R}/J_{n+1}\tilde{R}$. ☐

4. Weighted homogeneous D–curves

Continue the setup of the last section, but assume at the start that $R = R'$ is a positively graded ring, so that C is a weighted homogeneous curve with a D–action. We can choose a (not necessarily minimal) set of homogeneous generators x_i, $i = 1, \dots, m$, of R, of weights ℓ_i (necessarily with greatest common divisor 1), so that D acts on x_i via a character χ_i. The value semigroup $\mathcal{S}(C)$ is then generated by the elements (ℓ_i, χ_i).

The x_i embed $C \subset \mathbb{C}^m$. By scaling the x_i if necessary we may assume $(1, 1, \dots, 1) \in C$. Then one of the r branches of C is the monomial curve $\{(u^{\ell_1}, \dots, u^{\ell_m}) : u \in \mathbb{C}\}$.

There are two subgroups of $(\mathbb{C}^*)^m$ which act on C via multiplication in the entries: D, embedded via $(\chi_1, \dots, \chi_m)$; and $\mathbb{C}^*$, via $u \mapsto (u^{\ell_1}, \dots, u^{\ell_m})$.

Definition 4.1. $G \subset (\mathbb{C}^*)^m$ is the subgroup generated by D and $\mathbb{C}^*$, embedded as above.

D acts transitively on the branches of C, so G acts simply transitively on $C - \{0\}$. Thus, we may view $G = C - \{0\}$, while C is the closure of G in $\mathbb{C}^m$.

Consider next $H := D \cap \mathbb{C}^*$, the subgroup of D which stabilizes the branch given by the monomial curve through $(1, 1, \dots, 1)$. As a subgroup of $\mathbb{C}^*$, it is cyclic, and the quotient group $F := D/H$ has order r, the number of branches of C. Using the inclusion map of H to D, and the negative of its inclusion map to $\mathbb{C}^*$, we deduce exact sequences:

$$1 \to H \to \mathbb{C}^* \times D \to G \to 1\,, \tag{4}$$

$$1 \to \mathbb{C}^* \to G \to F \to 1\,. \tag{5}$$

Since $\mathbb{C}^*$ is a divisible group, the exact sequence (5) splits (but not canonically). This means that there exists a subgroup of G (though not necessarily of D), isomorphic to F, which acts on the curve and acts simply transitively on the set of branches. Dualizing (5), we get a natural exact sequence of groups of characters

$$0 \to \hat{F} \to \hat{G} \to \mathbb{Z} \to 0\,, \tag{6}$$

which splits non-canonically to give an abstract isomorphism

$$\hat{G} \cong \mathbb{Z} \oplus \hat{F}\,.$$

Dualizing (4) gives a natural inclusion

$$\hat{G} \subset \mathbb{Z} \oplus \hat{D}\,.$$

The inclusion $G \subset (\mathbb{C}^*)^m$ induces a surjection $\mathbb{Z}^m \to \hat{G}$, giving m natural characters $\bar{\chi}_k \in \hat{G}$, $k = 1, \dots, m$, which generate $\hat{G}$. The image of $\bar{\chi}_k$ in $\mathbb{Z} \oplus \hat{D}$ is (ℓ_k, χ_k). So any character of G may be written additively as[1]

$$\hat{I} := \sum_{k=1}^{m} i_k \bar{\chi}_k,$$

where $I = (i_1, \dots, i_m) \in \mathbb{Z}^m$, and I and I' represent equal characters $\hat{I} = \hat{I}' \in \hat{G}$ if and only if their weights and D–characters are equal.

Definition 4.2. A character $X \in \hat{G}$ is called *non-negative* if it is a linear combination of the $\bar{\chi}_k$ with non-negative coefficients. Thus, the set of non-negative characters forms the value semigroup $\mathcal{S}(C)$ as a subsemigroup of $\hat{G} \subset \mathbb{Z} \times \hat{D}$. The value semigroup $\mathcal{S}(\hat{C})$ of the normalization is $\pi^{-1}(\mathbb{N}) \subset \hat{D}$. So $\delta(C)$ is the number of elements of $\pi^{-1}(\mathbb{N})$ that are not non-negative characters.

The x_i embed $C \subset \mathbb{C}^m$. Let $I = (i_1, \dots, i_m)$ be an m–tuple, with all $i_k \geq 0$. Each monomial $x^I := x_1^{i_1} x_2^{i_2} \dots x_m^{i_m}$ is homogeneous of weight $\sum i_k \mathrm{wt}(x_k)$ and transforms via the character $\chi_I := \prod \chi_k^{i_k} \in \hat{D}$. Introducing weighted coordinates $Y_1, \dots, Y_m$ in $\mathbb{C}^m$, we can summarise results of this and the previous section.

Proposition 4.3. *x^I equals a constant times $x^{I'}$ if and only if their weights and D–characters are equal. With the x_i scaled so that $(1, 1, \dots, 1) \in C' \subset \mathbb{C}^m$, the D–curve $C \subset \mathbb{C}^m$ is the closure of the G–orbit of $(1, 1, \dots, 1)$. The ideal of $C \subset \mathbb{C}^m$ is generated by all differences $Y^I - Y^{I'}$ of monomials with equal G–characters: $\hat{I} = \hat{I}'$.*

The finite abelian "group of components" $F = G/\mathbb{C}^*$ is dual (hence isomorphic) to the torsion subgroup of $\hat{G}$.

The delta-invariant $\delta(R) = \delta(R')$ is computed by summing, over all $n \geq 0$, r minus the number of monomial elements x^I of weight n and distinct D–characters. □

We will need the following proposition.

Proposition 4.4. *The inclusion map $D \subset G$ gives a surjection $\hat{G} \twoheadrightarrow \hat{D}$, whose kernel is the cyclic group generated by a distinguished character $\hat{Q}$, whose weight is the order of H, i.e. $|D|/|F|$. Its image in $\mathbb{Z} \oplus \hat{D}$ is $(|H|, 0)$.*

Proof. This is immediate from the sequence of kernels of the surjection of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \hat{G} \arrow[r] \arrow[d, two heads] & \mathbb{Z} \oplus \hat{D} \arrow[r] \arrow[d, two heads] & \hat{H} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \hat{D} \arrow[r] & \hat{D} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$ □

[1]We write $\hat{G}$ additively in the following even though it is more natural to think of $\hat{D}$ as a multiplicative group.

This proposition can be restated as follows: Suppose $\hat{I}$ represents a character of G for which the image in $\mathbb{Z} \oplus \hat{D}$ is of the form $(\ell', 0)$; then ℓ' is a multiple of $|D|/|F|$, and more precisely $\hat{I}$ is a multiple of $\hat{Q}$. It will be of special interest to us when $\hat{Q}$ is a non-negative character, i.e., $\hat{Q} \in \mathcal{S}$. In this case we can represent $\hat{Q}$ by a non-negative tuple Q, and the monomial Y^Q will play a special rôle for us.

The following Corollary of Proposition 4.3 clarifies the relationship between the value semigroup and weighted homogeneous D–curve.

Corollary 4.5. *A weighted homogeneous D–curve C is determined, up to isomorphism of weighted homogeneous curves, by its value semigroup $\mathcal{S}(C)$. Its graded ring R is the semigroup ring $\mathbb{C}[\mathcal{S}(C)]$. The D–action is determined by the surjection $\mathcal{S} \twoheadrightarrow \hat{D}$.*

A commutative semigroup $\mathcal{S}$ and surjection $\chi\colon \mathcal{S} \twoheadrightarrow \hat{D}$ determines a weighted homogeneous D–curve if and only if $\mathcal{S}$ satisfies the cancellation law and is not a group, and the induced homomorphism of its group of quotients to $\hat{D}$ has infinite cyclic kernel.

Proof. The first paragraph is just a reinterpretation of the first part of Proposition 4.3: The isomorphism $\mathbb{C}[\mathcal{S}(C)] \to R$ is given by $\hat{I} \mapsto Y^I$. The grading is determined by the map $\mathcal{S}(C) \to \mathbb{N}$ given by mapping $\mathcal{S}(C)$ into its group of quotients $\hat{G}$ and then factoring by the torsion subgroup of $\hat{G}$. D acts on elements $X \in \mathcal{S}(C)$—i.e., generators of $\mathbb{C}[\mathcal{S}(C)]$—by $g \cdot X = \chi(X)(g)X$.

We won't use the second part of the corollary so we leave it to the reader. $\square$

5. An instructive example

Let $n_1, \dots, n_m \geq 2$ be integers, and consider the curve $C \subset \mathbb{C}^m$ defined by

$$Y_1^{n_1} = Y_i^{n_i}, \quad i = 2, \dots, m.$$

Write $\mathcal{N} = n_1 \dots n_m$, $\mathcal{N}_i = \mathcal{N}/n_i$, $s = \mathrm{GCD}(\mathcal{N}_1, \dots, \mathcal{N}_m)$.

Example 5.1. With the above notation,

1. C is a reduced weighted homogeneous curve with (relatively prime) weights $\mathcal{N}_1/s, \dots, \mathcal{N}_m/s$; a particular branch C_1 is the irreducible monomial curve $(u^{\mathcal{N}_1/s}, \dots, u^{\mathcal{N}_m/s})$.
2. Let $G := C - \{0\} \subset (\mathbb{C}^*)^m$; G is a subgroup of $(\mathbb{C}^*)^m$, acting on C by coordinate-wise multiplication, and simply transitively on $C - \{0\}$.
3. The connected component of the identity of G is $C_1 - \{0\}$, a copy of $\mathbb{C}^*$.
4. $G/\mathbb{C}^*$ is a finite abelian group whose order r (the number of components of C) equals s (the GCD above).
5. There is a (non-canonical) splitting $G = \mathbb{C}^* \times D'$, so that C is a D'–curve, and D' acts simply transitively on the set of branches.

