Separating sets, metric tangent cone and applications for complex algebraic germs

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Abstract. An explanation is given for the initially surprising ubiquity of separating sets in normal complex surface germs. It is shown that they are quite common in higher dimensions too. The relationship between separating sets and the geometry of the metric tangent cone of Bernig and Lytchak is described. Moreover, separating sets are used to show that the inner Lipschitz type need not be constant in a family of normal complex surface germs of constant topology.

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1. Introduction

Given a complex algebraic germ (X, x_0), a choice of generators $x_1, \dots, x_N$ of its local ring gives an embedding of (X, x_0) into $(\mathbb{C}^N, 0)$. It then carries two induced metric space structures: the "outer metric" induced from distance in $\mathbb{C}^N$ and the "inner metric" induced by arc-length of curves on X. In the Lipschitz category each of these metrics is independent of choice of embedding: different choices give metrics for which the identity map is a bi-Lipschitz homeomorphism. The inner metric, which is given by a Riemannian metric off the singular set, is the one that interests us most here. It is determined by the outer metric, so germs that are distinguished by their inner metrics are distinguished by their outer ones.

These metric structures have so far seen much more study in real algebraic geometry than in the complex algebraic world. In fact, until fairly recently conventional wisdom was that bi-Lipschitz geometry would have little to say for normal germs of complex varieties. For example, it is easy to see that two complex curve germs with the same number of components are bi-Lipschitz homeomorphic (inner metric). So for curve germs bi-Lipschitz geometry is equivalent to topology. The same holds for outer bi-Lipschitz geometry of plane curves: two germs of complex curves in $\mathbb{C}^2$ are bi-Lipschitz homeomorphic for the outer metric if and only if they are topologically equivalent as embedded germs [18, 12]. However, it has recently become apparent that the bi-Lipschitz geometry of complex surface germs is quite rich; for example, they rarely have trivial geometry (in the sense of being bi-Lipschitz homeomorphic to a metric Euclidean cone). We give here an explanation which shows that the same holds in higher dimensions too. The particular bi-Lipschitz invariants we will discuss are "separating sets".

Let (X, x_0) be a germ of a k-dimensional semialgebraic set. A *separating set* of (X, x_0) (see Section 2) is a subgerm $(Y, x_0) \subset (X, x_0)$ of dimension less than k which locally separates X into two pieces A and B which are "fat" at x_0 while Y itself is "thin" (i.e., the k–dimensional densities at x_0 of A and B are nonzero and the $(k-1)$–dimensional density at x_0 of Y is zero).

There are trivial ways a separating set can occur—for example as the intersection of the components of a complex germ (X, x_0) which is the union of two irreducible components of equal dimension. The intersection of the two components clearly separates X and it is thin because its real codimension is at least 2. Fig. 1 illustrates schematically a codimension 1 example of a separating set. The

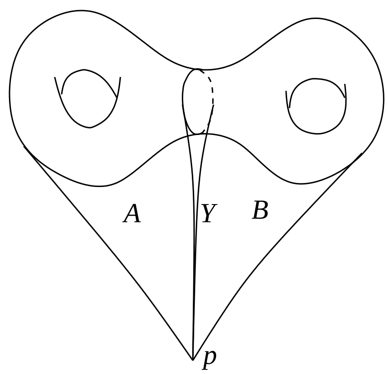

FIGURE 1. Separating set

interesting question is whether such separating sets exist "in nature"—for isolated singularities in particular.

For real algebraic singularities examples can be constructed (see [2]), but they do not seem to arise very naturally. But for normal complex surface singularities they had turned out to be surprisingly common: already the the simplest singularities, namely the Kleinian surfaces singularities $A_k = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^{k+1} = 0\}$, have separating sets at the origin when $k > 1$ (see [4]). This paper is devoted to the investigation of this phenomena in all complex dimensions ≥ 2. We restrict to isolated complex singularities.

Our first result (Theorem 3.1) is that *if X is a weighted homogeneous complex surface $\mathbb{C}^3$ with weights $w_1 \geq w_2 > w_3$ and if the zero set $X \cap \{z = 0\}$ of the* *variable z of lowest weight has more than one branch at the origin, then $(X,0)$ has a separating set.* This reproves that A_k has a separating set at the origin when $k>1$, and shows more generally that the same holds for the Brieskorn singularity

$$X(p,q,r) := \{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\}$$

if $p \le q < r$ and $\gcd(p,q) > 1$.

It also proves that for $t \neq 0$ the singularity X_t from the Briançon-Speder family [9]

$$X_t = \{(x,y,z) \in \mathbb{C}^3 \mid x^5 + z^{15} + y^7 z + txy^6 = 0\}$$

has a separating set at the origin. On the other hand, we show (Theorem 4.1) that X_t does not have a separating set when $t=0$. Thus *the inner bi-Lipschitz type of a normal surface germ is not determined by topological type, even in a family of singularities of constant topological type* (and is thus also not determined by the resolution graph).

We also show (see Theorem 5.4) that *if the tangent cone $T_{x_0}X$ of an isolated complex singularity (X,x_0) has a non-isolated singularity and the non-isolated locus is a separating set of $T_{x_0}X$, then (X,x_0) has a separating set.*

It follows, for instance (see [6]), that all quotient surface singularities $\mathbb{C}^2/G$ (with $G \subset \mathrm{GL}(2,\mathbb{C})$ acting freely on $\mathbb{C}^2 \setminus \{0\}$) have separating sets except the ones that are obviously conical ($\mathbb{C}^2/\mu_n$ with the group μ_n of n–th roots of unity acting by multiplication) and possibly, among the simple singularities, E_6, E_7, E_8 and the D_n series. Moreover, for any k one can find cyclic quotient singularities with more than k disjoint non-equivalent separating sets. Theorem 5.4 also easily gives examples of separating sets for isolated singularities in higher dimension.

