Massey Products, the Lower Central Series, and the Poincaré Conjecture

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Let's start with the Hept line:

\[ L: \quad \bigcirc \bigcirc \quad \text{One way to detect linkedness is to use the cup product in } H^*(S^3 - L). \]

Lefschetz duality: \[ H^1(S^3 - L, \mathbb{Z}) \cong H_2(S^3 - L; \mathbb{Z}) = \langle \alpha, \beta \rangle. \]

For an unlinked pair of circles, \[ \alpha \cup \beta = 0. \]

But in \( L \):

\[ Y = \alpha \cup \beta \in H^1(S^3 - L, \mathbb{Z}) \]

And \( Y \) is nontrivial:

\[ yuT = \pm 1, \text{ where } T \in H_2(S^3 - L; \mathbb{Z}). \]
Next consider the Borromean link:

As similar analysis as in the case of \( L \) shows that the cap product pairing in \( H^*(S^3-B; \mathbb{Z}) \) is trivial.

But it turns out that "higher multiplicative structures" on \( H^*(S^3-B; \mathbb{Z}) \) do not vanish. This is the notion of a Massey Product. For simplicity, we'll only define the special case that we need.

**Def.** Let \((C^*, \partial)\) be a cochain complex with homology \( H^* \).

Suppose \( \alpha, \beta, \gamma \in H^1 \) satisfy \( \partial \alpha \beta = \partial \beta \gamma = \beta - \alpha = 0 \).

Pick \( \alpha, \beta, \gamma \in C^1 \) representing \( \alpha, \beta, \gamma \). Then by assumption, \( \partial (\alpha \beta \gamma) = 0 \).

Observe that \( S(\alpha \beta + \alpha \gamma) = 8 \alpha \beta \gamma - \alpha \beta \gamma - \alpha \gamma - \alpha \beta \gamma = (\alpha \beta \gamma) - \alpha \beta \gamma = 0 \).

So \( \alpha \beta \gamma \) determines a cohomology class.

As always, we must be cognizant of the choices made.

It turns out \([\alpha \beta \gamma]\) does depend on the choices of \( x, y \). (But not on choices of \( a, b, c, \ldots \)).

The **Massey triple product** \( \langle \alpha | \beta | \gamma \rangle = \frac{\partial}{\partial x, y} \int_{\alpha \beta \gamma} \)

as \( x, y \) range over all such choices.
Now let $\alpha, \beta, \gamma \in H_2(S^3 \setminus B, B) \cong H^1(S^3 \setminus B)$ be spanning disks for the components of $B$.

As remarked, $\alpha \cup \beta \cup \gamma = \gamma \cup \alpha = 0$. So let's compute $<\alpha, \beta, \gamma>$:

Voila:

$$3 = <\alpha, \beta, \gamma>.$$
So far, the LCS has been absent from our discussion.
To incorporate it, we'll have to frame a form into group cohomology.

\[ \text{Crash Course on } H^1(G, \mathbb{Z}), \ H^2(G, \mathbb{Z}) \]

**Def.** \[ H^2(G, \mathbb{Z}) := H^2(K(G,1), \mathbb{Z}). \]

- So \( H^1(G, \mathbb{Z}) = H^1(K(G,1), \mathbb{Z}) = \text{Hom}(G, \mathbb{Z}). \)

Recall \( H^2(X, \mathbb{Z}) = [X, K(\mathbb{Z},2)] = [X, \text{CP}^\infty]. \)

And \( \text{CP}^\infty = \text{Gr}(1, \mathbb{C}) \) parametrizes \( \mathbb{C} \)-line bundles.

Let \( x \in H^2(X, \mathbb{Z}) \) be such a bundle, giving rise to \( \mathbb{S}^1 \to X \to X_a \)

by taking unit circle bundle of associated \( \mathbb{C} \)-bundle. \( X \)

This gives rise to SES of groups \( 1 \to \mathbb{Z} \to \tilde{G}_x \to G \to 1 \)

(when \( X = K(G,1) \)). And \( Z = \pi_1 \mathbb{S}^1 \) is central.

**Fact:** This is a correspondence:

\[ H^2(G, \mathbb{Z}) \leftrightarrow \{ \text{Equivalence classes of central } \mathbb{Z} \text{-extensions} \}
\]

Our goal now will be to attempt to understand cup \( i \)-Milnor products in this setting.

(4)
Start with the cup product.

I want to define a map $H^1(G, \mathbb{Z}) \otimes H^1(G, \mathbb{Z}) \to H^2(G, \mathbb{Z})$.

Let $\alpha, \beta \in H^0(G, \mathbb{Z})$, i.e. $\alpha: G \to \mathbb{Z}$, $\beta: G \to \mathbb{Z}$ are homs.
Then $\alpha \times \beta: G \to \mathbb{Z}^2$ is given.

Thinking cohomologically, I can use this to pull back the "fundamental class" (i.e. generator) of $H^2(\mathbb{Z}^2, \mathbb{Z})$.

As a group extension:

$$
\begin{array}{ccc}
1 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
1 & \to & \mathbb{Z} \to \mathbb{Z}^2 \\
\downarrow & & \downarrow \\
1 & \to & \mathbb{Z} \to \widehat{G}_{\times \beta} \to G \to 1
\end{array}
$$

What group extension is this? It's the Heisenberg group!

Now let's construct Massey products. Suppose $\alpha, \beta \in H^1(G, \mathbb{Z})$ satisfy $\alpha \times \beta = 0$.

This means the following:

$$
\begin{array}{ccc}
1 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
1 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
1 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
1 & \to & \widehat{G}_{\times \beta} \to G \to 1
\end{array}
$$

There exists a section! $\uparrow$.

