4-manifolds can be surface bundles over surfaces in many ways.


* Preliminaries

- A surface bundle is a fiber bundle $\Sigma_g \to E \to B$

  Typically we like $g \geq 2$ and for $B$ to be a manifold ($\Rightarrow E$ as well)

- The theory of surface bundles has deep interactions with topology in dimensions 5, 2, 3, 4, as well as with algebraic, complex, symplectic, hyperbolic geometry, and geometric group theory.

* A (very) brief tour of fibered 3-manifolds

Recipe for 3-manifolds: Take $\phi \in \text{Diff}(\Sigma_g)$, form mapping torus

$$M^3 = \Sigma_g \times I / (x, 0) = (\phi(x), 0).$$

After Agol, every closed hyperbolic 3-manifold has a finite-sheeted cover of this form!
One basic feature of the theory is: it often is the case that $\phi$ is far from unique: distinct $\phi \in \text{Diff}^+ \Sigma_g, \psi \in \text{Diff}^+ \Sigma_h$ can be made diffeomorphic $M_\phi \simeq M_\psi$.

**Theorem (Thurston)** $\text{H}_2(M^3)$ admits a $\mathbb{Q}$-decomposition into finitely many rational cones. If some $x \in \text{H}_2(M^3, \mathbb{Z})$ corresponds to the fiber of a fibration over $S^1$, then every $y \in \text{H}_2(M^3, \mathbb{Z})$ also does. Thus $M$ is realized as a surface bundle over $S^1$ with fiber of genus $g$ for in finitely many $g$, but finitely many for each genus.

**Backround on SBS with multiple fiberings**

- Move up a dimension, and consider surface bundles over surfaces (SBS). $\Sigma_g \to E \xrightarrow{\text{Theory}} \Sigma_h$. Theory is richest when $h > 2$.

- Here there's no obvious generalization of the Thurston norm.
Theorem (FEA Johnson) Every 4-mfd $E$ has only finitely many distinct $\Sigma_g \to E$ with $g, h \geq 2$.

Here "distinct" means one of two equivalent things:
1. Distinct $\Sigma_g \subseteq \Sigma_h$ with $\pi_1 \Sigma_g \cong \pi_1 \Sigma_h$
2. Fibers $F_1 \to E$, $F_2 \to E$ have nonempty transverse intersection.

The obvious question: How many fibrings can a single $E^n$ admit?
More precise version: Define

$$N(d) = \max \left\{ \text{Number of distinct fibrings of } E^n, \chi(E) \leq 4d^2 \right\}$$

FEAJ's argument shows $N(d) \leq \Theta(d)(d+1)^{2d+6}$

(\Theta(d): Some of divisors function)

Problem: Find lower bounds on $N(d)$
- Products $\Sigma_g \times \Sigma_h$, as well as the Atiyah-Kodaira construction to be described shortly, show $2 \leq N(d)$.
For a long time, \( 2 \leq N(d) \leq \omega_0(d)(d+1)^{2d+6} \) were the best-known bounds.

I spent a long time trying to prove this was sharp: \( N(d) = 2 \).

Culminating theorem:

**Thm (S-)**: Let \( \Sigma \to E \) be SBS, with monodromy \( \rho: \pi_1 \Sigma \to K_3 \)

contained in the subgroup \( K_3 \leq \Sigma \) generated by all commuting Dehn twists. Then either \( E \cong \Sigma \times \Sigma \) is the trivial bundle, or else \( E \) has a unique surface bundle structure.

I presented this theorem at GTC in May (Thanks, Dan!) That afternoon, I found the first example of a SBS with \( >2 \) fiberings. The culmination of this is the following improvement in the lower bound on \( N(d) \):

**Thm (S-)**: \( 2^{(d+2)/6} \leq N(d) \leq \omega_0(d)(d+1)^{2d+6} \)

For the experts, the monodromy of these examples live in \( K_3 \).

Corollary: my theorem above is sharp wrt the Johnson filtration.
How do you make interesting surface bundles?

- For the remainder of the talk, I want to show you some methods for building SBS.

- The Atiyah-Kodaira construction:

  Based around the branched cover construction:

  A branched cover of surfaces is a map \( f: \Sigma_g \to \Sigma_h \) such that

  \( f \) is a covering map when finitely many pts are deleted from \( \Sigma_g \).

  Better to think of \( f \) as a quotient by a group action \( G/\Sigma_g \).