It is natural to ask if the converse to 5.4 holds, i.e., separating sets in (X,x_0) always correspond to separating sets in $T_{x_0}X$, but this is not so: the tangent cone of the Briançon-Speder singularity X_t, which has a separating set for $t \neq 0$, is $\mathbb{C}^2$. But in Theorem 5.7 we give necessary and sufficient conditions for existence of a separating set in terms of the "metric tangent cone" $\mathcal{T}_{x_0}X$, the theory of which was recently developed by Bernig and Lytchak [7].

$\mathcal{T}_{x_0}X$ is defined as the Gromov-Hausdorff limit as $t \to 0$ of the result of scaling the inner metric of the germ (X,x_0) by $\frac{1}{t}$. Another way of constructing $\mathcal{T}_{x_0}X$, and the one we actually use, is as the usual tangent cone of a "normal re-embedding" [8] of X (for a complex germ, such a normal re-embedding may only exists after forgetting complex structure and considering (X,x_0) as a real germ [5]).

2. Separating sets

Let $X \subset \mathbb{R}^n$ be a k-dimensional rectifiable subset. Recall that the inferior and superior k–densities of X at the point $x_0 \in \mathbb{R}^n$ are defined by:

$$\underline{\Theta}^k(X,x_0) = \lim_{\epsilon \to 0^+} \inf \frac{\mathcal{H}^k(X \cap \epsilon B(x_0))}{\eta \epsilon^k}$$

and

$$\overline{\Theta}^k(X,x_0) = \lim_{\epsilon\to 0^+} \sup \frac{\mathcal{H}^k(X\cap \epsilon B(x_0))}{\eta\epsilon^k}\,,$$

where $\epsilon B(x_0)$ is the n–dimensional ball of radius ϵ centered at x_0, η is the volume of the k–dimensional unit ball and $\mathcal{H}^k$ is k–dimensional Hausdorff measure in $\mathbb{R}^n$. If

$$\underline{\Theta}^k(X,x_0) = \theta = \overline{\Theta}^k(X,x_0)\,,$$

then Θ is called the k–dimensional density of X at x_0 (or simply k–density at x_o).

Remark 2.1. Recall that if $X \subset \mathbb{R}^n$ is a semialgebraic (even subanalytic) subset, then the above two limits are equal and the k–density of X is well defined for any point of $\mathbb{R}^n$. Moreover, the vanishing or non-vanishing of these densities is a bi-Lipschitz invariant invariant, since a bi-Lipschitz homeomorphism clearly changes them by a factor that is bounded by k and the Lipschitz constant. See [15].

Definition 2.2. Let $X \subset \mathbb{R}^n$ be a k-dimensional semialgebraic set and let $x_0 \in X$ be a point such that the link of X at x_0 is connected and the k–density of X at x_0 is positive. A $(k-1)$–dimensional closed rectifiable subset $Y \subset X$ with $x_0 \in Y$ is called a *separating set of X at x_0* if (see Fig. 1)

- for some small $\epsilon > 0$ the subset $\big(\epsilon B(x_0)\cap X\big)\setminus Y$ has at least two connected components A and B,
- the superior $(k-1)$–density of Y at x_0 is zero,
- the inferior k–densities of A and B at x_0 are nonzero.

More generally, we need only require that the above is true locally, in the sense that it holds after replacing X by the union of $\{x_0\}$ and a neighborhood of $Y\setminus\{x_0\}$ in $X\setminus\{x_0\}$. For simplicity of exposition we will leave to the reader to check that our results remain correct with this more general definition.

Proposition 2.3 (Lipschitz invariance of separating sets). *Let X and Z be two real semialgebraic sets. If there exists a bi-Lipschitz homeomorphism of germs $F\colon (X,x_0)\to(Z,z_0)$ with respect to the inner metric, then X has a separating set at $x_0\in X$ if and only if Z has a separating set at $z_0\in Z$.*

Proof. The result would be immediate if separating sets were defined in terms of the inner metrics on X and Z. So we must show that separating sets can be defined this way.

Let $X\subset\mathbb{R}^n$ be a connected semialgebraic subset. Consider the set X equipped with the inner metric and with the Hausdorff measure $\mathcal{H}^k_X$ associated to this metric. Let $Y\subset X$ be a k-dimensional rectifiable subset. We define the inner inferior and superior densities of Y at $x_0\in X$ with respect to inner metric on X as follows:

$$\underline{\Theta}^k(X,Y,x_0) = \lim_{\epsilon\to 0^+} \inf \frac{\mathcal{H}^k_X(Y\cap \epsilon B_X(x_0))}{\eta\epsilon^k}$$

and

$$\overline{\Theta}^k(X,Y,x_0) = \lim_{\epsilon\to 0^+} \sup \frac{\mathcal{H}^k_X(Y\cap \epsilon B_X(x_0))}{\eta\epsilon^k}\,,$$

where $\epsilon B_X(x_0)$ denotes the closed ball in X (with respect to the inner metric) of radius ϵ centered at x_0. The fact that separating sets can be defined using the inner metric now follows from the following proposition, completing the proof. $\square$