(6) The Milnor number μ and delta-invariant δ of C satisfy

$$\mu - 1 = 2\delta - r = (m-1)\mathcal{N} - \sum_{i=1}^{m} \mathcal{N}_i.$$

Proof. Statements (1)–(3) and (5) are obvious. We show (4). For any $a, a' \neq 0$, one has that the cardinality $|C \cap \{Y_1 = a\}| = n_2 \dots n_m = \mathcal{N}_1$, while $|C_1 \cap \{Y_1 = a'\}|$ is the weight $\mathcal{N}_1/s$ of Y_1. Since G acts transitively on $C - \{0\}$, it follows that any branch intersects $\{Y_1 = a'\}$ in $\mathcal{N}_1/s$ points, so that there must be s branches.

An alternative proof of (4) that does slightly more is as follows. Since $\mathbb{C}^*$ is a divisible group, one has a splitting $G = \mathbb{C}^* \times D'$, where $D' \subset G$ is a finite subgroup mapping isomorphically onto $G/\mathbb{C}^*$. Then C is a D'–curve, and D' acts simply transitively on the set of branches. As in Proposition (4.3), D' is isomorphic to the torsion subgroup of the group with generators $e_1, \dots, e_m$, and relations $n_1 e_1 = n_i e_i, i = 2, \dots, m$. This implies again that s is the number of branches, and also allows calculation of the elementary divisors of D' via the ideals of minors of the matrix associated to the relations.

The familiar formula [3] relating Milnor number and delta invariant ($\mu = 2\delta - r + 1$), plus the calculation of μ in e.g., [10] (or see Lemma 9.4), yields the last assertion. □

6. D–curves from rooted resolution diagrams

Let Γ be a resolution diagram which is a tree, with distinguished leaf $*$, called the *root*; we say $(\Gamma, *)$ is a *rooted tree*. These data will give rise to $C(\Gamma, *)$, a reduced and weighted homogeneous D–curve, where $D := D(\Gamma)$ is the discriminant group associated to Γ (i.e., the cokernel of the intersection matrix of Γ). We derive key properties of C by its inductive relationship with subtrees (with fewer nodes).

For the moment, assume Γ is not a string, i.e., has at least one vertex of valency ≥ 3 (this is not strictly necessary). Order the non-root leaves $w_1, \dots, w_m$, and index them by variables $Y_1, \dots, Y_m$. The m linking numbers (Definition 1.1) $\ell_i := \ell_{w_i,*}$ from the non-root leaves to $*$ give a set of (non-reduced) weights for the variables Y_i; one gets reduced weights by dividing by

$$s := \mathrm{GCD}(\ell_1, \dots, \ell_m). \tag{7}$$

These weights give a copy of $\mathbb{C}^*$ in the diagonal group $(\mathbb{C}^*)^m$. (The linking numbers, and hence s, can be read off from the splice diagram Δ associated to Γ.)

As described in subsection 1.2, every vertex v of Γ gives an element $e_v \in D$, and D is generated by the elements e_{w_i}, $i = 1, \dots, m$. Moreover, D has a natural non-degenerate $\mathbb{Q}/\mathbb{Z}$–valued bilinear form $(e, e') \mapsto e \cdot e'$, induced by the intersection pairing for Γ, and one has an embedding of D into $(\mathbb{C}^*)^m$, where the image of $e \in D$ is the m–tuple whose jth entry is the root of unity $e^{2\pi i\, e \cdot e_{w_j}}$. Recall from subsection 1.2 that for vertices w, w' of Γ, one has

$$e_w \cdot e_{w'} = -\ell_{ww'}/|D|\,, \tag{8}$$

which for nodes and leaves can be read off from $|D|$ and the splice diagram Δ with leaf weights.

As before, define $G \subset (\mathbb{C}^*)^m$ to be the subgroup generated by D and the $\mathbb{C}^*$.

Definition 6.1. The D–curve $C := C(\Gamma, *)$ associated to the rooted diagram $(\Gamma, *)$ is the closure of G in $\mathbb{C}^m$, or equivalently the closure in $\mathbb{C}^m$ of the G–orbit of $(1, 1, \dots, 1)$.

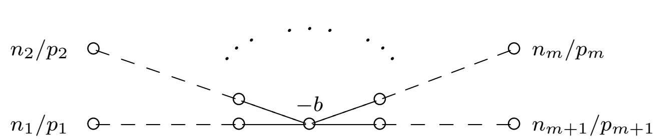
From the natural map $\mathbb{Z}^m \to \hat{G}$, any element of $\hat{G}$ may be represented as before by $\hat{I} = \sum_{j=1}^m i_j \bar{\chi}_j$, where this character gives the weight $\sum_{j=1}^m i_j \ell_{w_j}/s$, and sends an element $e \in D$ to $\prod_{j=1}^m e^{2\pi i\, i_j\, e\cdot e_{w_j}}$. If $I = (i_1, \dots, i_m)$ is non-negative $\hat{I}$ represents the weight and D–character of the monomial Y^I.

The ideal of C in $\mathbb{C}[Y_1, \dots, Y_m]$ is generated by $Y^I - Y^{I'}$ for all non-negative m–tuples I and I' with the same weight and D–character (i.e., giving the same image in $\hat{G}$). Denote by r the index in G of the stabilizer $H = \mathbb{C}^* \cap D$ of one branch of C. Thus $r = |G/\mathbb{C}^*|$ is the number of branches of C. We shall prove later that $r = s$ (s as in (7)), hence also this number can be read off the splice diagram.

The above applies also when Γ is a string, corresponding to a cyclic quotient singularity of some order $|D|$. Suppose there are two leaves, $*$ and w. Then D is the cyclic group generated by e_w, acting on $\mathbb{C}$ by $e^{2\pi i\, e_w\cdot e_w} = e^{-2\pi i \ell_{ww}/|D|}$. There is a single variable Y, with weight $\ell_{w,*} = 1$; we have $G = \mathbb{C}^*$, $D \subset G$ the cyclic subgroup of order $|D|$. The associated D–curve is simply a copy of $\mathbb{C}$ (and so the semigroup is $\mathbb{N}$). When there is only the one vertex $*$, it plays the role of both leaf w and root $*$, and all is the same, with $\ell_{w,*} = 1$ and D trivial.

For general $(\Gamma, *)$ our goal is to do an inductive comparison between the data of $(\Gamma, *)$ and data from rooted sub-diagrams. We start with the case of one node.

Proposition 6.2. *Consider a minimal resolution graph Γ with one node, given by the diagram*



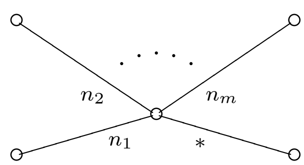
Strings of Γ are described by the continued fractions shown, starting from the node. Let $*$ denote the leaf on the lower right, C the associated D–curve, with discriminant group D of order $|D| = n_1 \dots n_{m+1}(b - \sum_{i=1}^{m+1} p_i/n_i)$. Then C and the group G are as in Example 5.1, and depend only on $n_1, \dots, n_m$. In particular, $r = s$, i.e., the number of branches is the GCD of the linking numbers to $*$.

Proof. It is shown in [16] that a weighted homogeneous surface singularity with resolution graph Γ is a splice quotient, with universal abelian cover defined by

$$\begin{aligned} Y_1^{n_1} - Y_2^{n_2} + a_2 Y_{m+1}^{n_{m+1}} &= 0 \\ \dots \quad \dots \quad &\dots \\ Y_1^{n_1} - Y_m^{n_m} + a_m Y_{m+1}^{n_{m+1}} &= 0 \end{aligned}$$

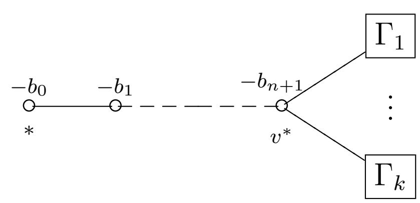
Here, the a_i are distinct and non-zero, giving the analytic type of the $m+1$ intersection points on the central curve. $D(\Gamma)$ acts on the coordinates Y_i according to the usual characters, and acts freely off the origin. The curve cut out by $Y_{m+1} = 0$ is the one in Example 5.1; its quotient by D is irreducible (see also [20, Theorem 7.2(6)]), so D acts transitively on the branches. By definition, the curve $C(\Gamma, *)$ in $\mathbb{C}^m$ consists of the monomial curve with weights $\mathcal{N}_i/s$ (which is a component of $\{Y_{m+1} = 0\}$) and its translates by D. It follows that $C(\Gamma, *)$ is equal to $\{Y_{m+1} = 0\}$. $\square$

Remark 6.3. Note that in this example, the rooted splice diagram



uniquely determines both the curve C and the group G, but *not* the group D.

We return to the inductive comparison of the data of $(\Gamma, *)$ and data from rooted sub-diagrams. Before starting, if any two nodes in Γ are adjacent, we blow up Γ once in between these, so that Γ has at least one vertex between any two adjacent nodes (cf. Definition 6.1 of [20]). Suppose $*$ is the leftmost leaf in the resolution diagram

$\Gamma =$ 

So, remove from Γ the leaf $*$, the adjacent node v^*, and all the vertices in between. This produces k new rooted resolution diagrams $(\Gamma_i, *_i)$. The new $*_i$ is the vertex of Γ_i closest to v^*, and it is now a leaf. We shall use notations like G_i, m_i, D_i, Δ_i (associated splice diagrams), r_i (number of branches), s_i (GCD of weights), etc., when referring to the new rooted diagrams. Note that the total number of non-root leaves is $m_1 + \ldots + m_k = m$.

In what follows, "leaf" shall mean non-root leaf. As before, we use notation $\ell_{ww'}$ and $\ell_w := \ell_{w,*}$ for linking numbers in Δ (as a splice diagram with leaf weights, so $\ell_{ww'}$ is also defined if $w = w'$ is a leaf). When computing linking numbers in Δ_i, we use notation $\tilde\ell_{ww'}$ and $\tilde\ell_w := \tilde\ell_{w,*_1}$. The weights around the distinguished node v^* are $|D_1|, \dots, |D_k|$ (by definition of the splice diagram) and c, the weight in the direction of $*$.