  Will fin. many fixed pts:

  \[ \begin{array}{c}
  \Sigma_g \\
  \downarrow \quad f \quad \downarrow \\
  \Sigma_h \\
  \end{array} \]

  A-K construction proceeds by taking an interesting fiber-wise branched cover of a product:

  \[ \begin{array}{c}
  \Sigma_g \\
  \downarrow \quad f \quad \downarrow \\
  \Sigma_h \\
  \end{array} \]

  - To specify branched cover, need to specify

    \[ \begin{array}{c}
    \Sigma_g \\
    \downarrow \quad f \quad \downarrow \\
    \Sigma_h \\
    \end{array} \]

    fiber over \( z \), the branch points. "Interesting" here means that branch points move round as fibers change.

    \[ \begin{array}{c}
    \Sigma_g \\
    \downarrow \quad f \quad \downarrow \\
    \Sigma_h \\
    \end{array} \]

    - Simplest example requires two distinct points in each fiber. One pt: Place at \( z \) in the fiber over \( z \). This is the diagonal section \( \Delta: \Sigma \to \Sigma \times \Sigma \).

    \[ \begin{array}{c}
    \Sigma_g \\
    \downarrow \quad f \quad \downarrow \\
    \Sigma_h \\
    \end{array} \]

    - If \( \Sigma \) is equipped with a free involution \( \tau: \Sigma \to \Sigma \) then \( \tau(z) \) and \( z \) are always different points.

(5)
Simplest $\Sigma: \Sigma_3$. In summary:

$\Sigma_3 \xrightarrow{\Delta} \Sigma_3$.

Want to take branched covers here. To do so, we need to remove $B$ and enlarge the base $B'$ to $B''$.

$$
\begin{array}{c}
\Sigma_{12q_1} \times \Sigma_3 \\
\downarrow \\
\Sigma_{12q_1} \\
\end{array}
\rightarrow
\begin{array}{c}
\Sigma_3 \times \Sigma_3 \\
\downarrow \\
\Sigma_3 \\
\end{array}
$$

Fiber of $E \rightarrow \Sigma_{12q_1} \times \Sigma_3 \rightarrow \Sigma_{12q_1}$ has genus 6, since we took a fiberwise double branched cover.

But also! $E \rightarrow \Sigma_{12q_1} \times \Sigma_3 \rightarrow \Sigma_3$ is a fibering, with fiber genus 321.

$E$ has amazing properties which have been important in 4-fold toric, alg. geom, complex geom, symplectic geom.
Now I'll explain my construction.

**Section sum:** Suppose $E_1 \to E_1 \downarrow \Sigma_1 \quad \Sigma_1 \to \Sigma_2 \downarrow \Sigma_2$

with $\sigma(\Sigma_1) \subset \sigma(\Sigma_2)$. Then $E_1 \setminus \Sigma_1$ can be glued to $E_1 \setminus \Sigma_1$. 

The result is a fibre-wise connect-sum.

The point of attachment in each fibre is specified by $\sigma_1, \sigma_2$.

\[
E_1 \begin{array}{c}
\sigma_1(x) \rightarrow \\
\circ \\
\end{array}
\quad \Rightarrow \quad \\
E_2 \begin{array}{c}
\sigma_2(x) \rightarrow \\
\circ \\
\end{array}
\]

**Example 1:** $\Sigma_2$ \hspace{1cm} $\Delta$: a single section. Do the section sum along $\Delta$.

It turns out that both horizontal \& vertical fibrings can be extended over the cylinder bundle, so that all fibrings are preserved.

Monodromy is a "cylinder drag."

Explain common philosophy:

Perform surgery operations fibrewise, using an interesting section to parameterize necessary data.

(?)
Four fibrations: 

\[ \text{(Appeal to } M_{12}\text{ condition)} \]

Pt of theorem: We'll move \( E_1^n \) with \( \chi(E_1^n) = 24n \cdot 8 \) admittng 2\(^n\) distinct fibrations

Stack! Need two distinct plans to attach: \( T \oplus \Sigma_2 \) shows up again. 2 choices (H or V) in \( n \) components

\[ \Rightarrow 2^n \text{ maps to } \Sigma_3. \]

And \( \chi(E_1^n) = \chi(\text{Base}) \chi(\text{Fiber}) \]

\[ -4 \quad 2 \cdot 6n. \]

Examples over bases of different genera:

Tame covers \[ \Sigma \xrightarrow{f_1} \Sigma_1 \]

\[ \Sigma \xrightarrow{f_2} \Sigma_2 \]

Build, \[ \Sigma \]

attatch \[ \Sigma \]

\[ \Sigma \]

\[ \Sigma \]

(8)