Proposition 2.4. *Let $X \subset \mathbb{R}^n$ be a semialgebraic connected subset. Let $W \subset X$ be a k-dimensional rectifiable subset and $x_0 \in X$. Then, there exist two positive constants κ_1 and κ_2 such that:*

$$\kappa_1 \underline{\Theta}^k(X, W, x_0) \leq \underline{\Theta}^k(W, x_0) \leq \kappa_2 \underline{\Theta}^k(X, W, x_0)$$

and

$$\kappa_1 \overline{\Theta^k}(X, W, x_0) \leq \overline{\Theta}^k(W, x_0) \leq \kappa_2 \overline{\Theta}^k(X, W, x_0)\,.$$

Proof. If we used the outer metric instead of the inner metric in the definition of $\overline{\Theta^k}(X, W, x_0)$ and $\underline{\Theta^k}(X, W, x_0)$ we'd just get $\overline{\Theta}^k(W, x_0)$ and $\underline{\Theta}^k(W, x_0)$. Thus the proposition follows immediately from the Kurdyka's "Pancake Theorem" ([14], [8]) which says that if $X \subset \mathbb{R}^n$ is a semialgebraic subset then there exists a finite semialgebraic partition $X = \bigcup_{i=1}^{l} X_i$ such that each X_i is a semialgebraic connected set whose inner metric and outer (Euclidean) metric are bi-Lipschitz equivalent. $\square$

The following Theorem shows that the germ of an isolated complex singularity which has a separating set cannot be *metrically conical*, i.e., bi-Lipschitz homeomorphic to the Euclidean metric cone on its link.

Theorem 2.5. *Let (X, x_0) be a $(n+1)$-dimensional metric cone whose base is a compact connected Lipschitz manifold (possibly with boundary). Then, X does not have a separating set at x_0.*

Proof. Let M be an n-dimensional compact connected Lipschitz manifold with boundary. For convenience of exposition we will suppose that M is a subset of the Euclidean sphere $S^{k-1} \in \mathbb{R}^k$ centered at 0 and with radius 1 and X the cone over M with vertex at the origin $0 \in \mathbb{R}^k$. Suppose that $Y \subset X$ is a separating set, so $X \setminus Y = A \cup B$ with A and B open in $X \setminus Y$; the n–density of Y at 0 is equal to zero and the inferior $(n+1)$–densities of A and B at 0 are unequal to zero. In particular, there exists $\xi > 0$ such that these inferior densities of A and B at 0 are bigger than ξ. For each $t > 0$, let $\rho_t \colon X \cap tD^k \to X$ be the map $\rho_t(x) = \frac{1}{t}x$, where tD^k is the ball about $0 \in \mathbb{R}^k$ of radius t. Denote $Y_t = \rho_t(Y \cap tD^k)$, $A_t = \rho_t(A \cap tD^k)$ and $B_t = \rho_t(B \cap tD^k)$. Since the n–density of Y at 0 is equal to zero, we have:

$$\lim_{t \to 0^+} \mathcal{H}^n(Y_t) = 0\,.$$

Also, since the inferior densities of A and B at 0 are bigger than ξ, we have that $\mathcal{H}^{n+1}(A_t) > \xi$ and $\mathcal{H}^{n+1}(B_t) > \xi$ for all sufficiently small $t > 0$.

Let r be a radius such that $X \cap rD^k$ has volume $\leq \xi/2$ and denote by X', A'_t, B'_t, Y'_t the result of removing from each of X, A_t, B_t, Y_t the intersection with the interior of the ball rB^k. Then X' is a Lipschitz $(n+1)$–manifold (with boundary),

A'_t and B'_t subsets of $(n+1)$–measure $> \xi/2$ separated by Y_t of arbitrarily small n–measure.

The following lemma then gives the contradiction to complete the proof. □

Lemma 2.6. *Let X' be a $(n+1)$-dimensional compact connected Lipschitz manifold with boundary. Then, for any $\xi > 0$ there exists $\epsilon > 0$ such that if $Y' \subset X'$ is a n-dimensional rectifiable subset with $\mathcal{H}^n(Y') < \epsilon$, then $X' \setminus Y'$ has a connected component A of $(n+1)$–measure exceeding $\mathcal{H}^{n+1}(X') - \xi/2$ (so any remaining components have total measure $< \xi/2$).*

Proof. If X' is bi-Lipschitz homeomorphic to a ball then this follows from standard isoperimetric results: for a ball the isoperimetric problem is solved by spherical caps normal to the boundary (Burago and Maz'ja [10] p. 54, see also Hutchins [13]). The isoperimetric problem is often formulated in terms of currents, in which case one uses also that the mass of the current boundary of a region is less than or equal to the Hausdorff measure of the topological boundary ([11] 4.5.6 or [16] Section 12.2).

Let $\{T_i\}_{i=1}^m$ be a cover of X' by subsets which are bi-Lipschitz homeomorphic to balls and such that

$$T_i \cap T_j \neq \emptyset \ \Rightarrow \ \mathcal{H}^{n+1}(T_i \cap T_j) > 0.$$

Without loss of generality we may assume

$$\xi/m < \min\{\mathcal{H}^{n+1}(T_i \cap T_j) \mid T_i \cap T_j \neq \emptyset\}\,.$$

Since T_i is bi-Lipschitz homeomorphic to a ball there exists ϵ_i satisfying the conclusion of this lemma for ξ/m. Let $\epsilon = \min(\epsilon_1, \dots, \epsilon_m)$. So if $Y' \subset X'$ is an n–dimensional rectifiable subset such that $\mathcal{H}^n(Y') < \epsilon$, then for each i the largest component A_i of $T_i \setminus Y'$ has complement B_i of measure $< \xi/2m$.