Lemma 6.4. *We have the following relations:*

(1) *For w a leaf in Δ_1,*
$$\ell_w = \tilde\ell_w \cdot |D_2| \dots |D_k| = \tilde\ell_w \cdot \ell_{v^*v^*}/c|D_1|.$$

(2) *For w a leaf in Δ_1, w' a leaf in Δ_i, $i > 1$,*
$$\ell_{v^*v^*}\ell_{ww'} = c^2\ell_w\ell_{w'}.$$

(3) *For w, w' leaves in Δ_1,*
$$\ell_{ww'}/|D| = \tilde\ell_{ww'}/|D_1| + \tilde\ell_w\tilde\ell_{w'}\ell_{v^*v^*}/|D||D_1|^2.$$

Proof. The first two claims are straightforward. For the third, we first need some notation. For any two vertices v and v', recall $\ell_{vv'}$ is the product of splice diagram weights adjacent to the path from v to v'; we will denote by $\ell'_{vv'}$ the product of splice diagram weights adjacent to the path from v to v', but excluding weights at v and v' (so $\ell'_{vv'} = 1$ if $v = v'$).

Let now v be the vertex of Γ that is closest to v^* on the path from w to w' (this is w itself if $w = w'$). Rewrite the equation to be proved first as
$$\ell_{ww'}|D_1| - \tilde\ell_{ww'}|D| = \tilde\ell_w\tilde\ell_{w'}\ell_{v^*v^*}/|D_1|\,.$$
If a, d_w, $d_{w'}$ are the weights in Δ at v towards v^*, w, w' respectively, R the product of the remaining weights at v, and $\tilde a$ the weight in Δ_1 at v towards v^*, then we can rewrite our equation
$$\ell'_{wv}\ell'_{w'v}aR|D_1| - \ell'_{wv}\ell'_{w'v}\tilde aR|D| = (\ell'_{wv}d_{w'}R\ell'_{vv*})(\ell'_{w'v}d_wR\ell'_{vv*})\ell_{v^*v^*}/|D_1|\,.$$
Cancelling common factors from this equation simplifies it to
$$a|D_1| - \tilde a|D| = d_wd_{w'}c(\ell'_{vv^*})^2\ell_{v^*v^*}/|D_1|\,.$$
This equation is what was proved in Lemma 12.7 of [20] (there v was a node, but that was not necessary to the proof). $\square$

We state the key results needed to do induction.

Theorem 6.5. *Enumerate the m non-root leaves of Γ so that the first m_1 of them yield exactly the non-root leaves of Γ_1. Compose the inclusion $G \subset (\mathbb{C}^*)^m$ with the projection onto $(\mathbb{C}^*)^{m_1}$ given by the first m_1 entries. Then the image is exactly G_1.*

Proof. We show the image of the projection map $\pi : G \subset (\mathbb{C}^*)^m \to (\mathbb{C}^*)^{m_1}$ is exactly G_1. Now, an inclusion $\mathbb{C}^* \subset \mathbb{C}^{*N}$ is described by a projective N–tuple of rational numbers:
$$[\alpha_1 : \alpha_2 : \dots : \alpha_N] \longleftrightarrow \{(u^{c\alpha_1}, \dots, u^{c\alpha_N}) \mid u \in \mathbb{C}^*\}$$

where c is any common denominator for the α_i's. Then Lemma 6.4 (1) implies that the image of the $\mathbb{C}^*$ in G is exactly the $\mathbb{C}^*$ in G_1, since the weights differ by a fixed multiple.

One must still show that image of D is in G_1, and (modulo $\mathbb{C}^*$) all of D_1 is in the image. So, choose a leaf w in Γ_1, and consider corresponding elements in the respective discriminant groups, i.e., $e_w \in D$ and $\tilde{e}_w \in D_1$. The following result suffices to complete the proof of Theorem 6.5. $\square$

Lemma 6.6. $\pi(e_w)\tilde{e}_w^{-1} \in \mathbb{C}^* \subset G_1$.

Proof. The left hand side is an element in $(\mathbb{C}^*)^{m_1}$, and we will show that the entry in the slot corresponding to a leaf w' in Δ_1 is $u^{\tilde{\ell}_{w'}}$ with $u = e^{-2\pi i(\tilde{\ell}_w \ell_{v^*v^*}/|D||D_1|^2)}$. This would establish the claim.

To verify the assertion, use the calculation of the entries of e_w and $\tilde{e}_w$ given by equation (8) (just before Definition 6.1). For a leaf w', the w'–entry of $\pi(e_w)\tilde{e}_w^{-1}$ is

$$e^{2\pi i(-\ell_{ww'}/|D|+\tilde{\ell}_{ww'}/|D_1|)}.$$

By Part 3 of Lemma 6.4, this expression is as claimed. $\square$

From Theorem 6.5, it is clear we have an inclusion

$$G \subset G_1 \times G_2 \cdots \times G_k,$$

whence a surjection on the level of characters

$$\hat{G}_1 \oplus \cdots \oplus \hat{G}_k \twoheadrightarrow \hat{G}.$$

Each of the character groups has its "reduced weight" map onto $\mathbb{Z}$ (see (6)). The map $\hat{G}_1 \to \hat{G}$ induces a map $\mathbb{Z} \to \mathbb{Z}$ which is multiplication by the ratio of the respective reduced weights, i.e., $(\ell_w/s)/(\tilde{\ell}_w/s_1)$ for any leaf w in Δ_1. Let us set $\mathcal{D} = |D_1||D_2|\dots|D_k|$, $\mathcal{D}_i = \mathcal{D}/|D_i|$. Then the aforementioned ratio is $s_1\mathcal{D}_1/s$. Moreover, it is easy to see that

$$s = \mathrm{GCD}(s_1\mathcal{D}_1, \dots, s_k\mathcal{D}_k). \tag{9}$$

We thus have a commutative diagram of surjections

$$\begin{tikzcd} \hat{G}_1 \oplus \cdots \oplus \hat{G}_k \arrow[r, two heads] \arrow[d] & \hat{G} \arrow[d] \\ \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \arrow[r, two heads] & \mathbb{Z} \end{tikzcd},$$

where the bottom horizontal map is given by dotting with $(s_1\mathcal{D}_1/s, \dots, s_k\mathcal{D}_k/s)$. In particular, the kernel of this map is a direct summand.

Recall (Proposition 4.4) that the characters of G_j vanishing on the discriminant group D_j form a cyclic group generated by a distinguished $\hat{Q}_j \in \hat{G}_j$, whose reduced weight is $|D_j|/r_j$. For each j, $2 \le j \le k$, consider the k–tuple

$$A_j := (\hat{Q}_1, 0, \dots, 0, -\hat{Q}_j, 0, \dots, 0) \quad \in \ \hat{G}_1 \oplus \cdots \oplus \hat{G}_k.$$

Theorem 6.7. (1) *The $k-1$ elements $\{A_j\}$ form a free basis of the kernel of the surjection*

$$\hat{G}_1 \oplus \cdots \oplus \hat{G}_k \to \hat{G}.$$

(2) *$r = s$, i.e., the number of branches of the curve $C(\Gamma, *)$ equals the GCD of the linking numbers.*

Proof. We proceed by induction on the number of nodes in Γ, the case of one node being easily checked using Proposition 6.2.

Thus, in the situation at hand, we may assume $r_i = s_i$, $i = 1, \dots, k$. An element in $\hat{G}_i$ (respectively $\hat{G}$) is represented by an m_i–tuple of integers I_i (respectively, m–tuple I), which are the coefficients of the basic characters corresponding to the non-root leaves in $(\Gamma_i, *_i)$ (resp. $(\Gamma, *)$). Recall that we denote the characters themselves by $\hat{I}_i$ and $\hat{I}$; we will denote the image of $(\hat{I}_1, \dots, \hat{I}_k)$ in $\hat{G}$ by $(I_1, \dots, I_k)\hat{}$.

Lemma 6.8. *Suppose $(I_1, \dots, I_k)\hat{} \in \hat{G}$ has weight 0. Then for w a leaf in Γ_i, one has an equality in $\mathbb{Q}/\mathbb{Z}$:*

$$(I_1, \dots, I_k)\hat{}\,(e_w) = \hat{I}_i(\tilde{e}_w).$$

Proof. We may as well assume $i = 1$, and write

$$I_1 = I, \ (I_2, \dots, I_k) = J,$$

where I is a tuple of integers indexed by the set S of non-root leaves of Γ_1, and J is indexed by the set S' of all the other non-root leaves of Γ. The non-reduced weight of $(I_1, \dots, I_k)\hat{}$ is 0 and is computed via linking numbers, yielding

$$\sum_{\alpha \in S} i_\alpha \ell_{w_\alpha} \ + \sum_{\beta \in S'} j_\beta \ell_{w'_\beta} \ = 0. \tag{10}$$

The left hand side of the equation in the Lemma is

$$(I_1, \dots, I_k)\hat{}\,(e_w) = \exp\Big(2\pi i \Big(\sum_{\alpha \in S} i_\alpha (e_{w_\alpha} \cdot e_w) \ + \sum_{\beta \in S'} j_\beta (e_{w'_\beta} \cdot e_w)\Big)\Big). \tag{11}$$

By (6.4) and (10), one has

$$\sum_{\beta \in S'} j_\beta (e_{w'_\beta} \cdot e_w) = \frac{-1}{|D|} \sum_{\beta \in S'} j_\beta \ell_{w'_\beta w} = \frac{-c^2 \ell_w}{|D| \ell_{v^* v^*}} \sum_{\beta \in S'} j_\beta \ell_{w'_\beta} \tag{12}$$

$$= \frac{c^2 \ell_w}{|D| \ell_{v^* v^*}} \sum_{\alpha \in S} i_\alpha \ell_{w_\alpha}. \tag{13}$$

We can therefore rewrite equation (11) as

$$(I_1,\dots,I_k)\hat{}\,(e_w)=\exp\Big(2\pi i\sum_{\alpha\in S}i_\alpha\Big((e_{w_\alpha}\cdot e_w)+\frac{c^2\ell_w\ell_{w_\alpha}}{|D|\ell_{v^*v^*}}\Big)\Big). \tag{14}$$

The lemma claims that this expression is equal to $\hat I_1(\tilde e_w)$, which is

$$\exp\Big(2\pi i\sum_{\alpha\in S}i_\alpha(\tilde e_{w_\alpha}\cdot\tilde e_w)\Big). \tag{15}$$

We need to check for each $\alpha\in S$ equality of the coefficient of i_α.