So we obtain a map $\sigma: G \to H$, $g \mapsto \left( \begin{smallmatrix} 1 & \alpha(g) & \beta(g) \\ 0 & 1 & \beta(g) \\ 0 & 0 & 1 \end{smallmatrix} \right)$.

Similarly, if $\beta \times \gamma = 0$, there is a map $G \to H$, $g \mapsto \left( \begin{smallmatrix} 1 & \beta(g) & \alpha(g) \\ 0 & 1 & \alpha(g) \\ 0 & 0 & 1 \end{smallmatrix} \right)$.

(5)
Now let \( \tilde{N} \) be the nilpotent group \( \{ (1_0 a b c), a, b, c \in \mathbb{Z} \} \).

\[
\mathbb{Z}(\tilde{N}) = \mathbb{Z} = \{ (1_{10} 0_{01}^{0}) \}, \quad \text{and} \quad N = \tilde{N}/\mathbb{Z}(\tilde{N}) = \{(1_{10} a b c) \}.
\]

There is a central extension
\[
1 \to \mathbb{Z} \to \tilde{N} \to N \to 1.
\]

Now note that the maps \( g \mapsto \begin{pmatrix} 1 & \alpha(0) & \xi(0) \\ 1 & \beta(0) & \omega(0) \\ 1 & \gamma(0) & 1 \end{pmatrix} \) \( g \mapsto \begin{pmatrix} 1 & \beta(0) & \omega(0) \\ 1 & \gamma(0) & 1 \end{pmatrix} \)

can be combined to give a map \( G \to N \),

\[
g \mapsto \begin{pmatrix} 1 & \alpha(0) & \xi(0) \\ 1 & \beta(0) & \omega(0) \\ 1 & \gamma(0) & 1 \end{pmatrix} \text{ Thus we obtain a class } \langle \alpha, \beta, \gamma \rangle \in H^2(G, \mathbb{Z}) \text{ associated to the pullback of } 1 \to \mathbb{Z} \to \tilde{N} \to N \to 1.
\]

This is the Massey product in group cohomology!

This also gives an indication of how to define Massey products of higher orders, and how this is related to \( n \)-step nilpotent quotients of \( G \).
Poincaré Conjecture:

Glib statement: Any simply-connected homology sphere is $S^3$ itself.

Let's study the fundamental groups of homology spheres.

Key fact: Every homology sphere is constructed by taking the standard Heegard splitting for $S^3$, and post-composing the gluing map $\Sigma g \to \Sigma g$ by some $\phi \in I_g$. $S^g = S^{\phi}$

- $\phi$ and $\psi \in I_g$, give rise to the same $M$ if $\Sigma g \phi = 3^i \phi^3 z_i$, for $z_i \in N_g$, (handle body group).
- $S^g = \langle a_i, \ldots, a_g | P(\phi(a_i)) \rangle$, with $P : \Sigma g \to F_g = \langle a_i, \ldots, a_g \rangle$.

This suggests an approach to P.C.: Show that if $P_{\alpha}(\phi) = \{1\}$, then $\phi \in N_g$.

To do this, we need to get a handle on when $\{ P(\phi(a_i)) \}$ give rise to a set of relations that don't generate the trivial group.

This is where the Johnson filtration comes in.

$\text{Mod } \Sigma g = I_g(0) \supset I_g(1) = I_g(2) \supset I_g(3) \supset I_g(4) \supset \cdots$

There are homomorphisms $T_k : I_g(k) \to \text{Hom}(H_1 F_g, \mathbb{Z}/p)$, which measure the action of $\phi \in I_g(k)$ on the $k$th nilpotent truncation of $\pi_1 \Sigma g$. 

\[(7)\]
These $T_z$'s can also be defined in terms of Massey products in the mapping torus $M_\phi$.

So here's how a proof of PC would go in this framework:

- Identity, for all $z$,\[ \text{im}(\overline{T_z}(N_{g_1} \cap I_{g_1}(z))) \subseteq W_z \subseteq \text{Hom}(H, \mathcal{F}_{g_2}(\mathbb{F}_{g_1})) \]
- Identify the entire image $V_z = \text{im}(\overline{T_z})$
- Show that if $\phi \in I_{g_1}(z)$ satisfies $T_z(\phi) \notin W_z$, then $\Pi_1(S_\phi) = 3$.\[ \phi \]

Together this shows that any counterexample to PC
lies in $\cap_{g_1 \neq g_2} \left( \bigcap I_{g_1}(z) \right) = N_{g_1}$ and so isn't a counterexample at all.

Example: Step 1. ($z = 2$) (Johnson)

Fact 1: $F^{(3)} / \mathcal{F}_{g_1}^{(2)} \subseteq \Lambda^2 H$, and $\text{im} \overline{T_z} \subseteq \text{Hom}(H, \Lambda^2 H) \cong \Lambda^3 H$.

Fact 2: (Hir. &) $\text{im}(\overline{T_z}(N_{g_1} \cap I_{g_1})) = W_y = \langle x_i \Lambda x_j \Lambda x_k, x_i \Lambda y_j \Lambda y_k \rangle$

So if $T(y) \notin W_y$, it contains a sum of the form $x_i \Lambda x_j \Lambda x_k$.

Now by unraveling the definition of $T_z$, it is possible to show that in these circumstances, $\Pi_1(S_\phi) = 3$.

Sketch: $x_i \Lambda x_j \Lambda x_k < m > p(\phi(x_i)) = x_i(x_j \Lambda x_k) \mod \mathcal{F}_{g_1}^{(3)}$ (also 1.2).

Now write in $F_{g_2} / \mathcal{F}_{g_1}^{(3)}$ to see that it is impossible to write $x_i$ as a word in $\mathcal{F}_{g_1}^{(3)}(\phi(x_i))$.\[ (8) \]