We claim $\bigcup_{i=1}^m A_i$ is connected. It suffices to show that

$$T_i \cap T_j \neq \emptyset \ \Rightarrow \ A_i \cap A_j \neq \emptyset\,.$$

So suppose $T_i \cap T_j \neq \emptyset$. Then $B_i \cup B_j$ has measure less than ξ, which is less than $\mathcal{H}^n(T_i \cap T_j)$, so $T_i \cap T_j \not\subset B_i \cup B_j$. This is equivalent to $A_i \cap A_j \neq \emptyset$.

Thus there exists a connected component A of $X' \backslash Y'$ which contains $\bigcup_{i=1}^m A_i$. Its complement B is a subset of $\bigcup_{i=1}^m B_i$ and thus has measure less than $\xi/2$. □

3. Separating sets in normal surface singularities

Theorem 3.1. *Let $X \subset \mathbb{C}^3$ be a weighted homogeneous algebraic surface with respect to the weights $w_1 \geq w_2 > w_3$ and with an isolated singularity at 0. If $\big(X \setminus \{0\}\big) \cap \{z = 0\}$ is not connected, then X has a separating set at 0.*

Example. This theorem applies to the Brieskorn singularity

$$X(p,q,r) := \{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0\}$$

if $p \leq q < r$ and $\gcd(p,q) > 1$. In particular it is not metrically conical. This was known for a different reason by [3]: a weighted homogeneous surface singularity (not necessarily hypersurface) whose two lowest weights are distinct is not metrically conical.

Proof of Theorem 3.1. Since X is weighted homogeneous, the intersection $X \cap S^5$ is transverse and gives the singularity link. By assumption, $\big(X \cap S^5\big) \cap \{z = 0\}$ is the disjoint union of two nonempty semialgebraic closed subsets $\widetilde{A}$ and $\widetilde{B} \subset \big(X \cap S^5\big) \cap \{z = 0\}$. Let $\widetilde{M}$ be the conflict set of $\widetilde{A}$ and $\widetilde{B}$ in $X \cap S^5$, i.e.,

$$\widetilde{M} := \{p \in X \cap S^5 \mid d(p, \widetilde{A}) = d(p, \widetilde{B})\}\,,$$

where $d(\cdot,\cdot)$ is the standard metric on S^5 (euclidean metric in $\mathbb{C}^3$ gives the same set). Clearly, $\widetilde{M}$ is a compact semialgebraic subset and there exists $\delta > 0$ such that $d(\widetilde{M}, \{z = 0\}) > \delta$. Let $M = \overline{\mathbb{C}^*\widetilde{M} \cup \{0\}}$ (the closure of the union of $\mathbb{C}^*$–orbits through $\widetilde{M}$). Note that the $\mathbb{C}^*$–action restricts to a unitary action of S^1, so the construction of $\widetilde{M}$ is invariant under the S^1–action, so $M = \mathbb{R}^*\widetilde{M}$, and is therefore 3–dimensional. It is semi-algebraic by the Tarski-Seidenberg theorem. We will use the weighted homogeneous property of M to show $\dim(T_0M) \leq 2$, where T_0M denotes the tangent cone of M at 0, from which will follow that M has zero 3–density. In fact, we will show that $T_0M \subset \{x = 0, y = 0\}$.

Let $T \colon \widetilde{M} \times [0, +\infty) \to M$ be defined by:

$$T((x,y,z),t) = (t^{\frac{w_1}{w_3}}x, t^{\frac{w_2}{w_3}}y, tz).$$

Clearly, the restriction $T|_{\widetilde{M}\times(0,+\infty)} \colon \widetilde{M} \times (0, +\infty) \to M \setminus \{0\}$ is a bijective semialgebraic map. Let $\gamma \colon [0, \epsilon) \to M$ be a semianalytic arc; $\gamma(0) = 0$ and $\gamma'(0) \neq 0$. We consider $\phi(s) = T^{-1}(\gamma(s))$ for all $s \neq 0$. Since ϕ is a semialgebraic map and M is compact, $\lim\limits_{s\to 0} \phi(s)$ exists and belongs to $\widetilde{M} \times \{0\}$. For the same reason, $\lim\limits_{s\to 0} \phi'(s)$ also exists and is nonzero. Therefore, the arc ϕ can be extended to $\phi \colon [0, \epsilon) \to \widetilde{M} \times [0, +\infty)$ such that $\phi(0) \in \widetilde{M} \times \{0\}$ and $\phi'(0)$ exists and is nonzero. We can take the $[0, \infty)$ component of ϕ as parameter and write $\phi(t) = ((x(t), y(t), z(t), t)$. Then $\gamma(t) = (t^{w_1/w_3}x(t), t^{w_2/w_3}y(t), tz(t))$, so

$$\begin{aligned} \lim_{t\to 0^+} \frac{\gamma(t)}{t} &= \left(\lim_{t\to 0} \frac{t^{\frac{w_1}{w_3}}}{t}x(t)\,,\ \lim_{t\to 0} \frac{t^{\frac{w_2}{w_3}}}{t}y(t)\,,\ \lim_{t\to 0} z(t)\right) \\ &= (0, 0, z(0))\,. \end{aligned}$$

This is a nonzero vector (note $|z(0)| > \delta$) in the set $\{x = 0, y = 0\}$, so we obtain that

$$T_0M \subset \{x = 0, y = 0\}.$$

Since M is a 3-dimensional semialgebraic set and $\dim(T_0M) \leq 2$, we obtain that the 3-dimensional density of M at 0 is equal to zero ([15]).