Using Equation (8) one needs to check that

$$-\ell_{w_\alpha w}/|D|\ +c^2\ell_w\ell_{w_\alpha}/|D|\ell_{v^*v^*}=-\tilde\ell_{w_\alpha w}/|D_1|.$$

This is a simple computation using parts (1) and (3) of Lemma 6.4. □

Continue the proof of Theorem 6.7. Recall $\hat Q_j\in\hat G_j$ has reduced weight $|D_j|/r_j=|D_j|/s_j$, so its non-reduced weight (using linking numbers in $(\Gamma_j,*_j)$) is $|D_j|$. Computing linking numbers in $(\Gamma,*)$ multiplies this weight by the other $|D_{k'}|$ (Lemma 6.4 (1)), resulting in non-reduced weight $|D_1|\dots|D_k|=\mathcal D$ (independent of j). It follows that the image of A_j in $\hat G$ has weight 0. Since $\hat Q_j$ vanishes on D_j, applying Lemma 6.8, the image of A_j in $\hat G$ vanishes at all e_w, hence is the trivial character.

Conversely, if $(I_1,\dots,I_k)$ represents the trivial character of G, then it certainly has weight 0, hence by Lemma 6.8 each character $\hat I_j$ vanishes on the corresponding discriminant group D_j. By Proposition 4.4, it follows that $\hat I_j$ is equivalent to a multiple $n_j\hat Q_j$. But now the condition of Γ–weight equal 0 means that $\sum_{j=1}^k n_j=0$. That the A_j form a basis of this space is now easy. This completes the proof of the first assertion of the Theorem.

We now have a commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{j=2}^k\mathbb Z\cdot A_k \arrow[r]\arrow[d] & \hat G_1\oplus\cdots\oplus\hat G_k \arrow[r]\arrow[d] & \hat G \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & K \arrow[r] & \mathbb Z\oplus\cdots\oplus\mathbb Z \arrow[r] & \mathbb Z \arrow[r] & 0\,.
\end{tikzcd}$$

The right two vertical maps are surjective, with kernels finite groups of order $r_1\dots r_k$ and r, respectively. The left vertical map is injective, and the order of the cokernel is (since K is a direct summand) the order of the torsion subgroup of the quotient of $\mathbb Z^k$ by the image of the A_j's. So, one considers a $(k-1)\times k$ matrix whose non-0 entries are of the form $\pm|D_i|/s_i$. The maximal minors are of the form $s_j\mathcal D_j/(s_1\dots s_k)$; so the order of this group is the GCD of these minors, which is $s/s_1\dots s_k$. The snake sequence for the diagram then yields that $r=s$. This completes the proof of the Theorem. □

We restate the key part of the Theorem:

Corollary 6.9. *Every element of $\hat G$ may be written in the form*

$$X_1 + X_2 + \dots\ + X_k, \quad X_i \in \hat G_i\,,$$

and this representation is unique modulo the equations $\hat Q_i = \hat Q_j$, all i, j. □

Corollary 6.10. *Consider as above the curves $C = C(\Gamma, *)$ and $C_i = C(\Gamma_i, *_i)$ coming from Γ and its rooted subtrees Γ_i, $i = 1, \dots, k$. For each i, let $Y_{i,j}$ be variables corresponding to the non-root leaves of Γ_i, and let $\mathcal{J}_i$ be the ideal in the variables $Y_{i,j}$ defining C_i. Assume that each $\hat Q_i \in \mathcal{S}_i$, so we can choose a non-negative m_i–tuple Q_i representing $\hat Q_i$, and hence a monomial $Y_i^{Q_i}$ in the variables $Y_{i,j}$.*

Then the ideal defining C in all the variables $Y_{i,j}$ is generated by $\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_k$ and $Y_1^{Q_1} - Y_i^{Q_i}$, $i = 2, \dots, k$.

Proof. According to Proposition 4.3, the ideal of C (resp. C_i) is generated by all differences of pairs of monomials whose (non-negative) exponents have the same image in the character group $\hat G$ (resp. $\hat G_i$). Suppose $I = (I_1, \dots, I_k)$ and $I' = (I'_1, \dots, I'_k)$ are non-negative exponent tuples giving the same element of $\hat G$. We will first simplify the relation $Y^I = Y^{I'}$ modulo the ideals $\mathcal{J}_i$.

According to Theorem 6.7, the images of I and I' in $\hat G_1 \oplus \dots \oplus \hat G_k$ differ by a sum $\sum_{j=2}^k p_j A_j$ of the A_j's. Putting $-p_1 = \sum_{j=2}^k p_j$, this means

$$\hat I_i - \hat I'_i = -p_i \hat Q_i \text{ in } \hat G_i,\ i = 1, \dots, k,\ .$$

For each $p_i \geq 0$, the tuples $I_i + p_i Q_i$ and I'_i are non-negative tuples giving the same element in $\hat G_i$. So, modulo relations coming from $\mathcal{J}_i$, one can subtract I_i from each exponent tuple I, I' in the i–slot, leaving 0 and $p_i Q_i$ in the new i–slots. For $p_i < 0$ one subtracts instead I'_i modulo relations from $\mathcal{J}_i$ to get $-p_i Q_i$ and 0 respectively in these slots. After doing this for each i one has either 0 or a positive multiple of Q_i in each slot of both I and I' and the total coefficient sum is the same for each, namely $p := \sum_{p_i>0} p_i$. At this point we have simplified our relation $Y^I = Y^{I'}$ to the point where each side of it is equivalent to $Y_1^{pQ_1}$ using the relations $Y_1^{Q_1} = Y_i^{Q_i}$. This proves the corollary. □

7. Semigroups and delta-invariants of D–curves from $(\Gamma, *)$

Let us maintain the same basic setup as the last section: the rooted tree $(\Gamma, *)$; the discriminant group $D = D(\Gamma)$, of order $|D|$; the group G and its character group $\hat G$; the D–curve $C = C(\Gamma, *)$; the reduced weight $|X|$ of an element X of $\hat G$, giving a surjection $\hat G \to \mathbb{Z}$, with kernel of order r; a surjection $\hat G \to \hat D$, with kernel the cyclic group generated by a distinguished element Q of weight $|D|/r$. r is both the number of components of the curve and the GCD of the non-reduced weights (i.e., linking numbers) of the leaves. Non-negative characters of G give a semigroup $\mathcal{S} \subset \hat G$, which is the value semigroup of the corresponding D–curve. We are interested in the

δ–invariant of the curve (which is the number of gaps of the semigroup $\mathcal{S}$), and the question of whether $Q \in \mathcal{S}$. All elements of $\hat{G}$ of sufficiently high weight are in $\mathcal{S}$, while there are $r-1$ gaps of weight 0.

As before, we compare data of $(\Gamma, *)$ with those of the subtrees $(\Gamma_i, *_i)$. Recall we have defined

$$\mathcal{D} := |D_1||D_2|\dots|D_k|, \ \mathcal{D}_i := \mathcal{D}/|D_i|.$$

Corollary 6.9 indicates that every element of $\hat{G}$ may be written

$$X_1 + X_2 + \dots\ + X_k,$$

where $X_i \in \hat{G}_i$, and this representation is unique modulo all the equations $\hat{Q}_i = \hat{Q}_j$. Using $|X_i|_i$ to denote weight in $\hat{G}_i$, one easily checks that

$$|X_i| = (r_i/r)\mathcal{D}_i|X_i|_i. \tag{16}$$

(Recall every $\hat{Q}_i$ has the same weight $\mathcal{D}/r$ in $\hat{G}$.)

Theorem 7.1. *Consider the data $(\Gamma, *)$, $\hat{G}$, $\hat{Q}$, $|D|$, r, $\mathcal{S}$, δ as before, with corresponding notation for the subtrees. Then:*

(1) *The number of gaps of the semigroups satisfies*

$$2\delta - r \le \sum_{i=1}^{k} \mathcal{D}_i(2\delta_i - r_i) \ + \ (k-1)\mathcal{D}.$$

(2) *If equality holds, then $\hat{Q}_i \in \mathcal{S}_i$, for all i.*

Proof. To show the inequality, we count the gaps in $\mathcal{S}$. We write an element of $\hat{G}$ as $X = X_1 + \dots + X_k$, where $X_i \in \hat{G}_i$, and the representation is unique up to repeated alterations of the form: add $\hat{Q}_i$ to X_i and subtract $\hat{Q}_j$ from X_j for some i, j. Denote

$$q_i := |\hat{Q}_i|_i = |D_i|/r_i.$$

We claim that every $X \in \hat{G}$ of weight ≥ 0 has a representation $X = X_1 + \dots + X_k$ in one of the following classes:

C_0: $|X_1|_1 < 0$; $0 \le |X_i|_i < q_i$ for $i \ge 2$ (and $|X| \ge 0$ by assumption).

C_1: $0 \le |X_1|_1$ and $X_1 \notin \mathcal{S}_1$; $0 \le |X_i|_i < q_i$ for $i \ge 2$.

C_2: $X_1 \in \mathcal{S}_1$ and $X_1 - s\hat{Q}_1 \notin \mathcal{S}_1$ for $s \ge 1$; $0 \le |X_2|_2$ and $X_2 \notin \mathcal{S}_2$; $0 \le |X_j|_j < q_j$ for $j \ge 3$.

… ....

C_i: For $1 \le j \le i-1$, $X_j \in \mathcal{S}_j$ and $X_j - s\hat{Q}_j \notin \mathcal{S}_j$ for $s \ge 1$; $0 \le |X_i|_i$ and $X_i \notin \mathcal{S}_i$; $0 \le |X_j|_j < q_j$ for $j \ge i+1$.

… ....

