On the non-realizability of braid groups by diffeomorphisms

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Basic setup

$X$ - topological space.

$\text{Diff}(X)$: group of $C^k$ diffeomorphisms of $X$ (rel. boundary)

$(S \subset X \text{ subset: } \text{Diff}(X,S) \text{ diffeos rel. } S)$

$\text{Diff}_0(X,S)$: diffeos isotopic to id. rel. $S$

We study $\text{Mod}(X,S) = \text{Diff}(X,S)/\text{Diff}_0(X,S)$
Example: $X = \Sigma_g$

$\text{Mod}(\Sigma_g)$: mapping class group of surface

Main concern today: $X = D^2$, $S = \{n \text{ points}\}$

$\text{Mod}(X,S) = B_n$ (braid group on $n$ strands)
Does the map $\pi : \text{Diff}(X, S) \to \text{Mod}(X, S)$ admit a section, i.e. a map

$$\sigma : \text{Mod}(X, S) \to \text{Diff}(X, S)$$

for which $\pi \circ \sigma = \text{id}$?

We say that $\text{Mod}(X, S)$ is (not) realizable by diffeomorphisms if the answer is yes (resp. no).
The case of $\text{Mod}(\Sigma_g)$ is well-understood.

Theorem (S. Morita, 1987):

$\text{Mod}(\Sigma_g)$ is not realizable by $C^2$ diffeomorphisms, for all $g \geq 18$
The idea

Use cohomology.

Existence of section implies a diagram

\[
H^*(\text{Mod}(\Sigma_g)) \xrightarrow{\pi^*} H^*(\text{Diff}(\Sigma_g)) \xrightarrow{\sigma^*} H^*(\text{Mod}(\Sigma_g))
\]

for which \( \sigma^* \circ \pi^* = 1 \)

In particular, \( \pi^* \) is an injection.
Morita constructs an element $e_3 \in H^6(\text{Mod}(\Sigma_g))$

He applies the \textit{Bott vanishing theorem} from foliation theory to show that $\pi^*(e_3) = 0$
Is $B_n$ realizable by diffeomorphisms?

Following Morita, would like to understand

$$\ker \pi^* : H^*(B_n) \to H^*(\text{Diff}(D^2, S))$$

Problem!

Theorem (Nariman, 2015):

$$\pi^* : H^*(B_n) \to H^*(\text{Diff}(D^2, S))$$ is an injection.
Many proofs of Morita’s theorem by now.

Bestvina-Church-Souto:
Milnor-Wood inequality.

Franks-Handel:
Use dynamics to produce fixed points.
Use these fixed points to construct homomorphisms that can’t exist.
This last approach inspires our method.

Theorem (S., Tshishiku, 2015):

For $n \geq 5$ the braid group is not realizable by diffeomorphisms. More generally, the “surface braid group” $B_n(\Sigma^b_g)$ is not realizable by diffeomorphisms for $n \geq 5$ if $b \geq 1$, and for $n \geq 6$ otherwise.
By exploiting the hyperelliptic mapping class group, we obtain a new and quite simple proof of Morita’s theorem in the best possible range:

**Theorem (S., Tshishiku, 2015):**

For all $g \geq 2$, the mapping class group $\text{Mod}(\Sigma_g)$ is not realizable by $C^1$ diffeomorphisms.
Diffeos realizing elements of $B_n$ must fix points in S.

Exploit this to manufacture homomorphisms $f : B_n \rightarrow A$ with A abelian.

Use dynamics/geometry to show these are highly non-degenerate.

Exhibit subgroups $G$ of $B$ with $H^1(G, \mathbb{Z}) = 0$
Derivative map: $D_x : B_n \to GL_2^+(\mathbb{R})$

Analyze centralizers: image must be abelian.

If trivial, use *Thurston stability*:

Theorem (Thurston, 1974):

*Let $G$ be f.g. acting via $C^1$ diffeomorphisms on $\mathbb{R}^n$ with a global fixed point $x$. If $D_x : G \to GL_n(\mathbb{R})$ is trivial, there is a map $f : G \to \mathbb{R}$ with nontrivial image.*
Fact: For $n \geq 5$, the commutator subgroup $[B_n, B_n]$ is finitely generated, and $H^1([B_n, B_n], \mathbb{Z}) = 1$

Proof: Just do it! Vaguely reminiscent of other arguments in the theory of diffeomorphism groups.
To prove Theorem 2, exploit the map

\[ f : B_{2g+2} \rightarrow \text{Mod}(\Sigma_g) \]
Outline of proof (II)

To use previous methods: need fixed points

\( \text{im}(f) \) commutes with hyperelliptic involution \( \iota \)

\( \sigma(\iota) \) has \( 2g+2 \) fixed points (Lefschetz)
Basic principle of group actions:
$\text{im}(f)$ preserves these points.

Obtain map $\phi : B_{2g+2} \rightarrow S_{2g+2}$

We show this map has to be the standard one.

Thus we can find fixed points, proceed as before.
Questions

- Study other subgroups of mapping class groups. When is a general $G \leq \text{Mod}(\Sigma_g)$ realizable by diffeomorphisms?
- In particular, are there surface subgroups of $\text{Mod}(\Sigma_g)$ that are not realizable? 3-manifold subgroups?
- Are there dynamical approaches to other lifting problems, e.g. for finding a section of
  \[
  \text{Mod}(\Sigma_{g,*}) \to \text{Mod}(\Sigma_g)
  \]
Thanks!