Now, we have the following decomposition:

$$X \setminus M = A \cup B\,,$$

where $\widetilde{A} \subset A$, $\widetilde{B} \subset B$, A and B are $\mathbb{C}^*$–invariant and $A \cap B = \emptyset$. Since A and B are semialgebraic sets, the 4–densities $\mathrm{density}_4(A,0)$ and $\mathrm{density}_4(B,0)$ are defined. We will show that these densities are nonzero. It is enough to prove that $\dim_{\mathbb{R}}(T_0A) = 4$ and $\dim_{\mathbb{R}}(T_0B) = 4$. Let $\Gamma \subset A$ be a connected component of $A \cap \{z = 0\}$. Note that $\bar{\Gamma} = \Gamma \cup \{0\}$ is a complex algebraic curve. We will show that T_0A contains the set $\{(x,y,v) \mid (x,y,0) \in \bar{\Gamma}, v \in \mathbb{C}\}$ if $w_1 = w_2$ (note that $\bar{\Gamma}$ is the line through $(x,y,0)$ in this case) or either the y–z or the x–z plane if $w_1 < w_2$.

Given a smooth point $(x,y,0) \in \Gamma$ and $v \in \mathbb{C}$, we may choose a smooth arc $\gamma\colon [0,\epsilon) \to A$ of the form $\gamma(t) = (\gamma_1(t), \gamma_2(t), t^m\gamma_3(t))$ with $(\gamma_1(0), \gamma_2(0)) = (x,y)$ and $\gamma_3(0) = v$. Then, using the $\mathbb{R}^*$–action, we transform this arc to the arc $\phi(t) = t^j\gamma(t)$ with j chosen so $jw_3 + m = jw_2$. Now $\phi(t) = (t^{jw_1}\gamma_1(t), t^{jw_2}\gamma_2(t), t^{jw_2}\gamma_3(t))$ is a path in A starting at the origin. Its tangent vector ρ at $t = 0$,

$$\rho = \lim_{t\to 0+} \frac{\phi(t)}{t^{jw_2}}\,,$$

is $\rho = (x,y,v)$ if $w_1 = w_2$ and $\rho = (0,y,v)$ if $w_1 > w_2$. If $w_1 > w_2$ and $y = 0$ then the same argument, but with j chosen with $jw_3 + m = jw_1$, gives $\rho = (x,0,v)$. This proves our claim and completes the proof that T_0A has real dimension 4. The proof for T_0B is the same. $\square$

4. The Briançon-Speder example

For each $t \in \mathbb{C}$, let $X_t = \{(x,y,z) \in \mathbb{C}^3 \mid x^5 + z^{15} + y^7z + txy^6 = 0\}$. This X_t is weighted homogeneous with respect to weights $(3,2,1)$ and has an isolated singularity at $0 \in \mathbb{C}^3$.

Theorem 4.1. *X_t has a separating set at 0 if $t \neq 0$ but does not have a separating set at 0 if $t = 0$.*

Proof. As already noted, for $t \neq 0$ Theorem 3.1 applies, so X_t has a separating set. So from now on we take $t = 0$. Denote $X := X_0$. In the following, for each sufficiently small $\epsilon > 0$, we use the notation

$$X^\epsilon = \{(x,y,z) \in X \mid \epsilon|y| \le |z| \le \frac{1}{\epsilon}|y|\}.$$

We need a lemma.

Lemma 4.2. *X^ϵ is metrically conical at the origin with connected link.*

Proof. Note that the lemma makes a statement about the germ of X^ϵ at the origin. We will restrict to the part of X^ϵ that lies in a suitable closed neighborhood of the origin.

Let $P\colon \mathbb{C}^3 \to \mathbb{C}^2$ be the orthogonal projection $P(x,y,z) = (y,z)$. The restriction P_X of P to X is a 5-fold cyclic branched covering map branched along $\{(y,z) \mid z^{15} + y^7z = 0\}$. This is the union of the y–axis in $\mathbb{C}^2$ and the seven curves $y = \zeta z^2$ for ζ a 7–th root of unity. These seven curves are tangent to the z–axis.

Let

$$C^\epsilon = \{(y,z) \in \mathbb{C}^2 \mid \epsilon|y| \le |z| \le \frac{1}{\epsilon}|y|\}.$$

Notice that no part of the branch locus of P_X with $|z| < \epsilon$ is in C^ϵ. In particular, if D is a disk in $\mathbb{C}^2$ of radius $< \epsilon$ around 0, then the map P_X restricted to X^ϵ has no branching over this disk. We choose the radius of D to be $\epsilon/2$ and denote by Y the part of part of X^ϵ whose image lies inside this disk. Then Y is a covering of $C^\epsilon \cap D$, and to complete the proof of the lemma we must show it is a connected covering space and that the covering map is locally K–bi-Lipschitz for a fixed K.