C_k: For $1 \le j \le k-1$, $X_j \in \mathcal{S}_j$ and $X_j - s\hat{Q}_j \notin \mathcal{S}_j$ for $s \ge 1$; $0 \le |X_k|_k$ and $X_k \notin \mathcal{S}_k$.

C_{k+1}: For $1 \le j \le k-1$, $X_j \in \mathcal{S}_j$ and $X_j - s\hat{Q}_j \notin \mathcal{S}_j$ for $s \ge 1$; $X_k \in \mathcal{S}_k$.

To see this, given X with $|X| \geq 0$, start by arranging the weights of X_i, $i \geq 2$, to be less than the weight q_i of the corresponding $\hat{Q}_i$. If the representation is then not in the class C_0 or C_1, then $X_1 \in \mathcal{S}_1$. So subtract $\hat{Q}_1$'s as necessary from X_1 and simultaneously add $\hat{Q}_2$'s to X_2 until $X_1 \in \mathcal{S}_1$ but $X_1 - \hat{Q}_1 \notin \mathcal{S}_1$. If now the representation is not in C_2, then $X_2 \in \mathcal{S}_2$, so subtract $\hat{Q}_2$'s as necessary from X_2 and add $\hat{Q}_3$'s to X_3 to make $X_2 \in \mathcal{S}_2$ but $X_2 - \hat{Q}_2 \notin \mathcal{S}_2$. Repeat this procedure until the representation is in some C_i. If one fails all the way to C_k, then the final representation is in C_{k+1}.

Elements in C_{k+1} are clearly not gaps, so adding up the sizes of the other classes gives an upper bound for δ, the number of gaps. Recall that the number of elements of $\hat{G}_i$ of a given weight is equal to r_i. But $q_i r_i = |D_i|$, so the number of elements in class C_1 is $\delta_1 \mathcal{D}_1$. Next, every $(\hat{Q}_i)$–coset in $\hat{G}_i$ contains a unique minimal representative in $\mathcal{S}_i$, i.e., which is not in $\mathcal{S}_i$ after subtracting any positive multiple of $\hat{Q}_i$. Thus, there are $[\hat{G}_i : (\hat{Q}_i)] = |\hat{D}_i| = |D_i|$ such elements. So, class C_j contains $\delta_j \mathcal{D}_j$ elements for $j = 2, \dots, k$. Thus, δ is bounded above as follows:

$$\delta \leq |C_0| + \sum_{i=1}^{k} \delta_i \mathcal{D}_i \,.$$

To count elements of C_0, we need to know the number N of allowed degrees $x_i := |X_i|_i$; then the total count of these elements would be $r_1 \dots r_k N$, where (thanks to (16))

$$N = \#\{(x_1, \dots, x_k) \in \mathbb{Z}^k |\ x_1 < 0;\ 0 \leq x_i < q_i,\ \text{all } i \geq 2; \sum_{i=1}^{k} x_i/q_i\ \geq 0\}.$$

We use (but prove later) the following

Lemma 7.2. *Let* $\mathcal{Q} = q_1 \dots q_k,\ \mathcal{Q}_i = \mathcal{Q}/q_i$. *Then*

$$2N = (k-1)\mathcal{Q}\ - \sum_{i=1}^{k} \mathcal{Q}_i\ + h,$$

where $h = \mathrm{GCD}(\mathcal{Q}_1, \dots, \mathcal{Q}_k)$.

Using the Lemma we conclude that

$$\delta \leq \sum_{i=1}^{k} \delta_i \mathcal{D}_i + \frac{1}{2} r_1 \dots r_k \Big((k-1)\mathcal{Q}\ - \sum_{i=1}^{k} \mathcal{Q}_i\ + h \Big) \,.$$

But $r = \mathrm{GCD}(r_i \mathcal{D}_i)$, while $r_1 \dots r_k \mathcal{Q}_i = r_i \mathcal{D}_i$, so

$$r = r_1 \dots r_k \cdot \mathrm{GCD}(\mathcal{Q}_1, \dots, \mathcal{Q}_k)\ = r_1 \dots r_k h.$$

The inequality in the theorem follows from these.

If equality holds, then the classes $C_0, C_1, \dots, C_k$ consist only of gaps, and there is no overlap between them. Suppose $\hat{Q}_i \notin \mathcal{S}_i$ for some i. Then $X = 0+\cdots+\hat{Q}_i+\cdots+0$ is in C_i, hence a gap. It is equal to $0+\cdots+\hat{Q}_j+\cdots+0$ for each j, so $\hat{Q}_j \notin \mathcal{S}_j$, so we have a common element in all the classes $C_1, \dots, C_k$. This contradicts that the C_i don't overlap, which proves the second claim of the theorem. $\square$

Proof of Lemma 7.2. To prove the Lemma, consider a one node resolution diagram with $-b$ in the center, and $k+1$ length 1 strings emanating, with weights $-q_1, \dots, -q_k$ and $-b'$. Here b is any integer at least $k+1$, and $b' > 0$ is arbitrary. View the $-b'$ vertex as the root. This gives a $(\Gamma, *)$, and k subdiagrams $(\Gamma_i, *_i)$, each of which consists of only one vertex. So, each $G_i = \mathbb{C}^*$, $r_i = 1$, $\delta_i = q_i$, $\mathcal{S}_i = \mathbb{N}$, $\hat{G}_i = \mathbb{Z}$. The gaps of $\hat{G}$ can be counted as above, but they are exactly those of the first type C_0. Thus, the desired quantity N is exactly the number of gaps δ (i.e., the delta-invariant of the corresponding curve). But the curve is the complete intersection defined by $X_1^{q_1} = X_i^{q_i},\ i = 2, \dots, k$, as in Example 5.1 and Proposition 6.2. This curve has $s = h$ branches, using the notation above. Using Example 5.1 gives

$$2N = (k-1)\mathcal{Q}\ - \sum_{i=1}^{k} \mathcal{Q}_i + h.$$

This proves the lemma. $\square$

There is another invariant of $(\Gamma, *)$ whose relationship to those of its subdiagrams $(\Gamma_i, *_i)$ is similar to the situation for $2\delta - r$, as revealed by Theorem 7.1. Namely, let

$$\nu(\Gamma, *) := \sum_{v \neq *} (d_v - 2)\ell_{*,v}\,,$$

the sum being over all vertices of Γ except for $*$. The following statements are easy to verify:

(1) ν depends only on the splice diagram $(\Delta, *)$.
(2) $\nu(\Gamma, *) = \sum_{i=1}^{k} \mathcal{D}_i \nu_i\ +\ (k-1)\mathcal{D}$.
(3) If $(\Gamma, *)$ has one node (Proposition 6.2), then $\nu = 2\delta - r$.

This brings us to our major result.

Theorem 7.3. *Let $C = C(\Gamma, *)$ be the D–curve constructed from a rooted resolution diagram. Then $2\delta(C) - r \le \nu(\Gamma, *)$, and equality implies the following:*

(1) *C is a complete intersection curve*
(2) *The splice diagram Δ satisfies the semigroup and congruence conditions at any node in the direction away from the leaf $*$.*

We recall that the semigroup and congruence conditions at a node can be formulated that for all outgoing edges at the node, monomials of appropriate weight can be found which transform equivariantly (i.e., with the same character) with respect to the D-action.

Proof. We do induction on the number of nodes of Γ. For one node, the inequality in question is an equality, and the claims follow from Example 5.1 (6) and Proposition 6.2. The semigroup and congruence conditions are automatic.

In the general case, we as usual compare $(\Gamma, *)$ with its subtrees $(\Gamma_i, *_i)$, $i = 1, \dots, k$. Theorem 7.1(1), the second statement about ν, and induction give the general inequality. Also, equality for $(\Gamma, *)$ implies equality for all the $(\Gamma_i, *_i)$, and that each $Q_i \in \mathcal{S}_i$. The induction assumption says that the curves C_i are complete intersections; applying Corollary 6.10, it follows that C is as well (compare number of defining equations to number of variables). Further, the monomials involved in the added equations $Y_1^{Q_1} - Y_i^{Q_i}$, $i = 2, \dots, k$ are monomials whose characters in $\hat{G}$ are equal by Theorem 6.7 and have the correct weight. This gives the semigroup and congruence condition at the node closest to $*$ in the directions away from $*$. For nodes further from $*$ it was proved during the induction, except that equivariance was proved in a subgraph Γ_i and hence proved for D_i rather than D. But D–equivariance follows from the fact the the D–action induces the D_i–action on the variables coming from the subgraph Γ_i (Theorem 6.5). □

8. Proof of the End Curve Theorem

Let (X, o) be a normal surface singularity with $\mathbb{Q}$HS link Σ. Recall that we say a knot or link $K \subset \Sigma$ is *cut out by the analytic function* $z\colon (X, o) \to (\mathbb{C}, o)$ if the pair (Σ, K) is topologically the link of the pair $(X, \{z = 0\})$ (i.e., the reduced germ $(X, \{z = 0\}, o)$ is homeomorphic, preserving orientations, to the cone on (Σ, K)). In [15] it is shown that the link of a surface–curve germ pair (X, B, o) determines the minimal good resolution of this pair.

Let v_i, $i = 1, \dots, t$ be the leaves of the resolution graph Γ. Suppose that for each i we have an end-curve function z_i, vanishing d_i times along the end-curve $B_i \subset X$. The following lemma tells us that a d_i–th root x_i of z_i is a well-defined analytic function on the universal abelian cover $(V, 0)$ of (X, o), and that x_i vanishes to order 1 on its zero-set.