Since it is a Galois covering with group $\mathbb{Z}/5$, to show it is a connected cover it suffices to show that there is a closed curve in $C^\epsilon \cap D$ which does not lift to a closed curve in Y. Choose a small constant $c \le \epsilon/4$ and consider the curve $\gamma\colon [0,1] \to C^\epsilon \cap D$ given by $\gamma(t) = (ce^{2\pi i t}, c)$. A lift to Y has x–coordinate $(c^{15} + c^8 e^{14\pi i t})^{1/5}$, which starts close to $c^{8/5}$ (at $t = 0$) and ends close to $c^{8/5}e^{(14/5)\pi i}$ (at $t = 1$), so it is not a closed curve.

To show that the covering map is locally K–bi-Lipschitz, we note that locally Y is the graph of the implicit function $(y,z) \mapsto x$ given by the equation $x^5 + z^{15} + y^7 z = 0$, so it suffices to show that the derivatives of this implicit function are bounded. Implicit differentiation gives

$$\frac{\partial x}{\partial y} = -\frac{7y^6 z}{5x^4}\,, \quad \frac{\partial x}{\partial z} = -\frac{15z^{14} + y^7}{5x^4}\,.$$

It is easy to see that there exists $\lambda > 0$ such that

$$|15z^{14} + y^7| \le \lambda |z|^4,\ |y^7| \le \lambda |z^{14} + y^7| \text{ and } |15z^{14} + y^7| \le \lambda |z^{14} + y^7|,$$

for all $(y,z) \in C^\epsilon \cap D$. We then get

$$\left|\frac{\partial x}{\partial y}\right|^5 = \frac{7^5 |y^{30} z^5|}{5^5 |z^{14} + y^7|^4 |z|^4} \le \frac{7^5}{5^5}\lambda^4 |y^2 z| < \frac{7^5 \lambda^4 \epsilon^3}{5^5 2^3}$$

and

$$\left|\frac{\partial x}{\partial z}\right|^5 = \frac{|15z^{14} + y^7|^5}{5^5 |z^{14} + y^7|^4 |z|^4} \le \frac{\lambda^5}{5^5}\,,$$

completing the proof. □

We now complete the proof of Theorem 4.1. Let us suppose that X has a separating set. Let $A, B, Y \subset X$ be subsets satisfying:

- for some small $\epsilon > 0$ the subset $[\epsilon B(x_0) \cap X] \setminus Y$ is the union of relatively open subsets A and B,
- the 3-dimensional density of Y at 0 is equal to zero,
- the 4-dimensional inferior densities of A and B at 0 are unequal to zero.

Set

$$N^\epsilon = \{(x,y,z) \in \mathbb{C}^3 \mid |z| \le \epsilon|y| \text{ or } |y| \le \epsilon|z|\}.$$

For each subset $H \subset \mathbb{C}^3$ we denote

$$H^\epsilon = H \cap [\mathbb{C}^3 \setminus N^\epsilon].$$

We are going to prove that $\Theta^4(X\cap N^\epsilon)\to 0$ as $\epsilon\to 0$. We recall the following statement of K. Kurdyka and G. Raby (Theorem 3.8 in [15]).

Proposition 4.3. *Let Z be a subanalytic subset of $\mathbb{R}^n$ of dimension k. Given $y\in\bar{Y}$, let*

$$C_1\dots,C_l$$

be the components of the tangent cone T_yY. Let $n_1,\dots,n_l\in\mathbb{N}$ be the multiplicities of Z at q along $C_1,\dots,C_l$ respectively. Then

$$\Theta^k(Z,y)=\sum_{i=1}^{l} n_i\frac{vol_k(C_j\cap B(0,1))}{\sigma^k}$$

where σ^k denotes the volume of the k-dimensional Euclidean ball.

The tangent cone of $X\cap N^\epsilon$ at 0 is contained in the following set

$$T_1^\epsilon=\{(0,y,z)\in\mathbb{C}^3\ :\ |z|\le\epsilon|y|\}\text{ and }T_2^\epsilon=\{(0,y,z)\in\mathbb{C}^3\ :\ |y|\le\epsilon|z|\}.$$

Since $\mathcal{H}^4(T_i^\epsilon\cap B(0,1))\to 0$ as $\epsilon\to 0$ ($i=1,2$), by the above proposition, it follows that $\Theta^4(X\cap N^\epsilon)\to 0$ as $\epsilon\to 0$.

Let $0<2\delta<\min\{\underline{\Theta}^4(A,0),\underline{\Theta}^4(B,0)\}$. Since $\Theta^4(X\cap N^\epsilon)\to 0$ as $\epsilon\to 0$, we take $\epsilon>0$ small enough such that

$$\Theta^4(X\cap N^\epsilon,0)<\delta.$$

In particular, for all $r>0$ small enough, we have

$$\mathcal{H}^4(X\cap N^\epsilon\cap B(0,r))\le\delta r^4. \tag{4.1}$$

Then,

$$\underline{\Theta}^4(A,0)=\liminf_{r\to 0^+}\left(\frac{\mathcal{H}^4(A^\epsilon\cap B(0,r))}{r^4}+\frac{\mathcal{H}^4(A\cap N^\epsilon\cap B(0,r))}{r^4}\right)$$

and, by (4.1),

$$2\delta<\underline{\Theta}^4(A,0)\le\underline{\Theta}^4(A^\epsilon,0)+\delta,$$

hence $\underline{\Theta}^4(A^\epsilon,0)>\delta$. In a similar way, we show that $\underline{\Theta}^4(B^\epsilon,0)>\delta$. These facts are enough to conclude that Y^ϵ is a separating set of X^ϵ. But in view of Lemma 4.2 this contradicts Theorem 2.5. □

5. Metric Tangent Cone and separating sets

Given a closed and connected semialgebraic subset $X\subset\mathbb{R}^m$ equipped with the inner metric d_X, for any point $x\in X$, we denote by $\mathcal{T}_xX$ the metric tangent cone of X at x; see Bernig and Lytchak [7]. Recall that the metric tangent cone of a metric space X at a point $x\in X$ is defined as the Gromov-Hausdorff limit

$$\mathcal{T}_xX=\lim_{t\to 0+}(B(x,t),\frac{1}{t}d_X)$$

where $\frac{1}{t}d_X$ is the distance on X divided by t. Bernig and Lytchak show that for a semialgebraic set the metric tangent cone exists and is semialgebraic. Moreover, a semialgebraic bi-Lipschitz homeomorphism of germs induces a bi-Lipschitz equivalence of their metric tangent cones (with the same Lipschitz constant).

Recall that a connected semialgebraic set $X \subset \mathbb{R}^m$ is called *normally embedded* if the inner d_X and the outer d_e metrics on X are bi-Lipschitz equivalent.

Theorem 5.1 ([8]). *Let $X \subset \mathbb{R}^m$ be a connected semialgebraic set. Then there exist a normally embedded semialgebraic set $\widetilde{X} \subset \mathbb{R}^q$ and a semialgebraic homeomorphism $p\colon \widetilde{X} \to X$ which is bi-Lipschitz with respect to the inner metric. $\widetilde{X}$, or more precisely the pair $(\widetilde{X}, p)$, is called a normal embedding of X.*

The following result relates the metric tangent cone of X at x and the usual tangent cone of the normally embedded set.

Theorem 5.2 ([7, Section 5]). *Let $X \subset \mathbb{R}^m$ be a closed and connected semialgebraic set and $x \in X$. If $(\widetilde{X}, p)$ is a normal embedding of X, then $T_{p^{-1}(x)}X$ is bi-Lipschitz homeomorphic to the metric tangent cone $\mathcal{T}_x X$.*

Theorem 5.3. *If (X_1, x_1) and (X_2, x_2) are germs of semialgebraic sets which are semialgebraically bi-Lipschitz homeomorphic with respect to the induced outer metric, then their tangent cones $T_{x_1}X_1$ and $T_{x_2}X_2$ are semialgebraically bi-Lipschitz homeomorphic.*

Proof. This is proved in [5]. Without the conclusion that the bi-Lipschitz homeomorphism of tangent cones is semi-algebraic it is immediate from Bernig and Lytchak [7], since, as they point out, the usual tangent cone (which they call the subanalytic tangent cone) is the metric cone with respect to the outer (Euclidean) metric. □

Recall that a partition $\{X_i\}_1^k$ of X is called an *L-stratification* if each X_i is a Lipschitz manifold and for each X_i and for each pair of points $x_1, x_2 \in X_i$ there exist two neighborhoods U_1 and U_2 and a bi-Lipschitz homeomorphisms $h\colon U_1 \to U_2$ such that for each X_j one has $h(X_j \cap U_1) = X_j \cap U_2$. An L-stratification is called *canonical* if any other L-stratification can be obtained as a refinement of this one. In [1] it is proved, by a slight modification of Parusinski's Lipschitz stratification [17], that any semialgebraic set admits a canonical semialgebraic L-stratification. The collection of k-dimensional strata of the canonical L-stratification of X is called the *k-dimensional L-locus of X*. By Theorem 5.2, the metric tangent cone of a semialgebraic set admits a canonical L-stratification.

Let $M \subset \mathbb{R}^n$ be a semialgebraic subset of the unit sphere centered at the origin $0 \in \mathbb{R}^n$. Let $C(M)$ be the straight cone over M with the vertex at the origin $0 \in \mathbb{R}^n$. We say that a subset is a *separating subcone* of $C(M)$ if:

- it is a straight cone over a closed subset $N \subset M$ with vertex at the origin $0 \in \mathbb{R}^n$;
- $M \setminus N$ is not connected.

Example. Consider the Brieskorn singularity defined by:

$$X(a_1,\dots,a_n) := \{(z_1,\dots,z_n)\in\mathbb{C}^n \mid z_1^{a_1}+\dots+z_n^{a_n}=0,,$$

with $a_1=a_2=a\geq 2$ and $a_k>a$ for $k>2$. The tangent cone at the origin is the union of the a complex hyperplanes $\{z_1=\xi z_2\}$ with ξ an a-th root of -1. These intersect along the $(n-2)$–plane $V=\{z_1=z_2=0\}$. Thus, V is a separating subcone of the tangent cone $T_0X(a_1,\dots,a_n)$. The following theorem shows that V is the tangent cone of a separating set in $X(a_1,\dots,a_n)$ (a special case of this is again the A_k surface singularity for $k>1$).

Theorem 5.4. *Let X be an n–dimensional closed semialgebraic set and let $x_0\in X$ be a point such that the link of X at x_0 is connected and the n–density of X at x_0 is positive. Any semialgebraic separating subcone of codimension ≥ 2 in the tangent cone $T_{x_0}X$ contains the tangent cone of a separating set of X at x_0.*

Proof. As usual we can suppose that the point x_0 is the origin. Recall $rB(0)$ means the ball of radius r about 0. Observe that the function

$$f(r)=d_{\text{Hausdorff}}(T_0X\cap rB(0), X\cap rB(0))$$

is semialgebraic, continuous and $f(0)=0$. By the definition of the tangent cone one has $f(r)=ar^\alpha+o(r^\alpha)$ for some $a>0$ and $\alpha>1$.