Lemma 8.1. *Let $z\colon (X, o) \to (\mathbb{C}, 0)$ be an analytic function that vanishes to order d on its reduced zero set $B \subset X$. Then the multivalued function $z^{1/d}$ on X lifts to a single valued function x on the universal abelian cover $(V, 0)$ of (X, o), and x vanishes to order 1 on its zero set (there are d such lifts that differ by d–th roots of unity). If B is irreducible then the zero-set of x has $|D|/d'$ components, where d' is the order of the the class of K (the link of B) in $H_1(\Sigma; \mathbb{Z})$.*

Proof. Take the branched cover $X' \to X$ given on the local ring level by adjoining t satisfying $t^d - z = 0$, and then normalizing. At any point of $B - \{o\}$, choose local analytic coordinates u, v, with B given by $u = 0$; we may further assume that locally $z = u^d$. Over such a point, X' is given by normalizing $t^d - u^d = 0$, yielding a smooth and unramified d-fold cyclic cover. Thus, $X' \to X$ is unramified away from the singular point. Clearly, t is still a single-valued function z' on this cover, and z' vanishes to order 1 on its zero-set. z' is well defined up to the covering transformations of $X' \to X$, which multiply z' by d–th roots of unity.

We may assume X' is connected (if X' has k components then replace X' by one of its components, z by $(z')^{d/k}$, which is then well defined on X, and d by d/k). Since $X' \to X$ is a connected abelian cover, it is covered by the universal abelian cover $V \to X$ so we get our desired lift x of z to V by composing z' with the projection $V \to X'$.

Now assume B is connected and K is its link. Let d' be the order of the class of K in $H_1(\Sigma)$. Then each component of the inverse image $\tilde{K}$ of K in $\widetilde{\Sigma}^{ab}$ is an d'-fold cover of K, so there are $|D|/d'$ such components. Since $\tilde{K}$ is the link of $\{x = 0\}$, the final sentence of the lemma follows. (Although we don't need it, it is not hard to see that k above is d/d', so there is a d/d'–th root of z that is defined on X. Hence, d' is the least order of vanishing any function z as in the lemma.) □

Denote the zero set of x_i in V by C_i. This is a D-curve, where $D = D(\Gamma)$ is the discriminant group, and it has $|D|/d_i$ branches, where d_i is the order of the class of the end-knot K in $H_1(\Sigma)$. By [20, Corollary 12.11], $|D|/d_i = r_i$. We will concentrate for the moment on C_1. By Lemma 9.4 below, its δ–invariant is

$$\delta(C_1) = \frac{1}{2}\big(r_1 + \nu(\Gamma, v_1)\big) = \frac{1}{2}\big(r_1 + \sum_{v \neq v_1}(d_v - 2)\ell_{v_1,v}\big)\,. \tag{17}$$

Let $\hat{G}$ be the character group associated with C_1 and its associated graded, as in section 4 , and let $\mathcal{S} \subset \hat{G}$ be the value semigroup. The characters in $\hat{G}$ associated with the functions $x_2, \dots, x_t$ generate a subsemigroup $\mathcal{S}_0$ of $\mathcal{S}$, whence

$$\delta(\mathcal{S}_0) \geq \delta(\mathcal{S}) = \delta(C_1). \tag{18}$$

Lemma 8.2. *The subsemigroup $\mathcal{S}_0$ is equal to the value semigroup of the curve associated to the rooted resolution diagram (Γ, v_1) as in Section 6. In particular, their δ–invariants are equal.*

Proof. Since intersection number of curves in X is given by linking number of their links in Σ, the function x_i $(i > 1)$ has vanishing degree $\ell_{v_1 v_i}$ on the curve C_1, hence $\ell_{v_1 v_i}/r_1$ on each branch; this last quantity is thus the weight of x_i in the associated graded of C_1 (alternatively, this can be seen by considering the intersection number of the proper transform of C_1 with the zero set of x_i in the resolution). By Theorem 9.3 below, D acts on x_i via the character χ_i corresponding to the leaf $v_i \in \Gamma$. Thus, the $\hat{G}$–character of x_i is the generator $\overline{\chi}_i$ of $\mathcal{S}(C(\Gamma, v_1))$ as described in section 6. □

From the Lemma and Theorem 7.3, we have

$$\delta(\mathcal{S}_0) \leq \frac{1}{2}\big(r_1 + \nu(\Gamma, v_1)\big). \tag{19}$$

Comparing (17), (18), (19), the inequalities are actually equalities. Hence, $\mathcal{S}_0 = \mathcal{S}$ and this is the semigroup both for the associated graded of C_1 and the model curve $C(\Gamma, v_1)$, so these are isomorphic as D-curves (Corollary 4.5). Again by Theorem 7.3, each curve is a complete intersection with maximal ideal generated by $x_2, \dots, x_t$. Since this curve is the associated graded of C_1, it follows that C_1 is a complete intersection and $x_2, \dots, x_t$ generates its maximal ideal. We conclude that $(V, 0)$ is a complete intersection with maximal ideal generated by $x_1, \dots, x_t$.

Moreover, Theorem 7.3 gives us the semigroup and congruence conditions at all nodes in the directions away from the leaf v_1. Repeating the argument at all leaves gives all semigroup and congruence conditions.

It remains to show that V is defined by a system of D-equivariant splice equations using the functions $x_1, \dots, x_t$. Pick a v of Γ of valency δ. Denote by E_v the exceptional curve corresponding to v and by $E_1, \dots, E_\delta$ the δ exceptional curves that intersect E_v. Since the congruence and semigroup conditions are satisfied, we can find a system of admissible monomials $M_1, \dots, M_\delta$ of (unreduced) weight ℓ_{vv}, corresponding to the outgoing edges at v, which transform the same way with respect to the action of D. For $1 \le i < \delta$ the ratio M_i/M_δ is thus D-invariant, hence defined on $X = V/D$. In the proof of Theorem 10.1 of [20] (see also Theorem 4.1 of [21]) it is shown that this function is meromorphic on the exceptional curve E_v corresponding to v, with a simple zero at the point $E_v \cap E_i$, a simple pole at $E_v \cap E_\delta$ and no other zeroes or poles. It follows that there are $\delta - 2$ linear relations among the M_i/M_δ on E_v. If $a_1 M_1/M_\delta + \dots + a_\delta M_\delta/M_\delta = 0$ is one of these linear relations, write $L := a_1 M_1 + \dots + a_\delta M_\delta$. Then the order of vanishing of L^δ on E_v is greater than that of M_δ^d. Since the v-weight of a function f on X is measured by the order of vanishing of f^d on E_v, we see that L has v–weight greater than ℓ_{vv}, so that, as in the proof of Theorem 10.1 of [20], we can adjust L by something that vanishes to higher v-weight to get an equation of splice type that holds identically on V. Doing this for each of our linear relations and repeating at all nodes gives a system of splice equations that hold on V. As proved in [20], such a system of splice type equations defines a complete intersection singularity $(V', 0)$ whose D-quotient (X', o) has resolution graph Γ. Moreover since the local rings are subrings of the local rings of V and X, we have finite maps $V \to V'$ and $X \to X'$. The degree of the map $V \to V'$ can be computed by restricting to the curve $C_1 = \{x_1 = 0\}$ and then taking the associated graded of this curve. By Corollary 6.10 we see this way that the degree is 1, so the proof is complete. □

9. Topological computations.

9.1. **Linking numbers.** In this subsection we describe the interpretation of the numbers ℓ_{ij} as linking numbers. Recall first that if K_1 and K_2 are disjoint oriented knots in a $\mathbb{Q}$HS Σ then their *linking number* is defined as follows: Some multiple dK_1 bounds a 2–chain A in Σ and $\ell(K_1, K_2)$ is defined as $\frac{1}{d} A \cdot K_2 \in \mathbb{Q}$ (intersection number). A (standard) easy calculation shows that this is well-defined. Now suppose Σ bounds an oriented 4–manifold Y with $H_2(Y;\mathbb{Q})=0$. Then multiples d_1K_1 and d_2K_2 bound 2–chains A_1 and A_2 in X and $\ell(K_1,K_2)$ can be computed as $\frac{1}{d_1d_2}A_1\cdot A_2$ (see, e.g. Durfee [6]; the point is that it is again easy to see this is independent of choices, and if one chooses A_1 to lie in Σ and A_2 to be transverse to Σ one gets the previous definition). We can extend to the case that $H_2(Y)\neq 0$ by requiring that A_1 be chosen to have zero intersection with any 2–cycle (i.e., closed 2–chain) in Y; it clearly suffices to require this for 2–cycles representing a generating set of $H_2(Y)$. Again, the proof that this works only involves showing that it gives a well-defined invariant, which is as before.

Suppose now that $\Sigma=\partial X$ is a $\mathbb{Q}$HS singularity link and $Y\to X$ is a resolution of the singularity. Let K_v and K_w be knots in Σ represented by meridians of exceptional curves E_v and E_w in Y.

Proposition 9.1. $\ell(K_v,K_w)=\frac{1}{|D|}\ell_{vw}$.

Proof. Let D_v and D_w be transverse disks to E_v and E_w with boundaries K_v and K_w. Recall that the matrix (ℓ_{ij}) is $-|D|(E_i\cdot E_j)^{-1}$ (see Def. 1.1). It follows that a 2-chain A whose boundary is $|D|K_v$ and which dots to zero with each E_i is given by $A=|D|D_v+\sum_i \ell_{vi}E_i$. So $\ell(K_v,K_w)=\frac{1}{|D|}A\cdot D_w=\frac{1}{|D|}\ell_{vw}$. □

9.2. **Torsion linking form.** The torsion linking form is a non-degenerate bilinear $\mathbb{Q}/\mathbb{Z}$-valued pairing on the torsion of $H_1(M;\mathbb{Z})$ for any closed oriented 3–manifold M. We recall the definition. If $\alpha,\beta\in H_1(\Sigma;\mathbb{Z})$ are torsion elements, we represent α and β by disjoint 1–cycles a and b. Since some multiple da of a bounds, so we can find a 2–chain A with $\partial A=da$. Then $\ell(\alpha,\beta):=\frac{1}{d}A\cdot b\in\mathbb{Q}/\mathbb{Z}$ where $A.b$ is the algebraic intersection number.