For a semialgebraic set $W\subset\mathbb{R}^N$ with $0\in W$, let $U_W^{c,\alpha}$ be the α-horn like neighborhood of W, defined by:

$$U_W^{c,\alpha}=\{x\in\mathbb{R}^N \mid d_e(x,W)<c\|x\|^\alpha\}.$$

For some $c>0$ and sufficiently small $r>0$ one has $X\cap rB(0)\subset U_{T_0X}^{c,\alpha}\cap rB(0)$. We fix this r and replace X by $X\cap rB(0)$, so $X\subset U_{T_0X}^{c,\alpha}$

Let $Y\subset T_0X$ be a semialgebraic separating subcone of codimension ≥ 2. We then have a partition

$$T_0X=A\cup Y\cup B\,,$$

where A and B are disjoint open subsets of T_0X of positive n–density. We can assume $\bar A\cap\bar B=Y$ (if not, replace A by $A\cup(Y\setminus(\bar A\cap\bar B))$). Then $U_{T_0X}^{c,\alpha}=U_A^{c,\alpha}\cup U_B^{c,\alpha}$, so $X\subset U_A^{c,\alpha}\cup U_B^{c,\alpha}$. Let $Z=\overline{U_A^{c,\alpha}}\cap\overline{U_B^{c,\alpha}}$ and $Y'=X\cap Z$. Then $X\setminus Y'$ is the disjoint union of the open sets $A':=(\overline{U_A^{c,\alpha}}\cap X)\setminus Z$ and $B':=(\overline{U_B^{c,\alpha}}\cap X)\setminus Z$.

Now $T_0U_A^{c,\alpha}=\overline{A}$ and $T_0U_B^{c,\alpha}=\overline{B}$, so $T_0Z\subset\overline{A}\cap\overline{B}=Y$ so $T_0Y'\subset Y$. It follows that $T_0(A')=\overline{A}$ and $T_0(B')=\overline{B}$, so A' and B' have positive n–density. Thus $\partial Y'$ separates X into open sets of which at least two have positive n-density. Moreover $\partial Y'$ is an $(n-1)$–dimensional semialgebraic set (if c is chosen generically) and, since its tangent cone has dimension $\leq(n-2)$, its $(n-1)$–density is zero ([15]). So $\partial Y'$ is a separating set. $\square$

Remark 5.5. The Briançon-Speder example X_t presented in Section 4 has tangent cone equal to the yz–plane, which is nonsingular and thus does not have separating subcone in codimension 2, but X_t nevertheless has a semialgebraic separating set at 0 if $t\neq 0$.

Proposition 5.6. *Let X be a n-dimensional closed semialgebraic set and let $x \in X$ be a point such that the link of X at x is connected and the n-density of X at x is positive. If X is normally embedded and has a semialgebraic separating set at x, then the tangent cone T_xX contains a semialgebraic separating subcone of codimension ≥ 2.*

Proof. Suppose that X has a separating set $Y \subset X$ at x. Let $A, B \subset X$ such that

a. $A \cap B = \{x\}$;
b. $X \setminus Y = A \setminus \{x\} \cup B \setminus \{x\}$;
c. the n-densities of A and B at x are positive.

Recall the following notation:

$$S_xZ = \{v \in T_xZ \mid |v| = 1\}.$$

So $C(S_xZ) = T_xZ$. Since the $(n-1)$-density of Y at x is equal to zero, S_xY has codimension at least two in S_xX. Let us show that $S_xX \setminus S_xY$ is not connected. If $S_xX \setminus S_xY$ were connected, then $(S_xA \setminus S_xY) \cap (S_xB \setminus S_xY) \neq \emptyset$. Let $v \in (S_xA \setminus S_xY) \cap (S_xB \setminus S_xY)$. Since $v \in S_xA$ and $v \in S_xB$, there exist two semialgebraic arcs $\gamma_1 \colon [0, r) \to A$ and $\gamma_2 \colon [0, r) \to B$ such that

$$|\gamma_i(t) - x| = t \text{ and } |\gamma_i(t) - x| = t \ \forall\ t \in [0, r)$$

and

$$\lim_{t \to 0^+} \frac{\gamma_1(t) - x}{t} = v = \lim_{t \to 0^+} \frac{\gamma_2(t) - x}{t}.$$

Since $\gamma_1(t)$ and $\gamma_2(t)$ belong to different components of $X \setminus Y$, any arc in X connecting $\gamma_1(t)$ to $\gamma_2(t)$ meets Y. That is why

$$d_X(\gamma_1(t), \gamma_2(t)) \geq d_X(\gamma_1(t), Y).$$

Since X is normally embedded, we conclude that

$$\lim_{t \to 0^+} \frac{d_e(\gamma_1(t), Y)}{t} = 0.$$

Thus, $v \in S_xY$.

Finally, the n-densities of $T_xA \setminus T_xY$ and $T_xB \setminus T_xY$ are positive (e.g., by [7, Proposition 1.2]), so the proof is complete. □

Theorem 5.7. *Let X be a closed semialgebraic set and let $x \in X$ be a point such that the link of X at x is connected. Then X has a semialgebraic separating set at x if, and only if, the metric tangent cone $\mathcal{T}_xX$ is separated by an L-locus of codimension ≥ 2.*

Proof. This follows directly from Proposition 5.5 and Proposition 5.7. □

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