For a $\mathbb{Q}$HS Σ the linking form is a non-degenerate symmetric bilinear pairing

$$\ell\colon H_1(\Sigma;Z)\times H_1(\Sigma;\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$$

and hence gives an isomorphism of $D=H_1(\Sigma;Z)$ with its character group $\hat{D}=\operatorname{Hom}(D,\mathbb{C}^*)\cong\operatorname{Hom}(D,\mathbb{Q}/\mathbb{Z})$. We take the negative of this isomorphism and for $x\in D$ we call the character $e^{-2\pi i\ell(x,-)}\in\hat{D}$ the *character dual to* x. By the above Proposition 9.1, we have:

Proposition 9.2. *For a $\mathbb{Q}$HS link of a complex surface singularity, the torsion linking form is the negative of the form $e_v.e_w$ of Section 1.2.* □

We now want to return to the situation of Lemma 8.1, where we lift a root of an end-curve function on our singularity (X,o) to a function on the universal abelian cover. It is convenient now to restrict just to the link of the singularity (and of the curve $B\subset X$). The result we need, Theorem 9.3 below, is a general statement about rational homology spheres.

So we assume we have a $\mathbb{Q}$HS Σ and a knot (or link) $K \subset \Sigma$ and a smooth function $z\colon \Sigma \to \mathbb{C}$ which vanishes to order exactly d along K. The proof of Lemma 8.1 applies to see that the multivalued function $z^{1/d}$ on Σ can be lifted to a single-valued function x on the universal abelian cover $\widetilde{\Sigma}^{ab}$ which vanishes to order 1 on its zero set (i.e., 0 is a regular value). The covering transformation group for the universal abelian covering $\pi\colon \widetilde{\Sigma}^{ab} \to \Sigma$ is $D = H_1(\Sigma; \mathbb{Z})$.

Theorem 9.3. *Let $K \subset \Sigma$ and x, as above, a lift to $\widetilde{\Sigma}^{ab}$ of the d–th root of a function z that vanishes to order d along K. Then the action of D on $\widetilde{\Sigma}^{ab}$ transforms the function x by the character dual to the homology class $[K] \in D = H_1(\Sigma; \mathbb{Z})$. That is,*

$$x(hp) = e^{-2\pi i \ell([K],h)} x(p) \quad \textit{for } p \in \widetilde{\Sigma}^{ab} \textit{ and } h \in D\,.$$

Proof. The action of $D = H_1(\Sigma)$ on $\widetilde{\Sigma}^{ab}$ can be described as follows: If $h \in D$ and $p \in \widetilde{\Sigma}^{ab}$ and $\gamma\colon [0,1] \to \Sigma$ is any loop based at $\pi(p)$ in Σ whose homology class is h, then the lift $\widetilde{\gamma}$ of γ that ends at p starts hp.

For a regular value λ of the function $x/|x|$ on $\Sigma - K$ consider the set $A := (x/|x|)^{-1}(\lambda) \cup K$. This set can be considered as a smooth 2–chain in Σ with boundary dK, so $\ell(h,[K]) = \frac{1}{d}\,\gamma \cdot A$. Denote the inverse image of A in $\widetilde{\Sigma}^{ab}$ by $\widetilde{A}$. So

$$\widetilde{A} = \{p \in \widetilde{\Sigma}^{ab} \mid d \arg(x(p)) = \arg(\lambda)\}\,.$$

The intersection number $A \cdot \gamma$ equals the algebraic number of intersections of $\widetilde{A}$ with $\widetilde{\gamma}$. But the function x changes continuously along $\widetilde{\gamma}$, with value at the point hp of $\widetilde{\gamma}$ some power of $e^{2\pi i/d}$ times its value at p. The power in question is clearly, up to sign, the intersection number $\widetilde{A} \cdot \widetilde{\gamma}$, since, as one moves along $\widetilde{\gamma}$ from p to hp the change in argument of $x(\widetilde{\gamma}(t))$ can be followed by counting how this argument passes through values of the form $(\arg(\lambda) + 2\pi k)/d$. Since $\widetilde{\gamma}$ is oriented from hp to p, the sign is as stated in the theorem. ☐

9.3. **Milnor number and δ-invariant.** Let (X, o) be a normal surface singularity with $\mathbb{Q}$HS link Σ and $z\colon (X, o) \to (\mathbb{C}, 0)$ a holomorphic germ which vanishes with degree d on its zero set B. By Lemma 8.1, a d–th root of z lifts to a well-defined function $x\colon (V, 0) \to (\mathbb{C}, 0)$ on the universal abelian cover $(V, 0)$ of (X, o), which vanishes to degree 1 on its zero-set $C \subset V$. We compute the δ-invariant of (C, o); this is a topological computation.

The link of the pair (V, C) is a fibered link whose "Milnor fiber" F is diffeomorphic to $F = x^{-1}(\delta) \cap D^{2N}_\epsilon$ for some sufficiently small $0 < \delta \ll \epsilon \ll 1$, where D^{2N}_ϵ is the ϵ–ball about the origin for some embedding $(V, 0) \to (\mathbb{C}^N, 0)$. A standard formula

$$\delta(C) = \frac{1}{2}(r - \chi(F)) \tag{20}$$

relates the δ–invariant of C with the number of branches r and the Euler characteristic of its smoothing F ([3]). We thus want to compute $\chi(F)$.

Let Γ be the resolution graph for a simultaneous good resolution of (X, B, o) and v_1 the vertex corresponding to the exceptional curve E_{v_1} that the proper transform of C meets. So B meets E_{v_1} transversally in one point and meets no other exceptional curve.

Lemma 9.4. *With the above notation, $\chi(F) = -\nu(\Gamma, v_1)$ where*

$$\nu(\Gamma, v) := \sum_v (\delta'_v - 2)\ell_{v_1 v}\,,$$

the sum is over all vertices v of Γ, and $\delta'_v = \delta_v$ if $v \neq v_1$ and $\delta'_v = \delta_v + 1$ if $v = v_1$ (where δ_v is valency of v). In particular, if v_1 is a leaf then the summand for v_1 is zero and, by (20),

$$\delta(C) = \frac{1}{2}\big(r + \nu(\Gamma, v_1)\big) = \frac{1}{2}\big(r + \sum_{v \neq v_1}(\delta_v - 2)\ell_{v_1 v}\big)\,.$$

Proof. If $dC + \sum a_v E_v$ is the zero set of z on the resolution of X then the intersection equations $(dC + \sum a_v E_v).E_w = 0$ show that the order of vanishing a_v of z on a curve E_v of the resolution is $d(\ell_{v_1 v}/|D|)$. It follows by a standard argument (originally due to A'Campo [1]) that the Milnor fiber F_z of z has Euler characteristic

$$\chi(F_z) = \frac{d}{|D|} \sum_v (2 - \delta'_v)\ell_{v_1 v}\,.$$

Let X' and z' be as in the proof of Lemma 8.1. As there, we may assume X' is connected. Then d is a divisor of $|D|$.

We can think of the Milnor fibers F and F_z as the fibers of the fibered (multi-)links $(\tilde\Sigma^{ab}, L)$ and (Σ, dK), which are the links of the pairs (V, C) and (X, B) respectively. We have a diagram whose horizontal rows are Milnor fibrations and whose vertical arrows are covering maps:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \tilde\Sigma^{ab} - L \arrow[r] \arrow[d] & S^1 \arrow[d] \\ F_z \arrow[r] & \Sigma - K \arrow[r] & S^1 \end{tikzcd}$$

The degrees of the second and third vertical arrows are $|D|$ and d respectively, so the first vertical arrow has degree $|D|/d$. Thus $\chi(F) = |D|/d(\chi(F_z))$ and the lemma is proved. $\square$

9.4. **Topological meaning of $\hat G$.** This subsection is a digression, without proofs, about the topology underlying Section 6 (D-curves determined by rooted resolution diagrams). In that section, after selecting a root leaf of the resolution diagram Γ, the group $\hat G$ arose in terms of a $\mathbb{C}^*$–action that is not easily seen to be part of the topological data. But $\hat G$ has a very simple topological meaning.

If Σ is the link of our singularity and K the knot corresponding to the root leaf, then $\hat{G} = H_1(\Sigma_0)$, where Σ_0 is the knot exterior (complement of an open solid torus neighborhood of K). The homology class of a meridian curve M of K in $\partial\Sigma_0$ represents the element $\hat{Q}$ of Proposition 4.4, while the end-knots corresponding to the non-root leaves of Γ represent the elements $\overline{\chi}_j$ of Section 6.

In particular, the value semigroup is the subsemigroup of $H_1(\Sigma_0)$ generated by the classes of the end knots, and the semigroup and congruence conditions together mean that the homology class of the meridian of K is a positive linear combination of the homology classes of the end knots.

Finally, in the induction of Section 6, the resolution sub-diagrams Γ_i determine knot exteriors Σ_i which embed disjointly into Σ_0 (in an obvious way) with complement $D_k \times S^1$, where D_k is a k–holed disk. The meridian curves M_i for the Σ_i match fibers of this $D_k \times S^1$, and Theorem 6.7 describes a part of the homology exact sequence for the pair $(\Sigma_0, \bigcup_{j=1}^k \Sigma_k)$.

10. Corollaries and applications of the End Curve Theorem

As corollaries of the End Curve Theorem, one can explain systematically why all previously known examples of splice quotients are in fact of this type.

Corollary 10.1 ([16])**.** *The universal abelian cover of a weighted homogeneous singularity with $\mathbb{Q}HS$ link is a Brieskorn complete intersection, and the covering group acts diagonally on the coordinates.*

Proof. The minimal resolution graph Γ has one node, and the t leaves correspond to the $\mathbb{C}^*$-orbits with non-trivial isotropy. We will show that for every leaf of Γ, there exists a weighted homogeneous end-curve function. Via the End Curve Theorem, weighted roots of these functions generate the maximal ideal of the universal abelian cover; also, the defining equations begin with sums of admissible monomials, which are powers of these root functions. Because of quasihomogeneity, there can be no higher terms in these equations, so that the splice equations give a Brieskorn complete intersection.

Let X be the affine variety with $\mathbb{C}^*$-action. The $\mathbb{Q}$HS condition is equivalent to the fact that $X - \{0\}/\mathbb{C}^*$ is a rational curve. The $\mathbb{C}^*$-action on $X - \{0\}$ has finitely many orbits with non-trivial isotropy, and the closures of these orbits are end-curves. We shall show that these orbits—in fact, every orbit—can be cut out by a weighted homogeneous function on X. Consider a common multiple N of the orders of the isotropy groups and let μ_N be the cyclic subgroup of $\mathbb{C}^*$ of order N. Then every non-trivial $\mathbb{C}^*$ orbit of $X' = X/\mu_N$ is isomorphic to $\mathbb{C}^*/\mu_N$, so X' is a cone over a rational curve, thus a cyclic quotient singularity. It is thus easy to see that any non-trivial $\mathbb{C}^*$–orbit of X' is cut out by some weighted homogeneous function f, and composing $f\colon X' \to \mathbb{C}$ with the projection $X \to X'$ then gives the desired function on X. $\square$

Corollary 10.2 (Okuma [23])**.** *Rational singularities, and minimally elliptic singularities with $\mathbb{Q}HS$ link, are splice quotients. In particular, their universal abelian cover's are complete intersections.*

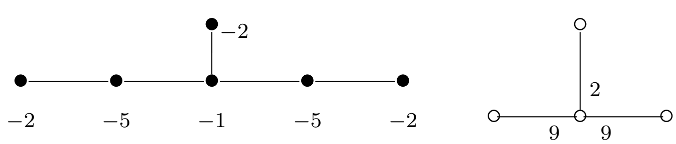
Proof. It was explained in Theorem 13.2 of [20]why the End Curve Theorem would imply this result. Specifically, standard results on rational singularities easily produce end-curve functions (or more generally, functions satisfying any topologically allowed vanishing). The existence of such functions in the minimally elliptic case is slightly harder, but is proved by M. Reid in [26], Lemma, p. 122. □

We remark that Okuma's proof is different from ours, with a key step the argument that the root functions generate the maximal ideal of the universal abelian cover. It uses a strong condition satisfied by the graphs of rational and minimally elliptic singularities, but not by splice-quotients in general. In [19], the first non-trivial case of this theorem was proved, showing that the universal abelian cover of a quotient-cusp (the simplest rational singularity whose graph has two nodes) is a complete intersection cusp singularity.

Corollary 10.3 ([21], Theorem 4.1)**.** *Let (X, o) be a normal surface singularity with integral homology sphere link, for which all the knots associated to leaves are links of hypersurface sections. Then the semigroup condition is fulfilled, and X is a complete intersection of splice type.*

"Equisingular" deformations of a splice quotient singularity whose link is an *integral* homology sphere should remain splice quotients; this is definitely true for positive weight deformations of weighted homogeneous singularities with $\mathbb{Z}HS$ links. On the other hand, it is not true even for fairly simple splice quotients (cf. [12]), and this becomes clear via the End Curve Theorem. The following example comes from E. Sell's Ph.D. thesis.

Example 10.4 ([27], 3.1.4)**.** The weighted homogeneous singularity X defined by $z^2 = x^4 + y^9$ has resolution dual graph and (reduced) splice diagram



Rewriting the equation as $(z - x^2)(z + x^2) = y^9$, one sees that the functions $z \pm x^2$ vanish 9 times along their zero-sets; they, together with x (whose zero-set is reduced) are end-curve functions. The discriminant group is cyclic of order 9. So, one can form the universal abelian cover by adjoining a U satisfying $U^9 = z - x^2$, and a V satisfying $V^9 = z + x^2$, along with x. Then $(UV)^9 = y^9$; but since the universal abelian cover is a normal domain, one must have (perhaps changing V by a 9–th root of 1) that

$y = UV$. Thus, the universal abelian cover is the hypersurface $V^9 = U^9 + 2x^2$, with discriminant group action

$$(U, V, x) \mapsto (\zeta U, \zeta^{-1} V, x),$$

where ζ is a primitive 9–th root of 1. The versal deformation of weight ≥ 0 of the universal abelian cover that is equivariant with respect to the group action is smooth of dimension 3, and defined by

$$V^9 = U^9 + 2x^2 + t_1(UV)^5 + t_2(UV)^6 + t_3(UV)^7.$$

Taking invariants to obtain the "versal splice quotient deformation" of X, and changing coordinates, gives the family

$$y^9 = (z - x^2 - \frac{1}{2}(t_1 y^5 + t_2 y^6 + t_3 y^7))(z + x^2 + \frac{1}{2}(t_1 y^5 + t_2 y^6 + t_3 y^7)),$$

which can be written (to show the deformation of the curve $x^4 + y^9 = 0$)

$$z^2 = x^4 + y^9 + t_1 x^2 y^5 + t_2 x^2 y^6 + t_3 x^2 y^7 + \text{quadratic terms in } t_i\text{'s}.$$

One important point is that the first equation for the versal splice quotient deformation shows explicitly how to lift the end-curve functions $z \pm x^2$ under deformation so that *they remain end-curve functions*, i.e., continue to vanish to order 9 along their zero-sets. A second point is that the positive weight deformation $z^2 = x^4 + y^9 + txy^7$ is *not* a splice-quotient deformation, even to first order.

Remark 10.5. More generally, in her thesis [27] E. Sell considers singularities $z^n = f(x, y)$ with $\mathbb{Q}HS$ link and f analytically irreducible. The splice and semigroup conditions depend only on n and the topological type (i.e., Puiseux pairs) of f, and it turns out that very rarely are they satisfied. However, in all cases where they are satisfied, there exist special f so that the singularity is a splice quotient; and in these cases the defining equation can be rewritten (as in the example above) to highlight the end curve functions. As above, only deformations which are both equisingular and preserve the order of vanishing of these functions give splice quotient deformations.

References

[1] Norbert A'Campo, La fonction zêta d'une monodromie, Comment. Math. Helv. **50** (1975), 233–248. 30

[2] Gabor Braun and Andras Némethi, Surgery formula for Seiberg–Witten invariants of negative definite plumbed 3-manifolds, arXiv:0704.3145. 2

[3] R.-O. Buchweitz and G.-M. Greuel, The Milnor number and deformations of complex curve singularities. Invent. Math. **58** (1980), 241–281. 14, 29

[4] Charles Delorme, Sous-monoïdes d'intersection complète de ℕ, Ann. Sci. École Norm. Sup. (4) **9** (1976), 145–154. 3

[5] B. de Smit and H.W. Lenstra, Jr, Finite complete intersection algebras and the completeness radical. J. Algebra **196** (1997), 520–531. 3

[6] Alan H Durfee, Fibered knots and algebraic singularities. Topology **13** (1974), 47–59. 28

[7] D. Eisenbud and W.D. Neumann, *Three-dimensional link theory and invariants of plane curve singularities.* Ann. Math. Stud. **110**, Princeton. Princeton Univ. Press (1985). 4, 8

[8] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, Manuscripta Math. **3** (1970), 175–193. 3

[9] J. Herzog and E. Kunz, Die Wertehalbgruppe eines lokalen Rings der Dimension 1, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, 2. Abh. (1971, Springer Verlag), 27–67. 3

[10] G.-M. Greuel and H. Hamm, Invarianten quasihomogener vollständiger Durchschnitte, Invent. Math. **49** (1978), 67-86. 14

[11] András Némethi and Liviu E. Nicolaescu, Seiberg–Witten invariants and surface singularities, Geometry and Topology **6** (2002), 269–328. 3

[12] I. Luengo-Velasco, A. Melle-Hernández, A. Némethi, Links and analytic invariants of superisolated singularities. J. Algebraic Geom. **14** (2005), 543–566. 3, 32

[13] Andras Némethi and Tomohiro Okuma, On the Casson Invariant Conjecture of Neumann–Wahl, arXiv:math/0610465 2

[14] ———, The Seiberg-Witten Invariant Conjecture for splice-quotients, preprint 2, 3

[15] W.D. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. Trans. Amer. Math. Soc. **268** (1981), 299–343. 25

[16] ———, Abelian covers of quasihomogeneous surface singularities, *Singularities, Arcata 1981*, Proc. Symp. Pure Math. **40** (Amer. Math. Soc. 1983), 233–243. 2, 16, 31

[17] W.D. Neumann and J. Wahl, Casson invariant of links of singularities. Comment. Math. Helv. **65** (1990), 58–78. 2

[18] ———, Universal abelian covers of surface singularities, *Trends on Singularities*, A. Libgober and M. Tibar, eds. (Birkhäuser Verlag, 2002), 181–190. 6

[19] ———, Universal abelian covers of quotient-cusps. Math. Ann. **326** (2003), 75–93. 32

[20] ———, Complete intersection singularities of splice type as universal abelian covers. Geom. Topol. **9** (2005), 699-755. 1, 2, 3, 5, 6, 7, 9, 16, 17, 26, 27, 32

[21] ———, Complex surface singularities with integral homology sphere links. Geom. Topol. **9** (2005), 757-811. 2, 3, 7, 8, 27, 32

[22] T. Okuma, Universal abelian covers of rational surface singularities. J. London Math. Soc. (2) **70** (2004), 307-324. 3

[23] ———, Universal abelian covers of certain surface singularities. Math. Ann. **334** (2006), 753-773. 2, 32

[24] ———, The geometric genus of splice-quotient singularities, arXiv:math/0610464 2

[25] H.M. Pedersen, Dissertation, Columbia University, in preparation. 5

[26] M. Reid, Chapters on Algebraic Surfaces, *Complex algebraic geometry (Park City UT, 1993)*, IAS/Park City Math. Ser. 3, Amer. Math. Soc. (Providence, RI, 1997), 3-159. 32

[27] E. Sell, Universal abelian covers for surface singularities $\{z^n = f(x,y)\}$, UNC PhD thesis (2007). 32, 33

[28] K.-i. Watanabe, Some examples of one dimensional Gorenstein domains, Nagoya Math. J. **49** (1973), 101-109. 3

Department of Mathematics, Barnard College, Columbia University, New York, NY 10027
E-mail address: neumann@math.columbia.edu

Department of Mathematics, The University of North Carolina, Chapel Hill, NC 27599-3250
E-mail address: jmwahl@email.unc.edu